The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 2, Issue 1, Article 6, pp. 1-12, 2005

ON A CRITERIA FOR STRONG STARLIKENESS

V. RAVICHANDRAN, M. DARUS, AND N. SEENIVASAGAN

Received 4 October, 2004; accepted 3 February, 2005; published 4 March, 2005.
Communicated by: N. Tuneski

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITI SAINS MALAYSIA, 11800 USM PENANG, MALAYSIA
vravi@cs.usm.my
URL: http://cs.usm.my/~vravi

SCHOOL OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCES AND TECHNOLOGY, UKM, BANGI 43600, MALAYSIA
maslina@pkrisc.cc.ukm.my

SINDHI COLLEGE, 123, P. H. ROAD, NUMBAL, CHENNAI 600 077 INDIA
vasagan2000@yahoo.co.in

ABSTRACT. In this paper, we are concerned with finding sufficient condition for certain normalized analytic function $f(z)$ defined on the open unit disk in the complex plane to be strongly starlike of order α. Also we have obtained similar results for certain functions defined by Ruscheweyh derivatives and Sălăgean derivatives. Further extension of these results are given for certain p-valent analytic functions defined through a linear operator.

Key words and phrases: Analytic functions, Starlike functions, Strongly starlike function, Subordination, Ruscheweyh derivative, Sălăgean derivative, Hadamard product (or Convolution), Linear operator.

2000 *Mathematics Subject Classification.* Primary 30C45, Secondary 30C80.

ISSN (electronic): 1449-5910

The authors are thankful to the referees for their comments. The research of V. Ravichandran and M. Darus are respectively supported by a post-doctoral research fellowship from Universiti Sains Malaysia and a grant (09-02-02-10029EAR) from IRPA, Malaysia.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of all *analytic* functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

defined on $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For two functions f and g analytic in Δ, we say that the function $f(z)$ is *subordinate* to $g(z)$ in Δ, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g\big(w(z)\big) \quad (z \in \Delta). \tag{1.1}$$

In particular, if the function g is *univalent* in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The class of *starlike functions of order* α, denoted by $S^*(\alpha)$, is defined by

$$S^*(\alpha) := \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha \quad (0 \leq \alpha < 1) \right\}$$

and the class of *Janowski starlike functions* is defined by

$$S^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1, z \in \Delta) \right\}.$$

In particular, we have $S^*[1 - 2\alpha, -1] = S^*(\alpha)$. The class $SS^*(\alpha)$ of *strongly starlike functions of order* α consists of functions $f \in \mathcal{A}$ satisfying

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| \leq \frac{\alpha\pi}{2}, \quad (0 < \alpha \leq 1, z \in \Delta)$$

or equivalently we have

$$SS^*(\alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\alpha}, \quad (0 < \alpha \leq 1, z \in \Delta) \right\}.$$

Obradovič and Owa [7], Silverman [16], Obradovič and Tuneski [8] and Tuneski [18] have studied the properties of classes of functions defined in terms of the ratio of

$$1 + \frac{zf''(z)}{f'(z)} \text{ and } \frac{zf'(z)}{f(z)}.$$

Also Ravichandran and Darus [13] have obtianed the following:

Theorem 1.1. *Let $h(z)$ be starlike in Δ and $h(0) = 0$. If $f \in A$ and*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + h(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec \left[1 - \int_0^z \frac{h(\eta)}{\eta} \, d\eta \right]^{-1}.$$

They have also studied similar problem for classes defined by Ruscheweyh derivatives and Sălăgean derivatives. Note that the *convolution* of two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The *Ruscheweyh derivative* of order $\delta > -1$ is defined by

$$D^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}.$$

The *Sălăgean derivative* of a function $f(z)$, denoted by $\mathcal{D}^m f(z)$ is defined by

$$\mathcal{D}^m f(z) = f(z) * (z + \sum_{n=2}^{\infty} n^m a_n z^n).$$

It is also easy to see that $\mathcal{D}^0 f(z) = f(z)$, $\mathcal{D}^1 f(z) = zf'(z)$ and $\mathcal{D}^n f(z) = z(\mathcal{D}^{n-1} f(z))'$.

Li and Owa [2], Lewandowski, Miller and Zlotkiewics [1] and Ramesha, Kumar, and Padmanabhan [11], Li and Owa [2] and Ravichandran et al. [12] have considered sufficient conditions for starlikeness in terms of $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$.

Ravichandran[14] have proved the following:

Theorem 1.2. *If $q(z)$ is convex univalent and $0 < \alpha \le 1$,*

$$\operatorname{Re}\left\{(1-\alpha)/\alpha + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0$$

and

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec (1-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z),$$

then $\frac{zf'(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

In this paper, we are concerned with finding sufficient condition for $f(z) \in \mathcal{A}$ to be strongly starlike of order α in terms of the argument of either the ratio $[zf'(z)/f(z)]/[1 + zf''(z)/f'(z)]$ or $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$. Also we have obtained similar results for certain functions defined by Ruscheweyh derivatives and Sălăgean derivatives. Further extension of these results are given for certain p-valent analytic functions defined through a linear operator.

In our present investigation, we need the following results:

Lemma 1.3. [13] *Let $h(z)$ be starlike in Δ and $h(0) = 0$. If $p(z)$ is analytic in Δ, $p(0) = 1$ and*

$$\frac{zp'(z)}{p(z)^2} \prec \frac{zq'(z)}{q(z)^2} = h(z),$$

then

$$p(z) \prec q(z) = \left[1 - \int_0^z \frac{h(\eta)}{\eta}\, d\eta\right]^{-1}.$$

In fact, we need only the following special case of Lemma 1.3 in our present investigation:

Lemma 1.4. *If $p(z)$ is analytic in Δ, $p(0)=1$ and*

$$\frac{zp'(z)}{p(z)^2} \prec \frac{2\alpha z}{(1+z)^{1+\alpha}(1-z)^{1-\alpha}} \quad (0<\alpha\leq 1),$$

then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0<\alpha\leq 1).$$

Lemma 1.5. (cf. Miller and Mocanu [3, Theorem 3.4h, p.132]) *Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$ with $\varphi(w)\neq 0$ when $w\in q(\Delta)$. Set*

$$Q(z) := zq'(z)\varphi(q(z)), \quad \text{and} \quad h(z) := \vartheta(q(z)) + Q(z).$$

Suppose that either

(1) *$h(z)$ is convex, or*
(2) *$Q(z)$ is starlike univalent in Δ.*

In addition, assume that

$$\Re\frac{zh'(z)}{Q(z)} > 0 \text{ for } z\in\Delta.$$

If $p(z)$ is analytic with $p(0)=q(0)$, $p(\Delta)\subseteq D$ and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.2}$$

then

$$p(z) \prec q(z) \tag{1.3}$$

and $q(z)$ is the best dominant.

2. A SUFFICIENT CONDITION FOR STRONG STARLIKENESS

By appealing to Lemma 1.4, we first prove the following:

Lemma 2.1. *Let $0<\alpha<1$. Let $0<\beta<1$ be given by*

$$\tan\left(\frac{\beta\pi}{2}\right)\left[\frac{\alpha}{1-\alpha}\sin\left(\frac{\alpha\pi}{2}\right) + \left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1+\alpha}{2}}\right] = \frac{\alpha}{1-\alpha}\cos\left(\frac{\alpha\pi}{2}\right). \tag{2.1}$$

Let $p(z)$ be analytic in Δ and satisfies

$$1+\frac{zp'(z)}{p(z)^2} \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Proof. Let the function $h(z)$ be defined by

$$h(z) := 1+\frac{2\alpha z}{(1+z)^{1+\alpha}(1-z)^{1-\alpha}}. \tag{2.2}$$

In view of Lemma 1.4, it is enough to show that the sector $|\arg w| < \frac{\beta\pi}{2}$, where β is given by (2.1), is contained in $h(\Delta)$. We first analyze the image of the unit circle $|z|=1$ under the mapping $h(z)$. For this purpose, let $z=e^{i\theta}$, $0\leq\theta\leq 2\pi$. Then

$$\frac{1+z}{1-z} = it \tag{2.3}$$

where $t = \cot(\theta/2)$. Since the function $h(z)$ has real coefficient and hence $h(\Delta)$ is symmetric with respect to real axis, it is enough to consider the case where $t \geq 0$. A computation shows that

$$\frac{z}{1-z^2} = \frac{i(1+t^2)}{4t}. \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), we have

$$\begin{aligned} h(e^{i\theta}) &= 1 + \frac{\alpha}{2}(1+t^2)t^{-(1+\alpha)}e^{(1-\alpha)\frac{\pi}{2}i} \\ &= 1 + \frac{\alpha}{2}(1+t^2)t^{-(1+\alpha)}\sin\left(\alpha\frac{\pi}{2}\right) + i\frac{\alpha}{2}(1+t^2)t^{-(1+\alpha)}\cos\left(\alpha\frac{\pi}{2}\right). \end{aligned} \tag{2.5}$$

From the equation (2.5), we have

$$\arg h(e^{i\theta}) = \arctan\left(\frac{\frac{\alpha}{2}(1+t^2)\cos\left(\alpha\frac{\pi}{2}\right)}{t^{1+\alpha} + \frac{\alpha}{2}(1+t^2)\sin\left(\alpha\frac{\pi}{2}\right)}\right). \tag{2.6}$$

Define the function $\phi(t)$ by

$$\phi(t) := \frac{\frac{\alpha}{2}(1+t^2)\cos\left(\alpha\frac{\pi}{2}\right)}{t^{1+\alpha} + \frac{\alpha}{2}(1+t^2)\sin\left(\alpha\frac{\pi}{2}\right)}. \tag{2.7}$$

A simple calculation shows that the function $\phi(t)$ attains its extremum at the roots of the equation

$$2t\left[t^{1+\alpha} + \frac{\alpha}{2}(1+t^2)\sin\left(\alpha\frac{\pi}{2}\right)\right] - (1+t^2)\left[(1+\alpha)t^{\alpha} + \alpha t\sin\left(\alpha\frac{\pi}{2}\right)\right] = 0$$

or at

$$t = 0 \text{ and } t = \sqrt{\frac{1+\alpha}{1-\alpha}}.$$

Yet another calculation shows that the minimum of the function $\phi(t)$ is attained at

$$t = \sqrt{\frac{1+\alpha}{1-\alpha}}$$

and the minimum of $\phi(t)$ is

$$\frac{\frac{\alpha}{1-\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}{\frac{\alpha}{1-\alpha}\sin\left(\frac{\alpha\pi}{2}\right) + \left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1+\alpha}{2}}} = \tan\frac{\beta\pi}{2}$$

provided β is given by (2.1). Thus we see that the hypothesis of our Theorem 2.1 implies the hypothesis of Lemma 1.4 and our result now follows from Lemma 1.4. ∎

As an application of our Lemma 2.1, we have the following:

Theorem 2.2. *Let $0 < \alpha < 1$ and β be given by (2.1). If $f \in \mathcal{A}$ satisfies*

$$\left|\arg\left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}\right)\right| < \frac{\beta\pi}{2},$$

then $f \in SS^*(\alpha)$.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$1+\frac{zp'(z)}{p(z)^2}=\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$$

and our result now follows from Lemma 2.1. ∎

Theorem 2.3. *Let* $0<\alpha<1$ *and* β *be given by (2.1). If* $f(z)\in\mathcal{A}$ *satisfies*

$$\left|\arg\left(\frac{(\delta+2)\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}-1}{\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}}-\delta\right)\right|<\frac{\beta\pi}{2},$$

then

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Proof. Define the function $p(z)$ by

$$p(z)=\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}.$$

Clearly $p(z)$ is analytic in Δ and $p(0)=1$. Using the familiar identity

$$z(D^{\delta}f(z))'=(\delta+1)D^{\delta+1}f(z)-\delta D^{\delta}f(z),$$

we have

$$\frac{zp'(z)}{p(z)}=(\delta+2)\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}-(\delta+1)p(z)-1$$

and hence

$$1+\frac{zp'(z)}{p(z)^2}=\frac{(\delta+2)\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}-1}{\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}}-\delta.$$

Our result now follows from Lemma 2.1. ∎

Now we give another result in terms of Sălăgean derivative $\mathcal{D}^m f(z)$:

Theorem 2.4. *Let* $0<\alpha<1$ *and* β *be given by (2.1). If* $f(z)\in\mathcal{A}$ *satisfies*

$$\left|\arg\left(\frac{\mathcal{D}^{m+2}f(z)\mathcal{D}^{m}f(z)}{(\mathcal{D}^{m+1}f(z))^2}\right)\right|\leq\frac{\beta\pi}{2},$$

then

$$\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^{m}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Proof. Define the function $p(z)$ by

$$p(z)=\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^{m}f(z)}.$$

Then a computation shows that

$$\frac{zp'(z)}{p(z)}=\frac{z(\mathcal{D}^{m+1}f(z))'}{\mathcal{D}^{m+1}f(z)}-\frac{z(\mathcal{D}^{m}f(z))'}{\mathcal{D}^{m}f(z)}=\frac{\mathcal{D}^{m+2}f(z)}{\mathcal{D}^{m+1}f(z)}-\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^{m}f(z)}.$$

Therefore

$$1+\frac{zp'(z)}{p(z)^2}=\frac{\mathcal{D}^{m+2}f(z)\mathcal{D}^{m}f(z)}{(\mathcal{D}^{m+1}f(z))^2}.$$

Our result now follows from Lemma 2.1. ∎

3. ANOTHER SUFFICIENT CONDITION FOR STRONG STARLIKENESS

We begin by proving the following:

Lemma 3.1. *Let α, β and γ be positive real numbers and $\rho_0 \in (0,1)$ be the largest root of*

$$\frac{\gamma}{\alpha}\rho_0 = \tan\left(\frac{\rho_0\pi}{2}\right).$$

For $\rho_0 < \rho \leq 1$, let t_0 be the unique root of the equation

$$\beta t^\rho \left[\gamma(1-\rho)\cos\left(\frac{\rho\pi}{2}\right)t^2 + 2\alpha\sin\left(\frac{\rho\pi}{2}\right)t - \gamma(1+\rho)\cos\left(\frac{\rho\pi}{2}\right)\right] + \alpha\gamma(t^2-1) = 0 \tag{3.1}$$

If $p(z)$ is analytic in Δ, $p(0)=1$ and

$$\left|\arg\left(\alpha p(z) + \beta p(z)^2 + \gamma z p'(z)\right)\right| \leq \frac{(1+\rho)\pi}{2} - \arctan\left(\frac{2\alpha t_0}{\gamma[(1+\rho)-(1-\rho)t_0^2]}\right), \tag{3.2}$$

then

$$|\arg(p(z))| \leq \frac{\rho\pi}{2}. \tag{3.3}$$

Proof. Our proof of Lemma 3.1 is essentially similar to the proof of Theorem 1 of Miller and Mocanu [4]. Let the functions $q(z)$ and $h(z)$ be defined by

$$q(z) := \left(\frac{1+z}{1-z}\right)^\rho$$

and

$$h(z) := \alpha q(z) + \beta q(z)^2 + \gamma z q'(z).$$

We first analyze the image of the unit circle $|z|=1$ under the mapping $h(z)$. For this purpose, as in the proof of Theorem 2.1, let $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Since the function $h(z)$ has real coefficient and hence $h(\Delta)$ is symmetric with respect to real axis, it is enough to consider the case $0 \leq \theta \leq \pi$. With $t = \cot(\theta/2)$, we have

$$\frac{1+z}{1-z} = it \quad (t \geq 0). \tag{3.4}$$

By using (3.4), we have

$$q(z) = (it)^\rho \text{ and } zq'(z) = -\frac{\rho}{2}(1+t^2)(it)^{\rho-1}$$

and therefore we have

$$\begin{aligned} h(e^{i\theta}) &= \alpha(it)^\rho + \beta(it)^{2\rho} - \frac{\gamma\rho}{2}(1+t^2)(it)^{\rho-1} \\ &= (it)^{\rho-1}\left[\alpha t i + \beta(it)^{\rho+1} - \frac{\gamma\rho}{2}(1+t^2)\right]. \end{aligned}$$

Therefore

$$h(e^{i\theta}) = (it)^{\rho-1}H(t)$$

where

$$H(t) := \alpha t i + \beta(it)^{\rho+1} - \frac{\gamma\rho}{2}(1+t^2).$$

If $\phi(\rho)$ is defined by

$$\phi(\rho) := \min_{t\geq 0}[\arg H(t)],$$

then

$$\arg h(e^{i\theta}) \geq \frac{(1+\rho)\pi}{2} + \phi(\rho).$$

Let

$$a := \cos\left(\frac{(1+\rho)\pi}{2}\right) = -\sin\left(\frac{\rho\pi}{2}\right) \text{ and } b := \sin\left(\frac{(1+\rho)\pi}{2}\right) = \cos\left(\frac{\rho\pi}{2}\right).$$

Then

$$\arg H(t) = \arctan\left(\frac{\alpha t + \beta b t^{\rho+1}}{\beta a t^{\rho+1} - \frac{\gamma\rho}{2}(1+t^2)}\right).$$

The minimum of $\arg H(t)$ is given by the unique root of the equation

$$G(t) := t^\rho K(t) + \frac{\alpha\gamma}{2}(t^2-1) = 0$$

where

$$K(t) := \beta\left[\frac{b\gamma(1-\rho)}{2}t^2 - a\alpha t - \frac{b\gamma(1+\rho)}{2}\right].$$

For $\rho_0 < \rho \leq 1$,

$$G(1) = K(1) = \beta\left[\frac{b\gamma(1-\rho)}{2} - a\alpha - \frac{b\gamma(1+\rho)}{2}\right] = -\beta\left[a\alpha + b\gamma\rho\right] > 0$$

and

$$G(0) = -\frac{\alpha\gamma}{2} < 0.$$

Since

$$K'(t) = \beta\left[\beta\gamma(1-\rho)t - a\alpha\right] \geq 0$$

for $t \geq 0$ and $K(1) > 0$, we have $K(t) > 0$ for $t \geq 1$ and therefore $G(t) > 0$ for $t \geq 1$. Also

$$G''(t) = \frac{(1+\rho)\beta}{2}t^{\rho-2}\left[\beta\gamma(1-\rho)(2+\rho)t^2 - 2a\alpha\rho t + \rho(1-\rho)\gamma b\right] + \alpha\gamma > 0$$

for $t > 0$. Therefore $G(t) = 0$ has a unique root in $(0,1)$ and the root is t_0 as given in the hypothesis of our Lemma 3.1. A straightforward computation shows that

$$\beta a t_0^{\rho+1} - \frac{\gamma\rho}{2}(1+t_0^2) = \frac{\gamma}{2\alpha}\left(\alpha + \beta b t_0^\rho\right)\left((1-\rho)t_0^2 - (1+\rho)\right)$$

and hence

$$\phi(\rho) = -\arctan\left(\frac{2\alpha t_0}{\gamma[(1+\rho)-(1-\rho)t_0^2]}\right).$$

Therefore if the condition (3.2) of Lemma 3.1 holds, then we have

$$\alpha p(z) + \beta p(z)^2 + \gamma z p'(z) \prec \alpha q(z) + \beta q(z)^2 + \gamma z q'(z). \tag{3.5}$$

Define the functions ϑ and φ by

$$\vartheta(w) := \alpha w + \beta w^2 \text{ and } \varphi(w) := \gamma. \tag{3.6}$$

Clearly the functions $\vartheta(w)$ and $\varphi(w)$ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Since $q(z)$ is convex univalent, $zq'(z)$ is starlike univalent and therefore the function $Q(z)$ defined by

$$Q(z) := zq'(z)\varphi(q(z)) = \gamma z q'(z) = \frac{2\alpha\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^\alpha$$

is starlike univalent in Δ. Define the function $h(z)$ by

$$h(z) := \vartheta(q(z)) + Q(z) = \alpha q(z) + \beta q^2(z) + \gamma z q'(z).$$

Since $q(\Delta)$ is the convex region $|\arg(q(z))| < \alpha\pi/2$ contained in the right half-plane, we see that

$$\Re\,\frac{zh'(z)}{Q(z)} = \Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma}q(z) + 1 + \frac{zq''(z)}{q'(z)}\right] > 0$$

for $z \in \Delta$. Since the subordination (3.5) is same as (1.2) for the choices of functions φ and ϑ given by (3.6), by an application of Lemma 1.5, we get $p(z) \prec q(z)$. This completes the proof of our Lemma 3.1. ∎

As an application of Lemma 3.1, we have the following:

Theorem 3.2. *Let $0 < \alpha < 1$ and $\rho_0 \in (0,1)$ be the largest root of*

$$\frac{\alpha}{1-\alpha}\rho_0 = \tan\left(\frac{\rho_0\pi}{2}\right).$$

For $\rho_0 < \rho \leq 1$, let t_0 be the unique root of the equation

$$t^\rho\left[\alpha(1-\rho)\cos\left(\frac{\rho\pi}{2}\right)t^2 + 2(1-\alpha)\sin\left(\frac{\rho\pi}{2}\right)t - \alpha(1+\rho)\cos\left(\frac{\rho\pi}{2}\right)\right] + (1-\alpha)(t^2-1) = 0.$$

Let β be given by

$$\beta = 1 + \rho - \frac{2}{\pi}\arctan\left(\frac{2(1-\alpha)t_0}{\alpha[(1+\rho)-(1-\rho)t_0^2]}\right).$$

If $f \in \mathcal{A}$ satisfies

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)\right]\right| \leq \frac{\beta\pi}{2}$$

then $f \in SS^*(\rho)$.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}$$

which shows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

Therefore, we have

$$\begin{aligned} \alpha\frac{z^2f''(z)}{f(z)} &= \alpha\frac{zf''(z)}{f'(z)}\frac{zf'(z)}{f(z)} \\ &= \alpha[\frac{zp'(z)}{p(z)} + p(z) - 1]p(z) \\ &= \alpha zp'(z) + \alpha p^2(z) - \alpha p(z) \end{aligned}$$

and hence we have

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)} = (1-\alpha)p(z) + \alpha p^2(z) + \alpha zp'(z).$$

By using Lemma 3.1, the proof our Theorem 3.2 is completed. ∎

The proof of the following two Theorems are similar to the proof of Theorem 3.2 and hence it is omitted.

Theorem 3.3. *For $0 < \alpha \leq 1$, let β ρ and ρ_0 be as in Theorem 3.2. If $f \in \mathcal{A}$ satisfies*

$$\left|\arg\left[\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\left(\alpha(\delta+2)\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} - \alpha\delta\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} + (1-2\alpha)\right)\right]\right| < \frac{\beta\pi}{2},$$

then

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\rho}.$$

Theorem 3.4. *For $0 < \alpha \leq 1$, let β, ρ and ρ_0 be as in Theorem 3.2. If $f \in \mathcal{A}$ satisfies*

$$\left|\arg\left[\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^{m}f(z)}\left(1-\alpha+\alpha\frac{\mathcal{D}^{m+2}f(z)}{\mathcal{D}^{m+1}f(z)}\right)\right]\right| < \frac{\beta\pi}{2},$$

then

$$\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^{m}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\rho}.$$

4. FURTHER RESULTS FOR p-VALENT FUNCTIONS

In this section, we apply Lemma 2.1 and Lemma 3.1 to certain p-valent analytic functions defined through a linear operator $L_p(a,c)$ which we define below. Let $\mathcal{A}_p$ be the class of all analytic functions $f(z)$ of the form

(4.1) $$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1,2,3\ldots\})$$

For two functions $f(z)$ given by (4.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k\, z^k \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

(4.2) $$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k\, b_k\, z^k =: (g * f)(z).$$

In terms of the Pochhammer symbol $(\lambda)_k$ or the *shifted* factorial given by

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_k := \lambda(\lambda+1)\cdots(\lambda+k-1) \quad (k \in \mathbb{N}),$$

we now define the function $\phi_p(a,c;z)$ by

(4.3) $$\phi_p(a,c;z) := z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k}\, z^{k+p}$$

$$\left(z \in \Delta;\ a \in \mathbb{R};\ c \in \mathbb{R}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,\ldots\}\right).$$

Corresponding to the function $\phi_p(a,c;z)$, Saitoh [15] introduced a linear operator $L_p(a,c)$ which is defined by means of the following Hadamard product (or convolution):

(4.4) $$L_p(a,c)\, f(z) := \phi_p(a,c;z) * f(z) \quad (f \in \mathcal{A}_p)$$

or, equivalently, by

(4.5) $$L_p(a,c)\, f(z) := z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k}\, a_{k+p}\, z^{k+p} \quad (z \in \Delta).$$

The definition (4.4) or (4.5) of the linear operator $L_p(a,c)$ is motivated essentially by the familiar Carlson-Shaffer operator

$$L(a,c) := L_1(a,c),$$

which has been used widely on such spaces of analytic and univalent functions in $\mathbb{U}$ as starlike and convex functions of order α (see, for example, [17]).

As an application of Lemma 2.1 and Lemma 3.1, we immediately obtain the following results:

Theorem 4.1. *Let $0 < \alpha < 1$ and β be given by (2.1). If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\arg\left(\left[(a+1)\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} - 1\right]\frac{L_p(a,c)f(z)}{L_p(a+1,c)f(z)} - (a-1)\right)\right| < \frac{\beta\pi}{2},$$

then

$$\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Theorem 4.2. *For $0 < \alpha \le 1$, let β, ρ and ρ_0 be as in Theorem 3.2. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\arg\left[\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\left(\alpha(a+1)\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} - \alpha(a-1)\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} + 1 - 2\alpha\right)\right]\right| < \frac{\beta\pi}{2},$$

then

$$\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\rho}.$$

The Ruscheweyh derivative of $f(z)$ of order $\delta + p - 1$ is defined by

$$D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (f \in \mathcal{A}(p,n);\ \delta \in \mathbb{R}\setminus(-\infty,-p]) \tag{4.6}$$

or, equivalently, by

$$D^{\delta+p-1} f(z) := z^p + \sum_{k=p+1}^{\infty} \binom{\delta+k-1}{k-p} a_k\, z^k \tag{4.7}$$

$$(f \in \mathcal{A}(p,n);\ \delta \in \mathbb{R}\setminus(-\infty,-p]).$$

In particular, if $\delta = l$ $(l + p \in \mathbb{N})$, we find from the definition (4.6) or (4.7) that

$$D^{l+p-1} f(z) = \frac{z^p}{(l+p-1)!}\frac{d^{l+p-1}}{dz^{l+p-1}}\left\{z^{l-1} f(z)\right\} \tag{4.8}$$

$$(f \in \mathcal{A}(p,n);\ l+p \in \mathbb{N}).$$

Our Theorems 4.1 and 4.2 can be specialized to obtaine results for p-valent functions defined by Ruscheweyh derivatives whcih are similar to Theorems 2.3 and 3.3, the details of which is omitted here.

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