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REVERSE OF MARTINS' INEQUALITY

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ABSTRACT. In this paper, it is proved that

$$\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < \left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r}$$

for all natural numbers n, and all real $r < 0$.

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1. INTRODUCTION

The inequality to which the title refers is

$$\left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1.1}$$

for all natural numbers n, and all real $r > 0$. For convenience, we call it Martins' inequality (see [4]).

We prove that Martins' inequality is reversed for $r < 0$.

Theorem. *For all natural numbers n, and all real $r < 0$, then*

$$\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r}. \tag{1.2}$$

2. LEMMAS

Lemma 1 (König's inequality [2, p. 149] and [3, p. 24])**.** *Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be decreasing nonnegative n-tuples such that*

$$\prod_{i=1}^{k} b_i \le \prod_{i=1}^{k} a_i, \quad 1 \le k \le n, \tag{2.1}$$

then, for $r > 0$,

$$\sum_{i=1}^{n} b_i^r \le \sum_{i=1}^{n} a_i^r. \tag{2.2}$$

The sign of equality in (2.2) *holds if and only if $a_i = b_i$ for all $1 \le i \le n$.*

Lemma 2 ([5])**.** *For all natural number n, then*

$$\frac{n}{n+1} < \frac{n+1}{n+2} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1. \tag{2.3}$$

3. PROOF OF THEOREM

Proof. Let $r = -s, s > 0$, then (1.2) is equivalent to

$$(n+1)\sum_{i=1}^{n}\left(\frac{1}{i\sqrt[n+1]{(n+1)!}}\right)^s < n\sum_{i=1}^{n+1}\left(\frac{1}{i\sqrt[n]{n!}}\right)^s. \tag{3.1}$$

Define

$$a_{pn+1} = a_{pn+2} = \cdots = a_{pn+n} = \frac{1}{(p+1)\sqrt[n]{n!}}$$

for $p = 0, 1, 2, \ldots, n$, and

$$b_{q(n+1)+1} = b_{q(n+1)+2} = \cdots = b_{q(n+1)+(n+1)} = \frac{1}{(q+1)\sqrt[n+1]{(n+1)!}}$$

for $q = 0, 1, 2, \ldots, n-1$.

Easy calculation reveals that

$$\sum_{i=1}^{n(n+1)} a_i^s = n \sum_{i=1}^{n+1} \left(\frac{1}{i \sqrt[n]{n!}} \right)^s,$$
$$\sum_{i=1}^{n(n+1)} b_i^s = (n+1) \sum_{i=1}^{n} \left(\frac{1}{i \sqrt[n+1]{(n+1)!}} \right)^s.$$

Then, (3.1) is equivalent to

$$\sum_{i=1}^{n(n+1)} b_i^s < \sum_{i=1}^{n(n+1)} a_i^s. \tag{3.2}$$

It is easy to see that both the sequences $\{a_i\}_{i=1}^{n(n+1)}$ and $\{b_i\}_{i=1}^{n(n+1)}$ are decreasing, and $a_{n(n+1)} = \frac{1}{(n+1)\sqrt[n]{n!}} < \frac{1}{n \sqrt[n+1]{(n+1)!}} = b_{n(n+1)}$. Therefore, by Lemma 1, to prove (3.2) it is sufficient to show that

$$\prod_{i=1}^{k} b_i \leq \prod_{i=1}^{k} a_i \quad for\ 1 \leq k \leq n(n+1). \tag{3.3}$$

Set $A_k = \prod_{i=1}^k a_i$ and $B_k = \prod_{i=1}^k b_i$, then

$$A_k = \left(\frac{1}{(p+1)\sqrt[n]{n!}} \right)^{k-pn} \prod_{j=0}^{p-1} \left(\frac{1}{(j+1)\sqrt[n]{n!}} \right)^n,$$

for $pn+1 \leq k \leq (p+1)n, p = 0,1,2,\ldots,n$, and

$$B_k = \left(\frac{1}{(q+1) \sqrt[n+1]{(n+1)!}} \right)^{k-q(n+1)} \prod_{j=0}^{q-1} \left(\frac{1}{(j+1) \sqrt[n+1]{(n+1)!}} \right)^{n+1},$$

for $q(n+1)+1 \leq k \leq (q+1)(n+1), q = 0,1,2,\ldots,n-1$.

Let $k \in \{1,2,\ldots,n(n+1)\}$, then there exists a uniquely determined number $i \in \{0,1,2,\ldots,n\}$ such that $in+1 \leq k \leq (i+1)n$. We consider three cases to show (3.3).

Case 1. $i = 0$. Then we have $1 \leq k \leq n$ which leads to

$$A_k = \left(\frac{1}{\sqrt[n]{n!}} \right)^k \quad \text{and} \quad B_k = \left(\frac{1}{\sqrt[n+1]{(n+1)!}} \right)^k.$$

Because of $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$, we have $B_k < A_k$.

Case 2. $i = n$. Then we have $n^2+1 \leq k \leq n(n+1)$ which leads to

$$A_k = \left(\frac{1}{(n+1)\sqrt[n]{n!}} \right)^{k-n^2} \prod_{j=0}^{n-1} \left(\frac{1}{(j+1)\sqrt[n]{n!}} \right)^n$$

and

$$B_k = \left(\frac{1}{n \sqrt[n+1]{(n+1)!}} \right)^{k-(n-1)(n+1)} \prod_{j=0}^{n-2} \left(\frac{1}{(j+1) \sqrt[n+1]{(n+1)!}} \right)^{n+1}.$$

Inequality $B_k \leq A_k$ is equivalent to

$$\left(\frac{n+1}{n} \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \right)^k \leq \frac{n!}{n^n} \left(\frac{n+1}{n} \right)^{n^2}. \tag{3.4}$$

Beause of $\frac{n+1}{n}\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} > 1$, we have to show that (3.4) holds for $k = n(n+1)$. Easy computation shows that the sign of equality in (3.4) holds for $k = n(n+1)$.

Case 3. $1 \le i \le n-1$. Then we have $in+1 \le k \le i(n+1)$ or $i(n+1)+1 \le k \le (i+1)n$. If $in+1 \le k \le i(n+1)$, then

$$A_k = \left(\frac{1}{(i+1)\sqrt[n]{n!}}\right)^{k-in} \prod_{j=0}^{i-1} \left(\frac{1}{(j+1)\sqrt[n]{n!}}\right)^{n}$$

and

$$B_k = \left(\frac{1}{i\sqrt[n+1]{(n+1)!}}\right)^{k-(i-1)(n+1)} \prod_{j=0}^{i-2} \left(\frac{1}{(j+1)\sqrt[n+1]{(n+1)!}}\right)^{n+1}.$$

For $i = 1$, then $k = n+1$, $B_k \le A_k$ is equivalent to

$$\frac{n+1}{\sqrt[n]{n!}} \ge 2,$$

which is true, since the sequence $\left\{\frac{n+1}{\sqrt[n]{n!}}\right\}_{n=1}^{\infty}$ is strictly increasing from (2.3).

For $2 \le i \leqslant n-1$. $B_k \le A_k$ can be written as

$$\begin{aligned} &\frac{\prod_{j=0}^{i-2}[(j+1)\sqrt[n+1]{(n+1)!}\,]^{n+1}}{\prod_{j=0}^{i-1}[(j+1)\sqrt[n]{n!}\,]^{n}} \\ &\ge \frac{[i\sqrt[n+1]{(n+1)!}\,]^{(i-1)(n+1)}}{[(i+1)\sqrt[n]{n!}\,]^{in}} \left(\frac{(i+1)\sqrt[n]{n!}}{i\sqrt[n+1]{(n+1)!}}\right)^{k}, 2 \le i \le n-1. \end{aligned} \tag{3.5}$$

It is easy to see from (2.3) that

$$\frac{i}{i+1} < \frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}, \quad 2 \le i \le n-1,$$

which implies

$$\frac{i+1}{i}\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} > 1, \quad 2 \le i \le n-1. \tag{3.6}$$

Because of (3.6), it suffices to prove that (3.5) is valid for $k = i(n+1)$. This means we have to show that

$$\frac{i+1}{\sqrt[i]{i!}} \le \frac{n+1}{\sqrt[n]{n!}}, \quad 2 \le i \le n-1,$$

which holds strictly, since the sequence $\left\{\frac{n+1}{\sqrt[n]{n!}}\right\}_{n=1}^{\infty}$ is strictly increasing from (2.3).

If $i(n+1)+1 \le k \le (i+1)n$, then we have

$$A_k = \left(\frac{1}{(i+1)\sqrt[n]{n!}}\right)^{k-in} \prod_{j=0}^{i-1} \left(\frac{1}{(j+1)\sqrt[n]{n!}}\right)^{n}$$

and

$$B_k = \left(\frac{1}{(i+1)\sqrt[n+1]{(n+1)!}}\right)^{k-i(n+1)} \prod_{j=0}^{i-1} \left(\frac{1}{(j+1)\sqrt[n+1]{(n+1)!}}\right)^{n+1}$$

such that $B_k \le A_k$ is equivalent to

$$\frac{i!}{(i+1)^i} \ge \left(\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)^k, \quad 2 \le i \le n-1. \tag{3.7}$$

Because of $\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1$, it is enough to prove inequality (3.7) for $k = i(n+1)+1$. This means we have to show that

$$\left(\frac{n+1}{\sqrt[n]{n!}} \Big/ \frac{i+1}{\sqrt[i]{i!}}\right)^i \ge \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{3.8}$$

From Lemma 2 we conclude that

$$\frac{n+1}{\sqrt[n]{n!}} \Big/ \frac{i+1}{\sqrt[i]{i!}} > 1 > \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}, \quad 2 \le i \le n-1, \tag{3.9}$$

which implies (3.8) with strictly inequality. This finishes the proof of (3.3), and thus of the theorem. ∎

The proof of Theorem has motivated by an article of Alzer [1].

REFERENCES

[1] H. ALZER, On an inequality of H. Minc and L. Sathre, *J. Math. Anal. Appl.*, **179** (1993), 396–402.

[2] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.

[3] H. KÖNIG, *Eigenvalue Distribution of Compact Operators*, Birkhaüser, Basel, 1986.

[4] J. S. MARTINS, Arithmetic and geometric means, an application to Lorentz sequence spaces, *Math. Nachr.* **139** (1988), 281–288.

[5] J.-S. SUN, On upper and lower bounds of Minc-Sathre inequality, *Bulletin of Mathematics*, **11** (2003), 40. (Chinese)