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MULTIVALUED HEMIEQUILIBRIUM PROBLEMS

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ABSTRACT. In this paper, we introduce and study a new class of equilibrium problems, known as multivalued hemiequilibrium problems. The auxiliary principle technique is used to suggest and analyze some new classes of iterative algorithms for solving multivalued hemiequilibrium problems. The convergence of the proposed methods either requires partially relaxed strongly monotonicity or pseudomonotonicity. As special cases, we obtain a number of known and new results for solving various classes of equilibrium and variational inequality problems. Since multivalued hemiequilibrium problems include hemiequilibrium, hemivariational inequalities, variational inequalities and complementarity problems as specials cases, our results still hold for these problems.

Key words and phrases: Equilibrium problems, variational inequalities, auxiliary principle, iterative methods, convergence.

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1. INTRODUCTION

Equilibrium problems theory is an interesting and fascinating branch of mathematical sciences with a wide range of applications in industry, physical, regional, social, pure and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields, see [1], [5], [6], [13], [14], [15], [16], [17], [18], [19], [20]. In recent years, equilibrium problems have been generalized and extended in different directions using the novel and innovative techniques. Inspired and motivated by the recent research going on in this area, we introduce and consider a new class of equilibrium problems, which is called *multivalued hemiequilibrium problem*. It is shown that multivalued hemiequilibrium problems include hemiequilibrium, hemivariational inequalities, variational inequalities, and complementarity problems as special cases. There are several numerical methods including projection methods, Wiener-Hopf equations, descent and decomposition for solving variational inequalities, see [5], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. Due to the nature of the hemiequilibrium problems, these methods can not be extended for solving these problems. To overcome these drawbacks, one usually uses the auxiliary principle technique to suggest some iterative methods for solving hemiequilibrium problems. Glowinski, Lions and Tremolieres [8] used this approach to study the existence of a solution of the mixed variational inequalities. In recent years, Noor [12], [13], [14], [15], [16] has used this technique to study some predictor-corrector methods for various classes of equilibrium and variational inequality problems. In this paper, we again use the auxiliary principle technique to suggest a class of three-step predictor-corrector iterative methods for multivalued hemiequilibrium problems. In particular, we show that one can obtain various forward-backward splitting, modified projection, and other methods as special cases from these methods. We also prove that the convergence of the suggested methods requires only the partially relaxed strongly monotonicity. Using the auxiliary principle technique, we also suggest and analyze an inertial proximal method for solving multivalued equilibrium problems. We show that the convergence of the inertial proximal method converges for pseudomonotone functions, which is a weaker condition than monotonicity. It is worth mentioning that inertial proximal method includes the classical proximal method as a special case. Consequently, our results represent an improvement and refinement of the previously known results. Our results can be considered as an important and significant extension of the previously known results for solving hemiequilibrium, hemivariational inequalities, variational inequalities and complementarity problems.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively. Let $C(H)$ be the family of all non-empty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a multivalued operator. Let K be a nonempty, closed and convex set in H. Let $A(.;.) : H \times H \longrightarrow H$ be a nonlinear continuous operator.

For a given single-valued trifunction $F(.,.,.) : H \times H \times H \longrightarrow C(H)$, we consider the problem of finding $u \in K, \nu \in T(u)$, such that

$$F(u, \nu, v) + A(u; v - u) \geq 0, \quad \forall v \in K, \tag{2.1}$$

which is called the *multivalued hemiequilibrium problem.* For example, if $J^0(u; v-u)$ denotes the Clarke generalized directional derivative of a locally Lipschitz continuous function $j(.)$ at u in the direction $v - u$, then clearly $J^0(u; v - u) \equiv A(u; v - u)$. It can be shown that a wide class of problems arising in various branches of pure and applied sciences can be studied in the general framework of multivalued equilibrium problems.

If $T : H \longrightarrow H$ is a single-valued operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$F(u, Tu, v) + A(u; v - u) \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is called the hemiequilibrium problem with trifunction and appears to be a new one.

If $F(u, \nu, v) \equiv F(\nu, v)$, then problem (2.1) is equivalent to finding $u \in K, \quad \nu \in T(u)$ such that

$$F(\nu, v) + A(u; v - u) \geq 0, \quad \forall v \in K, \tag{2.3}$$

which is known as the multivalued hemiequilibrium problem.

If $F(u, \nu, v) = \langle \nu, v - u \rangle$, then problem (2.1) is equivalent to finding $u \in K, \nu \in T(u)$ such that

$$\langle \nu, v - u \rangle + A(u; v - u) \quad \geq \quad 0, \qquad \forall v \in K. \tag{2.4}$$

The inequality of type (2.4) is called the *multivalued hemivariational inequality*. It is known that a wide class of problems arising in pure and applied sciences can be studied via the multivalued hemivariational inequalities (2.4), see [4].

We note that, if $T : H \longrightarrow H$ is a single-valued operator, then problem (2.4) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle + A(u; v - u) \quad \geq \quad 0, \quad \forall v \in K, \tag{2.5}$$

which is known as the hemivariational inequality. Special cases of these hemivariational inequalities (2.5) were introduced and studied by Panagiotopoulos [21], [22] in order to formulate variational principles associated with energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in enginnering and industrial sciences, see [4], [17], [21], [22] and the references therein.

If $A(.;.) = 0$, then problem (2.1) is equivalent to finding $u \in K, \nu \in T(u)$ such that

$$F(u, \nu, v) \geq 0, \quad \forall v \in K, \tag{2.6}$$

which is called the *multivalued equilibrium problem* studied by Noor and Oettli [20] and Noor [12], [15] using quite different techniques.

If $F(u, \nu, v) = \langle \nu, v - u \rangle$ and $A(.;.) = 0$, then problem (2.1) is equivalent to finding $u \in K, \nu \in T(u)$ such that

$$\langle \nu, v - u \rangle \quad \geq \quad 0, \qquad \forall v \in K. \tag{2.7}$$

The inequality of type (2.7) is called the *multivalued variational inequality.* It is known that a wide class of free, obstacle, moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequalities, see Noor [10]. If T is a single-valued operator, then we obtain the original variational inequality considered by Stampacchia [25] in 1964.

It is clear that problems (2.2)-(2.7) are special cases of the multivalued variational inequality (2.1). In brief, for a suitable and appropriate choice of the operators $F(.,.,.)$, T, $A(.;.)$ and the space H, one can obtain a wide class of equilibrium, variational inequalities and complementarity problems. This clearly shows that problem (2.1) is quite general and unifying one. Furthermore, problem (2.1) has many important applications in various branches of pure and applied sciences, see [1], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23].

We also need the following well known results and concepts.

Lemma 2.1. $\forall u, v \in H,$ *we have*

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{2.8}$$

Definition 2.1. The trifunction $F(.,.,.)$ and the operator T is said to be:

(1) (i) *partially relaxed strongly jointly monotone,* iff, there exists a constant $\alpha > 0,$ such that

$$F(u_1, w_1, u_2) + F(u_2, w_2, z) \leq \alpha\|z - u_1\|^2, \quad \forall u_1, u_2, z \in H, w_1 \in T(u_1), w_2 \in T(u_2).$$

(2) (ii) *jointly monotone,* iff,

$$F(u_1, w_1, u_2) + F(u_2, w_2, u_1) \leq 0, \quad \forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2).$$

(3) (iii) *jointly pseudomonotone,* iff,

$$F(u_1, w_1, u_2) \geq 0, \quad \text{implies} \quad F(u_2, w_2, u_1) \leq 0, \quad \forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2).$$

Definition 2.2. The multivalued operator $T : H \longrightarrow C(H)$ is said to be M*-Lipschitz continuous,* iff, there exists a constant $\delta > 0,$ such that

$$M(T(u_1), T(u_2)) \leq \delta||u_1 - u_2||, \quad \forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2),$$

where $M(.,.)$ is the Hausdorff metric on $C(H).$

We remark that, if $z = u_1$, then partially relaxed strongly monotonicity is exactly the jointly monotonicity of $F(.,.,.)$ and $T.$

Definition 2.3. The operator $A(.;.)$ is said to partially relaxed strongly monotone, iff, there exists a constant $\alpha > 0$ such that

$$A(u; v-u) + A(v; z-v) \leq \alpha\|u - z\|^2, \quad \forall u, v, z \in H.$$

Note that for $z = u,$ partially relaxed strongly monotonicity reduces to

$$A(u : v-u) + A(v; u-v) \leq 0,$$

that is, the operator $A(.;.)$ is monotone.

3. MAIN RESULTS

In this section, we suggest and analyze a class of iterative methods for solving the problem (2.1) by using the auxiliary principle technique.

For a given $u \in K, \nu \in T(u)$, consider the problem of finding a solution $w \in K,$ satisfying the auxiliary equilibrium problem

$$\rho F(u, \nu, v) + \langle w - u, v - w\rangle + \rho A(u; v - w) \quad \geq 0, \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant. We note that, if $w = u$, then clearly w is a solution of the multivalued equilibrium problem (2.1). This observation enables us to suggest the following predictor-corrector method for solving the multivalued equilibrium problem (2.1).

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho F(w_n, \eta_n, v) + \langle u_{n+1} - w_n, v - u_{n+1}\rangle + \rho A(w_n; v - u_{n+1}) \geq \quad 0, \forall \in K \tag{3.2}$$

$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)) \tag{3.3}$$

$$\beta F(y_n, \xi_n, v) + \langle w_n - y_n, v - w_n\rangle + \beta A(y_n; v - w_n) \geq 0, \quad \forall v \in K \tag{3.4}$$

$$\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leq M(T(y_{n+1}), T(y_n)) \tag{3.5}$$

and

(3.6) $$\mu F(u_n, \nu_n, g(v)) + \langle y_n - u_n, v - y_n\rangle + \nu A(u_n; v - y_n) \geq 0, \quad \forall v \in K.$$

(3.7) $$\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots$$

where $\rho > 0$ *,* $\mu > 0$ *and* $\beta > 0$ *are constants.*

Note that, if $A(.;.) = 0$, then Algorithm 3.1 reduces to the following predictor-corrector method for solving the multivalued equilibrium problem (2.6), see Noor [15].

Algorithm 3.2. *For a given* $u_0 \in H$*, compute* u_{n+1} *by the iterative schemes*

$$\begin{aligned}
&\rho F(w_n, \eta_n, v) + \langle u_{n+1} - w_n, v - u_{n+1}\rangle \quad \geq 0, \quad \forall v \in K \\
&\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(w_{n+1}), T(w_n)) \\
&\beta F(y_n, \xi_n, v) + \langle w_n - y_n, v - w_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leq M(T(y_{n+1}), T(y_n)) \\
&\mu F(u_n, \nu_n, v) + \langle y_n - u_n, v - y_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2 \ldots
\end{aligned}$$

If $F(u, \nu, v) = F(\nu, v)$, then Algorithm 3.1 reduces to the following algorithm for solving multivalued hemiequilibrium problem (2.3).

Algorithm 3.3. *For a given* $u_0 \in H$*, compute the approximate solution* u_{n+1} *by the iterative schemes*

$$\begin{aligned}
&\rho F(\eta_n, v) + \langle u_{n+1} - w_n, v - u_{n+1}\rangle + \rho A(w_n; v - u_{n+1}) \geq 0, \quad \forall v \in K \\
&\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(w_{n+1}), T(w_n)) \\
&\beta F(\xi_n, v) + \langle w_n - y_n, v - w_n\rangle + \beta A(y_n; v - w_n) \geq 0, \quad \forall v \in K \\
&\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leq M(T(y_{n+1}), T(y_n)) \\
&\mu F(\nu_n, v) + \langle y_n - u_n, v - y_n\rangle + \nu A(u_n; v - y_n) \geq 0, \quad \forall v \in K. \\
&\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots.
\end{aligned}$$

If $A(.;.) = 0$ and $F(\nu, v) = \langle \nu, v - u\rangle$, then Algorithm 3.3 reduces to

Algorithm 3.4. *For a given* $u_0 \in H$*, compute the approximate solution* u_{n+1} *by the iterative schemes*

$$\begin{aligned}
&\langle \rho\eta_n + u_{n+1} - w_n, v - u_{n+1}\rangle \quad \geq 0, \quad \forall v \in K, \\
&\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(w_{n+1}), T(w_n)) \\
&\langle \beta\xi_n + w_n - y_n, v - w_n\rangle \quad \geq \quad 0, \quad \forall v \in K \\
&\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leq M(T(y_{n+1}), T(y_n)) \\
&\langle \mu\nu_n + y_n - u_n, v - y_n\rangle \quad \geq 0, \quad \forall v \in K \\
&\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2 \ldots
\end{aligned}$$

or equivalently

$$\begin{aligned}
u_{n+1} &= P_K[w_n - \rho\eta_n] \\
\eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)) \\
w_n &= P_K[y_n - \beta\xi] \\
\xi_n \in T(y_n) &: \|\xi_{n+1} - \xi_n\| \leq M(T(y_{n+1}), T(y_n)) \\
y_n &= P_K[u_n - \mu\nu] \\
\nu_n \in T(u_n) &: \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)),
\end{aligned}$$

where P_K is the projection of H onto the closed convex set K.

Algorithm 3.4 is known as the predictor-corrector method for solving the multivalued variational inequalities (2.7), see [11].

If T is a single-valued operator, then Algorithms 3.3 and 3.4 reduce to:

Algorithm 3.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} \langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1}\rangle &\geq 0, \quad \forall v \in K \\ \langle \beta T y_n + (w_n - y_n, v - w_n\rangle &\geq 0, \quad \forall v \in K \\ \langle \mu T u_n + y_n - u_n, v - y_n\rangle &\geq 0, \quad \forall v \in K, \end{aligned}$$

which is called the predictor-corrector method for solving variational inequalities (2.5), see Noor [12], [15].

We remark that Algorithm 3.5 can be written in the following equivalent form as

Algorithm 3.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= P_K[u_n - \mu T u_n] \\ w_n &= P_K[y_n - \beta T(y_n)] \\ u_{n+1} &= P_K[w_n - \rho T(w_n)], \quad n = 0, 1, 2 \ldots \end{aligned}$$

which can be written in the following form,

$$u_{n+1} = P_K[I - \rho T]P_K[I - \beta T]P_K[I - \mu T]g(u_n), \quad n = 0, 1, 2 \ldots$$

Algorithm 3.6 is known as three-step forward-backward splitting algorithms. Algorithm 3.6 is similar to the so-called θ-scheme of Glowinski and Le Tallec [7], which they suggested by using the Lagrangian multiplier method. It has been shown in [7] that three-step schemes are numerically efficient and are reasonably easy to use for computations as compared with one-step and two-step iterative methods for solving nonlinear problems arising in elasticity and mechanics. The convergence analysis of Algorithm 3.6 has been considered by Noor [11], [12], [14].

We now rewrite Algorithm 3.4 in the following form:

Algorithm 3.7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} u_{n+1} &= (1 - \rho_n)u_n + \rho_n P_K[w_n - \rho_n \eta_n] \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)) \\ w_n &= (1 - \beta_n)u_n + \beta_n P_K[y_n - \beta_n \xi_n] \\ \xi_n \in T(y_n) &: \|\xi_{n+1} - \xi_n\| \leq M(T(y_{n+1}), T(y_n)) \\ y_n &= (1 - \mu_n)u_n + \mu_n P_K[y_n - \mu_n \nu_n] \\ \nu_n \in T(u_n) &: \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)), \end{aligned}$$

where the sequences $\{\rho_n\}, \{\beta_n\}, \{\mu_n\}$ satisfy some certian conditions.

Algorithm 3.7 is also known as three-step (Noor) iteration process. Clearly Ishikawa and Mann iterations are special cases of Noor (three-step) iterations. Clearly for $K = H$ and a single-valued operator T, Algorithm 3.7 collapses to the following three-step iterative method for solving nonlinear equation $Tu = 0$ which has been studied in the Banach spaces setting.

Algorithm 3.8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T w_n \\ w_n &= (1-\beta_n)u_n + \beta_n T y_n \\ y_n &= (1-\mu_n)u_n + \mu_n T u_n, \quad n = 0, 1, 2, \ldots \end{aligned}$$

Algorithm 3.8 is well known three-step (Noor iteration) iterative method which has been studied extensively in recent years. It is obvious that the three-step iterative method includes Ishikawa-Mann iterations as special cases. For the equivalence between the Mann-Ishikawa and multistep iterations in an arbitrary Banach Space, see Rhoades and Soltuz [24].

For a suitable choice of the operators and the space H, one can obtainvarious new and known methods for solving equilibrium, variational inequality and complementarity problems.

For the convergence analysis of Algorithm 3.1, we need the following result.

Theorem 3.9. *Let $u \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $F(.,.,.)$ and $A(.;.)$ are partially relaxed strongly monotone operators with constant $\alpha > 0$ and $\gamma > 0$, then*

$$\|u_{n+1} - u\|^2 \leq \|w_n - u\|^2 - (1 - 2\rho(\alpha+\gamma))\|u_{n+1} - w_n\|^2 \tag{3.8}$$

$$\|w_n - u\|^2 \leq \|y_n - u\|^2 - (1 - 2(\alpha+\gamma)\beta)\|y_n - w_n\|^2 \tag{3.9}$$

$$\|y_n - u\|^2 \leq \|u_n - u\|^2 - (1 - 2(\alpha+\gamma)\mu)\|y_n - u_n\|^2. \tag{3.10}$$

Proof. Let $u \in K, \quad \nu \in T(u)$ be a solution of (2.1). Then

$$\rho F(\nu, Tu, v) + \rho A(u; v-u) \geq 0, \quad \forall v \in K \tag{3.11}$$

$$\beta F(\nu, Tu, v) + \beta A(u; v-u) \geq 0, \quad \forall v \in K \tag{3.12}$$

$$\mu F(\nu, Tu, v) + \nu A(u; v-u) \geq 0, \quad \forall v \in K, \tag{3.13}$$

where $\rho > 0$, $\beta > 0$ and $\mu > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.11) and $v = u$ in (3.2), we have

$$\rho F(u, \nu, u_{n+1}) + \rho A(u; u_{n+1} - u) \geq 0 \tag{3.14}$$

and

$$\begin{aligned} \rho F(w_n, \eta_n, u,) &+ \langle u_{n+1} - w_n, u - u_{n+1} \rangle \\ &+\rho A(w_n; u - u_{n+1}) \geq 0. \end{aligned} \tag{3.15}$$

Adding (3.14) and (3.15), we have

$$\begin{aligned} \langle u_{n+1} - w_n, u - u_{n+1} \rangle &\geq -\rho\{F(w_n, \eta_n, u) + F(u, \nu, u_{n+1})\} \\ &\quad -\rho\{A(u; u_{n+1} - u) + A(w_n; u - u_{n+1})\} \\ &\geq -\alpha\rho\|u_{n+1} - w_n\|^2 - \rho\gamma\{\|u_{n+1} - w_n\|^2\}, \\ &= -\rho\{\alpha+\gamma\}\|u_{n+1} - w_n\|^2, \end{aligned} \tag{3.16}$$

where we have used the fact that $F(.,.,.)$ and $A(.;.)$ are partially relaxed strongly monotone operators with constants $\alpha > 0.$ and $\gamma > 0$ respectively.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - w_n$ in (2.8), we obtain

$$\begin{aligned} \langle u_{n+1} - w_n, u - u_{n+1} \rangle &= \frac{1}{2}\{\|u - w_n\|^2 - \|u - u_{n+1}\|^2 \\ &- \|u_{n+1} - w_n\|^2\}. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\|u_{n+1} - u\|^2 \leq \|w_n - u\|^2 - (1 - 2\rho(\alpha + \gamma))\|u_{n+1} - w_n\|^2,$$

the required (3.8).

Taking $v = u$ in (3.4) and $v = w_n$ in (3.12), we have

$$\beta F(u, \nu, w_n) + \beta A(u; w_n - u) \geq 0 \tag{3.18}$$

and

$$\beta F(y_n, \xi_n, u) + \langle w_n - y_n, u - w_n\rangle + \beta A(y_n; u - w_n) \geq 0. \tag{3.19}$$

Adding (3.18) and (3.19) and rearranging the terms, we have

$$\begin{aligned} \langle w_n - y_n \quad , \quad u - w_n\rangle &\geq -\beta\{F(y_n, \xi_n, u) + F(u, \nu, w_n)\} \\ &\quad -\beta\{A(y_n; u - w_n) + A(u; w_n - u)\} \\ &\geq -\beta(\alpha + \gamma)\|y_n - w_n\|^2, \end{aligned} \tag{3.20}$$

since $F(.,.,.)$ and $A(.;.)$ are partially relaxed strongly monotone operators with constants $\alpha > 0$ and $\gamma > 0$ respectively.

Now taking $v = w_n - y_n$ and $u = u - w_n$ in (2.8), (3.20) can be written as

$$\|u - w_n\|^2 \leq \|u - y_n\|^2 - (1 - 2\beta(\alpha + \gamma))\|y_n - w_n\|^2,$$

the required (3.9).

Similarly, by taking $v = u$ in (3.6) and $v = u_{n+1}$ in (3.13) and using the partially relaxed strongly monotonicity of the operators $F(.,.,.)$ and $A(.;.)$, we have

$$\langle y_n - u_n, u - y_n\rangle \geq -\mu(\alpha + \gamma)\|y_n - u_n\|^2. \tag{3.21}$$

Letting $v = y_n - u_n$, and $u = u - y_n$ in (2.8), and combining the resultant with (3.21), we have

$$\|y_n - u\|^2 \leq \|u - u_n\|^2 - (1 - 2\mu(\alpha + \gamma))\|y_n - u_n\|^2,$$

the required (3.10). ∎

Theorem 3.10. *Let H be a finite dimensional space. Let $0 < \rho < \frac{1}{2(\alpha+\gamma)}$, $0 < \beta < \frac{1}{2(\alpha+\gamma)}$ and $0 < \mu < \frac{1}{2(\alpha+\gamma)}$. Let $T : H \longrightarrow C(H)$ be M-Lipschitz continuous operator. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 3.1 converges to a solution u of (2.1).*

Proof. Let $u \in K$ be a solution of (2.1). Since $0 < \rho < \frac{1}{2(\alpha+\gamma)}, 0 < \beta < \frac{1}{2(\alpha+\gamma)}$, $0 < \mu < \frac{1}{2(\alpha+\gamma)}$, from (3.8)-(3.10), it follows that the sequences $\{\|u - u_n\|\}, \{\|u - y_n\|\}$, $\{\|u - w_n\|\}$ are nonincreasing and consequently $\{u_n\}, \{y_n\}$ and $\{w_n\}$ are bounded. Furthermore, we have

$$\begin{aligned} \sum_{n=0}^{\infty}(1 - 2(\alpha + \gamma)\rho)\|w_n - u_n\|^2 &\leq \|u - w_0\|^2 \\ \sum_{n=0}^{\infty}(1 - 2(\alpha + \gamma)\beta)\|y_n - w_n\|^2 &\leq \|u - y_0\|^2 \\ \sum_{n=0}^{\infty}(1 - 2(\alpha + \gamma)\mu)\|y_n - u_n\|^2 &\leq \|u - u_0\|^2 \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{n\to\infty} \|w_n - u_n\| &= 0 \\ \lim_{n\to\infty} \|y_n - w_n\| &= 0 \\ \lim_{n\to\infty} \|y_n - u_n\| &= 0. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{n\to\infty} \|u_{n+1} - u_n\| &\leq \lim_{n\to\infty} \|u_{n+1} - w_n\| + \lim_{n\to\infty} \|y_n - w_n\| \\ &+ \lim_{n\to\infty} \|y_n - u_n\| = 0. \end{aligned} \tag{3.22}$$

Let $\hat{u}$ be the limlit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in (3.2), (3.4) and (3.6), taking the limit $n_j \longrightarrow \infty$ and using (3.22), we have

$$F(\hat{\nu}, T\hat{u}, v) + A(\hat{u}; v - \hat{u}) \geq 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the multivalued equilibrium problems (2.1) and

$$\|u_{n+1} - \hat{u}\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

It remains to show that $\nu \in T(u)$. From (3.7) and using the M-Lipschitz continuity of T, we have

$$||\nu_n - \nu|| \leq M(T(u_n), T(u)) \leq \delta ||u_n - u||,$$

which implies that $\nu_n \longrightarrow \nu$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} d(\nu, T(u)) &\leq \|\nu - \nu_n\| + d(\nu, T(u)) \\ &\leq \|\nu - \nu_n\| + M(T(u_n), T(u)) \\ &\leq \|\nu - \nu_n\| + \delta\|u_n - u\| \quad \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

where $d(\nu, T(u)) = inf\{\|\nu - z\| : z \in T(u)\}$ and $\delta > 0$ is the M-Lipschitz continuity constant of the operator T. From the above inequality, it follows that $d(\nu, T(u)) = 0$. This implies that $\nu \in T(u)$, since $T(u) \in C(H)$. This completes the proof. ∎

We now use the auxiliary principle technique to suggest an inertial proximal method for solving multi-valued equilibrium problems, which was studied and considered by Noor [12], [15] for solving multivalued equilibrium problems (2.6). We remark that the inertial proximal method includes the proximal method as a special case.

For a given $u \in\in K$, consider the auxiliary problem of finding $w \in K, \quad \eta \in T(w)$ such that

$$\rho F(\eta, Tu, v) + \langle w - u - \alpha(u - u), v - w\rangle + \rho A(w; v - w) \geq 0, \quad \forall v \in K, \tag{3.23}$$

where $\rho > 0$ and $\alpha > 0$ are constants. Note that if $w = u$, then w is a solution of (2.1). We use this fact to suggest the following iterative method for solving (2.1).

Algorithm 3.11. *For a given $u_0 \in H$, compute the approximate solution by the iterative schemes:*

$$\begin{aligned} \rho F(\eta_{n+1}, Tu_{n+1}, v) &+ \langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle \\ & +\rho A(u_{n+1}; v - u_{n+1}) \geq 0, \quad \forall v \in K, \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)), \end{aligned}$$

where $\rho > 0$ and $\alpha_n > 0$ are constants. Algorithm 3.11 is known as the inertial proximal method.

Note that for $\alpha_n = 0,$ Algorithm 3.11 reduces to:

Algorithm 3.12. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(\eta_{n+1}, Tu_{n+1}, v) &+ \langle u_{n+1} - u_n, v - u_n\rangle + \rho A(u_{n+1}; v - u_{n+1}) \geq 0, \quad \forall v \in K \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)), \end{aligned}$$

which is called the proximal method for solving multivalued hemiequilibrium problem (2.1).

If $F(u, \nu, v) = \langle \nu, v - u\rangle$, then Algorithm 3.11 reduces to:

Algorithm 3.13. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} \langle \rho\eta_{n+1} &+ u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\quad +\rho A(u_{n+1}; v - u_{n+1}) \geq 0, \quad \forall v \in K, \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)). \end{aligned}$$

which can be written as for $A(.;.) = 0,$

$$\begin{aligned} g(u_{n+1}) &= P_K[g(u_n) - \rho\eta_{n+1} + \alpha_n(u_n - u_{n-1})], \\ \eta_n \in T(w_n) &: \|\eta_{n+1} - \eta_n\| \leq M(T(w_{n+1}), T(w_n)), \end{aligned}$$

which is known as an inertial proximal method for solving the multivalued hemivariational inequalities (2.4) and appears to be a new one.

Note for $\alpha_n = 0,$ Algorithm 3.13 reduces to the well known proximal method for solving multivalued hemivariational inequalities. In a similar way, for suitable and appropriate choices of the trifunction $F(.,.,.), T$ and the space H, one can obtain a number of new and known iterative methods for solving equilibrium and variational inequality problems. Using the techniques and ideas of Noor [14], one can study the convergence analysis of Algorithm 3.12.

4. REGULARIZED HEIMEQUILIBRIUM PROBLEMS

In this section we show that the ideas and techniques developed in the previous sections can be extended for nonconvex (regularized) hemiequilibrium problems which are defined over the uniformly prox-regular sets K in H. It is known [3] that the uniformly prox-regular sets are nonconvex and include the convex sets as a special case. For this purpose, we need the following concepts from nonsmooth analysis, see [2], [3].

Definition 4.1. The proximal normal cone of K at u is given by

$$N^P(K; u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N^P(K; u)$ has the following characterization.

Lemma 4.1. *Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v-u \rangle \leq \alpha \|v-u\|^2, \quad \forall v \in K.$$

Definition 4.2. The Clarke normal cone, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull.

Clarke et al. [3] have introduced and studied a new class of nonconvex sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have

Definition 4.3. For a given $r \in (0, \infty]$, a subset K is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K;u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, v-u \rangle \leq (1/2r)\|v-u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3]. It is clear that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K;u)$ is closed as a set-valued mapping. Thus, we have $N^C(K;u) = N^P(K;u)$. For sake of simplicity, we denote $N(K;u) = N^C(K;u) = N^P(K;u)$ and take $\gamma = \frac{1}{2r}$. Clearly $\gamma = 0$ if and only if $r = \infty$.

From now onward, the set K is γ-prox-regular set, unless otherwise specified.

For a given single-valued trifunction $F(.,.,.) : H \times H \times H \longrightarrow C(H)$, we consider the problem of finding $u \in K, \nu \in T(u)$, such that

$$F(u,\nu,v) + A(u;v-u) + \gamma\|v-u\|^2 \geq 0, \quad \forall v \in K, \tag{4.1}$$

which is called the *multivalued regularized hemiequilibrium problem*. Note that for $\gamma = 0$, problem (4.1) is equivalent to problem (2.1) considered and studied in Sections 2 and 3. Using essentially the techniques of the previous sections, we can analyze and suggest the three-step predictor-corrector and inertial type methods for solving the multivalued regularized hemiequilibrium problems (4.1).

Remark 4.1. We would like to mention that the techniques and ideas of this paper can be extended and generalized for solving mixed quasi general multivalued hemiequilibrium problems involving nonlinear bifunction. It is an open problem to consider the numerical applications of these methods.

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