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GENERAL OSCILLATIONS FOR SOME THIRD ORDER DIFFERENTIAL SYSTEMS WITH NONLINEAR ACCELERATION TERM

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ABSTRACT. We generate some general nonuniform hypotheses for third order differential systems of the form $X^{'''} + F(t, X^{''}) + BX^{'} + CX = P(t)$, in which B and C are not neccessarily constant matrices. Some results requiring sharp conditions on this system have recently been published by the author in [5]. This work however examines more closely crucial properties associated with the generalised nature of the nonlinear acceleration term F, which were largely overlooked in the earlier paper.

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1. INTRODUCTION

The main aim of this paper is to formulate nonuniform nonresonant hypotheses for the existence and possibly, the uniqueness of the solutions of generalised differential systems of the form

$$X^{'''} + F(t, X^{''}) + BX^{'} + CX = P(t) \ , \tag{1.1}$$

subject to

$$X(0) - X(T) = X^{'}(0) - X^{'}(T) = X^{''}(0) - X^{''}(T) = 0 \ , \tag{1.2}$$

and thereby generalising and improving upon our recent results published in [5].

Accordingly, we shall assume that B and C are constant real symmetric $n \times n$ nonsingular matrices, and $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $P : [0, T] \to \mathbb{R}^n$ are n-vectors, which are T- periodic in t. We shall assume further that F satisfies the Carathéodory conditions, that is, $F(\cdot, X^{''})$ is measurable for every $X^{''} \in \mathbb{R}^n$; $F(t, \cdot)$ is continuous for a.e. $t \in [0, T]$, and for each $r > 0$, there exists an integrable function $\gamma_r \in L^1([0, T], \mathbb{R})$ such that $||F(t, X^{''})|| \leq \gamma_r(t)$, for $||X^{''}|| \leq r$ and a.e. $t \in [0, T]$. The case when B and C are not neccessarily constant matrices is also examined.

Arising significantly from an indepth analysis, expounded in [5], of the spectrum of the linear differential operator $\mathcal{L} : dom\mathcal{L} \subset L^\infty \to L^1$ by

$$\mathcal{L}X := -X^{'''} - AX^{''} - BX^{'} - CX \tag{1.3}$$

where

$$dom\mathcal{L} = \{ X \in L^\infty : X \in C^2 \ , \text{ with } X^{''} \text{ absolutely continuous on } [0, T] \text{ and satisfying } (1.2) \} \ ,$$

we were able to generate sharp nonresonant relations of the forms

$$(k + 1)^{-2}\omega^{-2} < \Delta_b \leq \frac{\langle B^{-1}X^{'}, X^{'}\rangle}{\|X^{'}\|^2} \leq \delta_b^{-1} < k^{-2}\omega^{-2} \tag{1.4}$$

and

$$(k + 1)^{-2}\omega^{-2} < \Delta_c^{-1}\delta_a \leq \frac{\langle C^{-1}AX^{''}, X^{''}\rangle}{\|X^{''}\|^2} \leq \delta_c^{-1}\Delta_a < k^{-2}\omega^{-2} \ ,$$

where δ_d and Δ_d represent respectively, the least and greatest eigenvalues of any matrix D. Thus, for the associated eigenvalue problem

$$X^{'''} + BX^{'} + CX = -\lambda CX^{''} \ , \tag{1.5}$$

we deduce from the above analysis that

(i) any $\lambda \neq k^{-2}\omega^{-2}$, for each $k \in \mathbb{N}$, is not an eigenvalue; and

(ii) $\lambda = k^{-2}\omega^{-2}$ for some $k \in \mathbb{N}$, if and only if $B^{-1} = k^{-2}\omega^{-2}I$, B nonsingular.

We observe that (i) implies that any $\lambda > \omega^{-2}$ is not an eigenvalue, and by (ii), the eigenvalues are all contained in the interval $(0 \ , \ \omega^{-2}]$.

Each of the statements (i) and (ii) has an important bearing on the solvability of the PBVP for the linear nonhomogeneous system

$$X^{'''} + \lambda CX^{''} + BX^{'} + CX = P(t) \ , \tag{1.6}$$

with $P \in L^1$. For instance, (i) and the Fredholm's alternative imply that (1.6) has a solution if λ is such that

$$(k + 1)^{-2}\omega^{-2} < \lambda_1 \leq \Delta_c^{-1}\delta_a \leq \frac{\langle C^{-1}AX^{''}, X^{''}\rangle}{\|X^{''}\|^2} \leq \delta_c^{-1}\Delta_a \leq \lambda_2 < k^{-2}\omega^{-2} \ , \tag{1.7}$$

$k \in \mathbb{N}$, λ_1 , λ_2 constants, for $X'' \neq 0$ and C nonsingular. On replacing AX'' with $F(t, X'')$, the inequality translates into the sharp nonresonant criterion

$$(\mathcal{F}_1) \qquad (k+1)^{-2}\omega^{-2} < \Delta_c^{-1}\delta_f \leq \frac{\langle C^{-1}F(t,X''), X''\rangle}{\|X''\|^2} \leq \delta_c^{-1}\Delta_f < k^{-2}\omega^{-2} \ ,$$

uniformly for a.e $t \in [0, T]$ and $X'' \in \mathbb{R}^n$ with $\|X''\| \geq r_0 > 0$, where $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, generated in [5] for the existence of T-periodic solutions of (1.1)-(1.2), with uniqueness established under the analogous condition

$$(\mathcal{F}_2) \qquad (k+1)^{-2}\omega^{-2} < \lambda_1 \leq \frac{\|C^{-1}\big(F(t,X_1'') - F(t,X_2'')\big)\|}{\|X_1'' - X_2''\|} \leq \lambda_2 < k^{-2}\omega^{-2} \ ,$$

holding uniformly in X_1'' , $X_2'' \in \mathbb{R}^n$, with $X_1'' \neq X_2''$, and for a.e $t \in [0, T]$.

On the other hand, (ii) implies that a solution exists for (1.6) for only some classes of P which are orthogonal to the kernel of the linear differential operator $\mathcal{L}$. Furthermore, it suggests other generalisations of condition $\mathcal{F}_1$ which allow the ratio $\frac{\langle C^{-1}F(t,Z),Z\rangle}{\|Z\|^2}$ to touch, and even cross, the spectrum for many values of t, on subsets of $[0, T]$ of measure zero, as $\|Z\| \to \infty$. This leads to the so-called nonuniform or generalised conditions which is the central thrust of this paper.

We shall end this section with an introduction of the functional setting of our problem. For any pair $X, Y \in \mathbb{R}^n$, we shall denote the usual scalar product by $\langle X, Y\rangle$, so that in particular, $\langle X, X\rangle = ||X||^2$ is the usual Euclidean norm in $\mathbb{R}^n$.

It is standard result that if D is a real $n \times n$ symmetric matrix, then for any $X \in \mathbb{R}^n$,

$$(1.8) \qquad \delta_d||X||^2 \ \leq \ \langle DX, X\rangle \ \leq \Delta_d||X||^2,$$

where δ_d and Δ_d are respectively the least and greatest eigenvalues of D. In general, $\lambda_i(D)$ shall denote the eigenvalues of any matrix D.

The classical spaces of k times continuously differentiable functions shall be denoted by $C^k\left([0, T], \mathbb{R}^n\right), k \geq 0$ an integer, where $C^0 = C$ and $C^\infty = \cap_{k\geq 0} C^k$ with norm $\|X\|_{C^k}$; while $L^p = L^p([0, T]),\ 1 \leq p \leq \infty$, will denote the usual Lebesgue spaces, with their respective norms $\|X\|_{L^p}$.

We shall denote by $W_T^{3,1}\left([0, T], \mathbb{R}^n\right)$ the Sobolev space of $T-$periodic functions with norm $||X||_{W_T^{3,1}}$; while $H^1\left([0, T], \mathbb{R}^n\right)$ shall denote the Hilbert space of $T-$periodic functions with norm $\|X\|_{H^1}$. Let $\widetilde{H}^1(0, T) = \left\{\, X \in H^1(0, T) \mid \frac{1}{T}\int_0^T X(t)\,dt = 0 \,\right\}$.

2. SOME ASSOCIATED INEQUALITIES AND PRELIMINARY RESULTS

In order to establish nonuniform results, we shall require some background lemmas, which are adaptations and generalisations of analogous results found in [1], [2] and [4] to the present situation.

Lemma 2.1. *Consider the linear homogeneous system*

$$(2.1) \qquad X''' + A(t)X'' + BX' + CX = 0$$

where C is nonsingular and $A(t) \equiv (a_{ij}(t))$, with $a_{ij} \in L^1(0, T)$, is such that

$$(2.2) \qquad \begin{aligned} (k+1)^{-2}\omega^{-2} \ &\leq \min_{1\leq j,k\leq n}\big(\,\lambda_j(C^{-1})\lambda_k(A(t))\,\big) \leq \lambda_i\big(\,C^{-1}A(t)\,\big) \\ &\leq \quad \max_{1\leq j,k\leq n}\big(\,\lambda_j(C^{-1})\lambda_k(A(t))\,\big) \leq k^{-2}\omega^{-2} \end{aligned}$$

hold uniformly for a.e. $t \in [0,T]$, $i = 1, \ldots, n$, $k \in \mathbb{N}$*, with the strict inequality holding on subsets of* $[0,T]$ *of positive measure.*
Suppose further that for every B *nonsingular, relation (1.4) holds uniformly in* $X' \in \mathbb{R}^n$ *Then, (2.1) - (1.2) has only the trivial solution.*

Proof. Set $D(t) = C^{-1}A(t)$ and let the solution $X(t) = \overline{X}(t) + \widetilde{X}(t)$ have the Fourier expansion

$$X(t) \sim \sum_{i=1}^{n}\biggl(c_{0,i} + \sum_{k=1}^{\infty}(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t)\biggr),$$

such that

$$\overline{X} = \sum_{i=1}^{n}\biggl(c_{0,i} + \sum_{k=1}^{N}(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t)\biggr)$$

and

$$\widetilde{X} = \sum_{i=1}^{n}\sum_{k=N+1}^{\infty}\bigl(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t\bigr) ,$$

for some integer $N > 0$ with $(N+1)^{-2}\omega^{-2} < \lambda < N^{-2}\omega^{-2}$, where $\omega = \frac{2\pi}{T}$.

Then, multiplying (2.1) by $\overline{X}(t) - \widetilde{X}(t)$ and integrating over $[0,T]$ gives

$$\int_0^T \bigl\langle\, \overline{X}(t) - \widetilde{X}(t),\ C^{-1}X'''(t) + D(t)X''(t) + C^{-1}BX'(t) + X(t)\,\bigr\rangle\, dt = 0\ . \tag{2.3}$$

Using the orthogonality of $\overline{X}$ and $\widetilde{X}$ and their derivatives over $[0,T]$ noting that

$$\int_0^T \bigl\langle\, \overline{X}(t) - \widetilde{X}(t),\ C^{-1}X'''(t) + C^{-1}BX'(t)\,\bigr\rangle\, dt = 0\ ,$$

we obtain

$$\begin{aligned}&\textstyle\int_0^T \bigl[\bigl\langle\, D(t)\widetilde{X}'(t),\ \widetilde{X}'(t)\,\bigr\rangle - \|\widetilde{X}(t)\|^2\bigr]\, dt \\ &\qquad - \textstyle\int_0^T \bigl[\bigl\langle\, D(t)\overline{X}'(t),\ \overline{X}'(t)\,\bigr\rangle - \|\overline{X}(t)\|^2\bigr]\, dt = 0\ .\end{aligned} \tag{2.4}$$

Let μ be a constant defined by

$$\mu =: \frac{1}{2}\Bigl(\min \lambda_i\bigl(D(t)\bigr) + \max \lambda_i\bigl(D(t)\bigr)\Bigr) \tag{2.5}$$

for a.e. $t \in [0,T]$. Then by relation (2.2),

$$\begin{aligned}&(k+1)^{-2}\omega^{-2} \le \mu \le k^{-2}\omega^{-2}\ ,\ \ \text{for a.e } t \in [0,T]\ ,\ \text{and} \\ &(k+1)^{-2}\omega^{-2} < \mu < k^{-2}\omega^{-2}\ ,\ \text{on subsets of } [0,T] \text{ of positive measure .}\end{aligned} \tag{2.6}$$

Thus, (2.4) gives

$$0 \ge \int_0^T \Bigl[\mu\|\widetilde{X}'(t)\|^2 - \|\widetilde{X}(t)\|^2\Bigr]\, dt - \int_0^T \Bigl[\mu\|\overline{X}'(t)\|^2 - \|\overline{X}(t)\|^2\Bigr]\, dt = 0\ . \tag{2.7}$$

By Parseval's identity given by

$$\int_0^T \|X\|^2\, dt = \sum_{i=1}^{n}\biggl(c_{0,i}^2 T + \frac{T}{2}\sum_{k=1}^{\infty}(c_{k,i}^2 + d_{k,i}^2)\biggr)\ ,$$

(2.7) becomes

$$\frac{T}{2}\sum_{i=1}^{n}\Big[\sum_{k=N+1}^{\infty}(\mu k^2\omega^2-1)(c_{k,i}^2+d_{k,i}^2)+2c_{0,i}^2T+\sum_{k=1}^{N}(1-\mu k^2\omega^2)(c_{k,i}^2+d_{k,i}^2)\Big]=0\,. \tag{2.8}$$

It follows from (2.6) that $c_{k,i} = 0$ $(k = 0, 1, 2, \dots)$ and $d_{k,i} = 0$ $(k = 1, 2, \dots)$, for all $i = 1, \dots, n$. Thus, $X \equiv 0$, and the lemma follows. ∎

The following inequalities which are associated with system (2.1)-(1.2) are vector derivations of Lemma 1 and Lemma 4 of Mawhin and Ward [4] :

Lemma 2.2. *Let C be nonsingular, and assume that $M,\ N \in L^1([0,T],\boldsymbol{R}^{n^2})$ are nonsingular matrices which satisfy the following conditions*

$$(k+1)^{-2}\omega^{-2}\|Z\|^2 \le \langle C^{-1}M(t)Z,Z\rangle \le \langle C^{-1}N(t)Z,Z\rangle \le k^{-2}\omega^{-2}\|Z\|^2 \tag{2.9}$$

uniformly in $Z \in \boldsymbol{R}^n$, for a.e. $t \in [0,T]$, $k \in \boldsymbol{N}$, $\omega = \frac{2\pi}{T}$, and

$$(k+1)^{-2}\omega^{-2}\|Z\|^2 < \langle C^{-1}M(t)Z,Z\rangle\ ,\ \langle C^{-1}N(t)Z,Z\rangle < k^{-2}\omega^{-2}\|Z\|^2 \tag{2.10}$$

on subsets of $[0,T]$ of positive measure.

Then, there exist constants $\epsilon = \epsilon(M,N,C) > 0$ and $\eta = \eta(M,N,C) > 0$ uniformly a.e. on $[0,T]$, such that for all $D(t) \equiv C^{-1}A(t) \in L^1([0,T],\boldsymbol{R}^{n^2})$ satisfying

$$\langle C^{-1}M(t)Z,Z\rangle - \epsilon\|Z\|^2 \le \langle D(t)Z,Z\rangle \le \langle C^{-1}N(t)Z,Z\rangle + \epsilon\|Z\|^2 \tag{2.11}$$

uniformly in $Z \in \boldsymbol{R}^n$, a.e. on $[0,T]$, and all $X \in W_T^{3,1}([0,T],\boldsymbol{R}^n)$, one has

$$\|C^{-1}X^{'''} + C^{-1}A(\cdot)X^{''} + C^{-1}BX^{'} + X\|_{L^1} \ge \eta\|X\|_{W_T^{3,1}}\ . \tag{2.12}$$

Proof. Let us assume that the conclusion of the Lemma does not hold, that is, ϵ and η do not exist. Then, there exists a sequence $(X_n) \in W^{3,1}([0,T],\mathbf{R}^n)$ with $\|X_n\|_{W^{3,1}} = 1$, and a sequence $(A_n) \in L^1([0,T],\mathbf{R}^{n^2})$ of nonsingular matrices with

$$\langle C^{-1}M(t)Z,Z\rangle - \frac{1}{n}\|Z\|^2 \le \langle D_n(t)Z,Z\rangle \le \langle C^{-1}N(t)Z,Z\rangle + \frac{1}{n}\|Z\|^2\ ,\ n \in \mathbf{N}\ , \tag{2.13}$$

uniformly in $Z \in \mathbf{R}^n$, for a.e. $t \in [0,T]$, where $D_n(t) \equiv C^{-1}A_n(t)$, such that for all $X \in W^{3,1}$, one has

$$\int_0^T \|C^{-1}X_n^{'''}(t) + C^{-1}A_n(t)X_n^{''}(t) + C^{-1}BX_n^{'}(t) + X_n\|\,dt < \frac{1}{n}\ . \tag{2.14}$$

Let $\|D_n\|$ denote the norm of D_n. Then, by (3.13), there exists some $\alpha \in L^1([0,T],\mathbf{R})$ such that

$$\|D_n(t)\| \le \alpha(t)\ ,\ n = 1, 2, \dots \tag{2.15}$$

for a.e. $t \in [0,T]$, $n \in \mathbf{N}$. For example, one can take

$$\alpha(t) \equiv \frac{1}{\|Z\|^2}\Big[\|\langle C^{-1}M(t)Z,Z\rangle - \langle Z,Z\rangle\| + \|\langle C^{-1}N(t)Z,Z\rangle + \langle Z,Z\rangle\|\Big].$$

Now, by the compact embedding of $W^{3,1}([0,T],\mathbf{R}^n)$ into $W^{2,1}([0,T],\mathbf{R}^n)$ and the continuous embedding of $W^{2,1}([0,T],\mathbf{R}^n)$ into $C^1([0,T],\mathbf{R}^n)$ imply that by going to subsequences if neccessary, we can assume that

$$X_n \to X \text{ in } C^1([0,T],\mathbf{R}^n)\ ,\ X_n^{''} \to X^{''} \text{ in } L^\infty([0,T],\mathbf{R}^n) \subset L^1([0,T],\mathbf{R}^n)\ . \tag{2.16}$$

Moreover, by (2.15), we deduce that

$$D_n \rightharpoonup D \text{ in } L^1([0,T],\mathbf{R}^{n^2}) \tag{2.17}$$

so that by (2.13),

$$\langle C^{-1}M(t)Z,Z\rangle \le \langle D(t)Z,Z\rangle \equiv \langle C^{-1}A(t)Z,Z\rangle \le \langle C^{-1}N(t)Z,Z\rangle \tag{2.18}$$

for a.e. $t\in[0,T]$.

On the other hand, for every $\Phi\in L^\infty([0,T],\mathbb{R}^n)$, we have by Schwarz inequality

$$\begin{aligned} \| \textstyle\int_0^T \big\langle\, D_n(t)X_n''(t) \;&-\; D(t)X''(t),\ \Phi(t)\,\big\rangle\, dt\| \\ &\le\; \| \textstyle\int_0^T \big\langle\, D_n(t)\big(X_n''(t)-X''(t)\big),\ \Phi(t)\,\big\rangle\, dt\| \\ &\qquad +\; \| \textstyle\int_0^T \big\langle\, \big(D_n(t)-D(t)\big)X''(t),\ \Phi(t)\,\big\rangle\, dt\| \\ &\le\; \|\Phi\|_\infty \|\alpha\|_{L^1}\|X_n''-X''\|_\infty \\ &\qquad +\; \| \textstyle\int_0^T \big\langle\, \big(D_n(t)-D(t)\big)X''(t),\ \Phi(t)\,\big\rangle\, dt\|\ . \end{aligned} \tag{2.19}$$

The right hand side of (2.19) tends to zero by (2.16) and (2.17), and we deduce that

$$D_nX_n'' \rightharpoonup DX'' \text{ in } L^1([0,T],\mathbb{R}^n)\ . \tag{2.20}$$

By (2.14), (2.16) and (2.20), it follows that

$$\begin{aligned} X_n''' &= -A_n(\cdot)X_n''-BX_n'-CX_n \\ &\rightharpoonup -A(\cdot)X''-BX'-CX \text{ in } L^1([0,T],\mathbb{R}^n)\ . \end{aligned} \tag{2.21}$$

Since the operator $\frac{d^3}{dt^3}: W^{3,1}([0,T],\mathbb{R}^n)\subset L^1([0,T],\mathbb{R}^n)\to L^1([0,T],\mathbb{R}^n)$ is weakly closed, this implies (by (2.16) and (2.21)) that $X\in W_T^{3,1}([0,T],\mathbb{R}^n)$, and $X'''=-A(\cdot)X''-BX'-CX$, that is,

$$X'''(t)+A(t)X''(t)+BX'(t)+CX(t)=0\ , \tag{2.22}$$

for a.e. $t\in[0,T]$ and $X\in W^{3,1}([0,T],\mathbb{R}^n)$.

It follows from (2.9), (2.10), (2.18), (2.22) and Lemma 2.1 that $X\equiv 0$, that is, $X_n\to 0$ in $W^{3,1}([0,T],\mathbb{R}^n)$ as $n\to\infty$. But this clearly contradicts the initial assumption that $\|X_n\|_{W^{3,1}}=1$ for all n, and the proof is complete. ∎

Lemma 2.3. *Let C and $N\in L^1([0,T],\mathbb{R}^{n^2})$ be defined as in Lemma 2.2, such that*

$$\langle C^{-1}N(t)Z,Z\rangle \ge \omega^{-2}\|Z\|^2 \tag{2.23}$$

uniformly in $Z\in\mathbb{R}^n$ for a.e. $t\in[0,T]$, with the strict inequality holding on subsets of $[0,T]$ of measure zero.

Then, there exists a constant $\rho=\rho(C,N)>0$ such that for all $\widetilde{X}\in\widetilde{H}^1([0,T],\mathbb{R}^n)$, we have

$$B_N(\widetilde{X})\equiv \frac{1}{T}\int_0^T\Big(\big\langle\, C^{-1}N(t)\widetilde{X}'(t),\ \widetilde{X}'(t)\,\big\rangle-\|\widetilde{X}(t)\|^2\Big)\,dt\ \ge\ \rho\|\widetilde{X}\|^2_{\widetilde{H}^1}\ . \tag{2.24}$$

Proof. Using (2.23) and Wirtinger's inequality, we observe that for all $\widetilde{X}\in\widetilde{H}^1([0,T),\mathbb{R}^n)$ and $\omega=\frac{2\pi}{T}$

$$\frac{1}{T}\int_0^T\Big[\big\langle\, C^{-1}N(t)\widetilde{X}'(t),\ \widetilde{X}'(t)\,\big\rangle-\|\widetilde{X}(t)\|^2\Big]\,dt \ge \frac{1}{T}\int_0^T\Big[\omega^{-2}\|\widetilde{X}'(t)\|^2-\|\widetilde{X}(t)\|^2\Big]\,dt=0\ .$$

Moreover

$$\int_0^T\Big[\big\langle\, C^{-1}N(t)\widetilde{X}'(t),\ \widetilde{X}'(t)\,\big\rangle-\|\widetilde{X}(t)\|^2\Big]\,dt=\int_0^T\Big[\omega^{-2}\|\widetilde{X}'(t)\|^2-\|\widetilde{X}(t)\|^2\Big]\,dt\ =\ 0$$

if and only if $\widetilde{X}(t) = \Lambda\,\sin(\omega t + \vartheta)$, for $\Lambda \in \mathbf{R}^n,\ \vartheta \in \mathbf{R}$. In this case, we see that

$$0 = \int_0^T \left[\left\langle\, C^{-1}N(t)\widetilde{X}'(t),\ \widetilde{X}'(t)\,\right\rangle\ - \omega^{-2}\|\widetilde{X}'(t)\|^2 \right] dt \\ = \|\Lambda\|^2\omega^2 \int_0^T \left(\Gamma(t) - \omega^{-2}\right)\sin^2(\omega t + \vartheta)\, dt$$

where we set $\Gamma(t) \equiv \max_{1\le i\le n}\,\lambda_i(C^{-1}N(t)$. Then certainly by (2.23), $\Gamma(t) \geq \omega^{-2}$, with the strict inequality holding on subsets of $[0,T]$ of measure zero. Thus we must have $\Lambda = 0$, which implies that $\widetilde{X} = 0$, and therefore $X = 0$. It follows that for all $\widetilde{X} \in \widetilde{H}^1([0,T),\mathbf{R}^n)$, we have $B_N(\widetilde{X}) \geq 0$.

Let us assume now that the conclusion of the Lemma is false. Then we can find a sequence $(\widetilde{X}_n) \in \widetilde{H}^1([0,T],\mathbf{R}^n)$ and $\widetilde{X} \in \widetilde{H}^1([0,T],\mathbf{R}^n)$ such that

$$\|\widetilde{X}_n\|_{\widetilde{H}^1} = 1\ ,\ \widetilde{X}_n \to \widetilde{X} \text{ in } C([0,T],\mathbf{R}^n)\ ,\ \widetilde{X}_n \rightharpoonup \widetilde{X} \text{ in } H^1([0,T],\mathbf{R}^n) \tag{2.25}$$

and

$$0\ \leq\ B_N(\widetilde{X}_n) \equiv \frac{1}{T}\int_0^T \left(\left\langle\, C^{-1}N(t)\widetilde{X}_n'(t),\ \widetilde{X}_n'(t)\,\right\rangle - \|\widetilde{X}_n(t)\|^2\right) dt\ \leq\ \frac{1}{n}\ ,\ n \in \mathbf{N}\ .$$

Proceeding as in Lemma 1 of [4], we deduce that $B_N(\widetilde{X}) \leq 0$, so that by the first part of the proof, $\widetilde{X} = 0$, leading to $\|\widetilde{X}_n\|_{\widetilde{H}^1} \to 0$, a contradiction with the first equality in (2.25). ∎

3. NONUNIFORM NONRESONANCE RESULTS

The following result which holds under nonuniform nonresonance conditions can now be proved:

Theorem 3.1. *Let C be nonsingular and positive definite, and suppose that F is an L^1-Carathéodory function which satisfies*

$$(k+1)^{-2}\omega^{-}2\ \leq\ \frac{\langle C^{-1}M(t)Z,Z\rangle}{\|Z\|^2} \\ \leq\ \liminf_{\|Z\|\to\infty}\frac{\langle C^{-1}F(t,Z),Z\rangle}{\|Z\|^2} \leq\ \limsup_{\|Z\|\to\infty}\frac{\langle C^{-1}F(t,Z),Z\rangle}{\|Z\|^2} \\ \leq\ \frac{\langle C^{-1}N(t)Z,Z\rangle}{\|Z\|^2} \leq k^{-2}\omega^{-2} \tag{$\mathcal{F}_3$}$$

uniformly in $Z \in \boldsymbol{R}^n$ for a.e. $t \in [0,T]$, $k \in \boldsymbol{N}$, and $M,\ N \in L^1([0,T],\boldsymbol{R}^{n^2})$ are such that $(k+1)^{-2}\omega^{-2}\|Z\|^2 < \langle C^{-1}M(t)Z,Z\rangle$ and $\langle C^{-1}N(t)Z,Z\rangle < k^{-2}\omega^{-2}\|Z\|^2$ on subsets of $[0,T]$ of positive measure.

Then, the system (1.1)-(1.2) has at least one solution for every $P \in L^1([0,T],\boldsymbol{R}^n)$.

Proof. We choose ϵ as in Lemma 2.2. Then, by ($\mathcal{F}_3$), we can fix a constant vector $\xi = \xi(\epsilon)$ with each $\xi_i > 0$ such that

$$\langle C^{-1}M(t)Z,Z\rangle - \epsilon\|Z\|^2 \leq \langle C^{-1}F(t,Z),Z\rangle \leq \langle C^{-1}N(t)Z,Z\rangle + \epsilon\|Z\|^2 \tag{3.1}$$

for a.e. $t \in [0,T]$ and all $Z \in \mathbf{R}^n$ with $|z_i| \geq \xi_i$.

Let us now define $\gamma(t,Z) \equiv \big(\gamma_i(t,Z)\big)_{1\le i\le n} : [0,T]\times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\gamma_i(t,Z) = \begin{cases} z_i^{-1} f_i(t,Z)\ , & \text{if } |z_i| \ge \xi_i\ ; \\ z_i\xi_i^{-2} f_i(t,z_1,\dots,z_{i-1},\xi_i,z_{i+1},\dots,z_n) + (1-\frac{z_i}{\xi_i})\alpha(t)\ , & \text{if } 0 \le z_i < \xi_i\ ; \\ z_i\xi_i^{-2} f_i(t,z_1,\dots,z_{i-1},-\xi_i,z_{i+1},\dots,z_n) + (1+\frac{z_i}{\xi_i})\alpha(t)\ , & \text{if } -\xi_i \le z_i < 0\ . \end{cases}$$

for a.e. $t\in[0,T]$, where α is given by

$$\alpha(t) \equiv \frac{1}{\|Z\|^2}\Big[\|\langle C^{-1}M(t)Z,Z\rangle - \langle Z,Z\rangle\| + \|\langle C^{-1}N(t)Z,Z\rangle + \langle Z,Z\rangle\|\Big]\ , \tag{3.2}$$

so that by construction and (3.1), we deduce that

$$\langle C^{-1}M(t)Z,Z\rangle - \epsilon\|Z\|^2 \le \langle C^{-1}\gamma(t,Z),Z\rangle \le \langle C^{-1}N(t)Z,Z\rangle + \epsilon\|Z\|^2 \tag{3.3}$$

for a.e. $t\in[0,T]$ and $Z\in\mathbb{R}^n$.

The function $\widetilde{F} \equiv (\tilde f_i(t,Z))_{1\le i\le n}[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ defined by $\tilde f_i(t,Z) = \gamma_i(t,Z)z_i$ satisfies the Carathéodory conditions, by construction. Hence, setting $\chi(t,Z) = F(t,Z) - \widetilde{F}(t,Z)$, then $\chi(t,Z)$ is also L^1-Carathéodory with

$$\|\chi(t,Z)\| \le \sup_{|z_i|\le\xi_i} \|F(t,Z) - \widetilde{F}(t,Z)\| \le \varphi(t) \tag{3.4}$$

for a.e. $t\in[0,T]$ and $Z\in\mathbb{R}^n$, for some $\varphi\in L^1([0,T],\mathbb{R})$ depending only on $M,\ N$ and γ_r mentioned at the beginning in association with F. Then, the problem (1.1) is equivalent to

$$X'''(t) + \widetilde{F}(t,X''(t)) + \chi(t,X''(t)) + BX'(t) + CX(t) = P(t)\ . \tag{3.5}$$

By the Leray-Schauder technique (see Mawhin [3]), the proof of the Theorem now follows by showing that there is a constant $K>0$, independent of $\lambda\in(0,1)$, such that $\|X\|_{C^2} < K$, for all possible solutions X of the homotopy

$$X''' + (1-\lambda)N(t)X'' + \lambda\widetilde{F}(t,X'') + BX' + CX + \lambda\chi(t,X'') = \lambda P(t) \tag{3.6}$$

$\lambda\in[0,1]$, or equivalently

$$\begin{aligned} C^{-1}X''' + (1-\lambda)C^{-1}N(t)X'' + \lambda C^{-1}\widetilde{F}(t,X'') + C^{-1}BX' + X \ &+ \lambda C^{-1}\chi(t,X'') \\ &= C^{-1}\lambda P(t). \end{aligned} \tag{3.7}$$

We observe from (3.3) that

$$\begin{aligned} \langle C^{-1}M(t)Z,Z\rangle - \epsilon\|Z\|^2 \le \langle\, (1-\lambda)C^{-1}N(t)Z \ &+ \lambda C^{-1}\widetilde{F}(t,Z), Z\,\rangle \\ &\le \langle C^{-1}N(t)Z,Z\rangle + \epsilon\|Z\|^2 \end{aligned} \tag{3.8}$$

for a.e. $t\in[0,T],\ Z\in\mathbb{R}^n$ and $\lambda\in[0,1]$.

Thus, we may set $(1-\lambda)C^{-1}N(t)X'' + \lambda C^{-1}\widetilde{F}(t,X'') \equiv C^{-1}A(t)X''$, for a.e. $t\in[0,T],\ X''\in\mathbb{R}^n$ and $\lambda\in[0,1]$, where, by (3.8), $A(t)$ is such that

$$\langle C^{-1}M(t)X'',X''\rangle - \epsilon\|X''\|^2 \le \langle C^{-1}A(t)X'',X''\rangle \le \langle C^{-1}N(t)X'',X''\rangle + \epsilon\|X''\|^2 \tag{3.9}$$

for a.e. $t\in[0,T],\ X''\in\mathbb{R}^n$ and $\lambda\in[0,1]$.

Thus integrating (3.7) over $[0,T]$ gives

$$\begin{aligned} 0 \ge \textstyle\int_0^T \|C^{-1}X''' + C^{-1}A(t)X'' \ &+ C^{-1}BX'(t) + X\|\,dt \\ &- \ \delta_c^{-1}\big(\textstyle\int_0^T \|\chi(t,X'')\|\,dt - \int_0^T \|P(t)\|\,dt\big) \end{aligned} \tag{3.10}$$

where $\delta_c > 0$ is the least eigenvalue of C assumed positive definite. It follows from Lemma 2.2 that

$$0 \geq \eta \|X\|_{W^{3,1}} - \delta_c^{-1}\big(\|\varphi\|_{L^1} - \|P\|_{L^1}\big) \tag{3.11}$$

which yields a constant $K_0 > 0$ such that $\|X\|_{W^{3,1}} \leq K_0$. Hence, we obtain the required constant $K > 0$ such that $\|X\|_{C^2} < K$, following a standard procedure just as in [2], completing the proof. ∎

Remark 3.1. We can obtain a partial generalisation of Theorem 3.1, corresponding to $k = 1$, for nonlinear systems of the form

$$X''' + F(t, X'') + \frac{d}{dt}\text{grad}\, g(X) + CH(X) = P(t)\ , \tag{3.12}$$

under suitable assumptions on F satisfying some requirements in respect of the first (possible) eigenvalue of the eigenvalue problem (2.1)-(1.2). Here, $g : \mathbb{R}^n \to \mathbb{R}$ is a C^2-function, $H : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, while $C,\ P$ are as specified earlier.

Theorem 3.2. *Let C be nonsingular and positive definite, and suppose that F is an L^1-Carathéodory function which satisfies*

$$(\mathcal{F}_4) \qquad \limsup_{\|Z\|\to\infty} \langle C^{-1}F(t,Z), Z\rangle \ \geq\ \langle C^{-1}N(t)Z, Z\rangle \ \geq\ \omega^{-2}\|Z\|^2$$

uniformly in $Z \in \mathbb{R}^n$ for a.e. $t \in [0,T]$, and $N \in L^1([0,T], \mathbb{R}^{n^2})$ is such that $\langle C^{-1}N(t)Z, Z\rangle > \omega^{-2}\|Z\|^2$ on subsets of $[0,T]$ of measure zero.

Furthermore, suppose that H is continuous and, for every $X \in \mathbb{R}^n$, satisfies

$$(\mathcal{H}) \qquad \langle\, H(X),\ X\,\rangle \ \geq\ \Delta_0\|X\|^2\ ,\ \Delta_0 \leq 0\ .$$

Then, for all arbitrary $g \in C^2$, (3.12)-(1.2) has at least one solution for every $P \in L^1([0,T], \mathbb{R}^n)$.

Proof. Proceeding as in the preceding proof, for some $\epsilon > 0$, there exists $\xi = \xi(\epsilon) > 0$ such that

$$\langle C^{-1}F(t,Z), Z\rangle \leq \langle C^{-1}N(t)Z, Z\rangle + \epsilon\|Z\|^2 \tag{3.13}$$

for a.e. $t \in [0,T]$ and all $Z \in \mathbb{R}^n$ with $|z_i| \geq \xi_i$.

Then, define $\widetilde{F}(t,Z)$ and $\chi(t,ZX)$ as before, so that the relations

$$(1-\lambda)C^{-1}N(t)Z + \lambda C^{-1}\widetilde{F}(t,Z) \ \leq\ C^{-1}N(t)Z + \epsilon Z \tag{3.14}$$

and

$$\|\chi(t,Z)\| \leq \varphi(t) \tag{3.15}$$

hold, for a.e. $t \in [0,T]$ and every $Z \in \mathbb{R}^n$.

It suffices to establish the appropriate a-priori bounds for the λ-dependent family of systems

$$\begin{aligned} X''' + (1-\lambda)(N(t)X'' \ &+ CX) + \lambda\widetilde{F}(t,X'') \\ &+ \lambda\tfrac{d}{dt}\text{grad}\, g(X) + \lambda CH(X) + \lambda\chi(t,X'') = \lambda P(t) \end{aligned} \tag{3.16}$$

$\lambda \in [0,1]$, or equivalently

$$\begin{aligned} C^{-1}X''' + \ &(1-\lambda)C^{-1}N(t)X'' + \lambda C^{-1}\widetilde{F}(t,X'') + (1-\lambda)X \\ &+ \ C^{-1}\lambda\tfrac{d}{dt}\text{grad}\, g(X) + \lambda H(X) + \lambda C^{-1}\chi(t,X'') = C^{-1}\lambda P(t)\ . \end{aligned} \tag{3.17}$$

Let X be a solution of (3.17)-(1.2). Taking the scalar product of (3.17) with $X(t)$ and integrating by parts over $[0,T]$ using (3.14) gives

$$\begin{aligned}(3.18)\qquad \tfrac{(1-\lambda)}{T}\textstyle\int_0^T\big[\langle\|X(t)\|^2- & \ C^{-1}N(t)X'(t),\ X'(t)\,\rangle\big]\,dt\\ & - \ \tfrac{\epsilon}{T}\textstyle\int_0^T\|X'(t)\|^2\,dt+\tfrac{\lambda}{T}\int_0^T\langle\, H(X),X\,\rangle\,dt\\ & \geq \ \tfrac{\lambda}{T}\textstyle\int_0^T C^{-1}\langle\, P(t)-\chi(t,X''),\ X\,\rangle\,dt\end{aligned}$$

noting that

$$\int_0^T\langle\, C^{-1}\frac{d}{dt}\text{grad}\,g(X),\ X\,\rangle\,dt=0\,.$$

Applying now the hypotheses on H, P, χ, we get

$$\begin{aligned}(3.19)\qquad 0\geq \textstyle\int_0^T\big[\langle\|C^{-1}N(t)X'(t),\ X'(t)\,\rangle- & \ (1+\Delta_0)X(t)\|^2\big]\,dt-\tfrac{\epsilon}{T}\textstyle\int_0^T\|X'(t)\|^2\,dt\\ & -\tfrac{\lambda}{T}\textstyle\int_0^T C^{-1}\langle\,\chi(t,X'')-P(t),\ X\,\rangle\,dt\,.\end{aligned}$$

Since $\Delta_0\leq 0$ by definition, we deduce from Lemma 2.3 and the Cauchy-Schwartz inequality that

$$(3.20)\qquad 0\ \geq\ \rho\|X\|^2_{H^1}-\frac{\epsilon}{T}\int_0^T\|X'(t)\|^2\,dt-\delta_c^{-1}\big(\|\varphi\|_{L^1}+\|P\|_{L^1}\big)\|X\|_\infty\,.$$

That is

$$(3.21)\qquad \rho\|X\|^2_{H^1}\ =\ \frac{\rho}{T}\|X'\|^2_{L^2}\ \leq\ \frac{\epsilon}{T}\int_0^T\|X'(t)\|^2\,dt+\delta_c^{-1}\big(\|\varphi\|_{L^1}+\|P\|_{L^1}\big)\|X\|_\infty\,.$$

Now observe that

$$\int_0^T\Big[(1-\lambda)C^{-1}N(t)X''+\lambda\widetilde{F}(t,X'')\Big]\,dt\leq\int_0^T(\,C^{-1}N(t)X''+\epsilon X''\,)\,dt=0\,.$$

Hence, taking the average of (3.17) on $[0,T]$, we obtain by the Mean Value Theorem,

$$\begin{aligned}(3.22)\qquad \big\|\ (1-\lambda)X(t_0)+ & \ \lambda H(X(t_0))\ \big\|\\ & = \ \big\|\ (1-\lambda)\big(\tfrac{1}{T}\textstyle\int_0^T X(t)\,dt\big)+\lambda\big(\tfrac{1}{T}\int_0^T H(X(t))\,dt\big)\ \big\|\\ & \leq \ \|C^{-1}\|\ \big(\tfrac{1}{T}\|\varphi\|_{L^1}+\tfrac{1}{T}\|P\|_{L^1}\big)\ :=\ \kappa_1\end{aligned}$$

for some $t_0\in[0,T]$.

Notice by the continuity of H that for any $k>0$, there exists a $q=q(k)>0$ such that $\|H(X)\|>k$ for every $X\in\mathbb{R}^n$ with $\|X\|>\max\{\,k,q\,\}$. Hence, for any $\lambda\in(0,1]$, we have

$$\|\,(1-\lambda)X+\lambda H(X)\,\|\ \geq\ (1-\lambda)k+\lambda k=k$$

for every $\|X\|>\max\{\,k,q\,\}$. Thus, choosing $k>\kappa_1$, it follows that

$$(3.23)\qquad \|X(t_0)\|\leq\max\{\,k,q\,\}\ :=\ \kappa_2\,.$$

Hence from the relation $X(t)=X(t_0)+\int_{t_0}^t X'(s)\,ds$, we obtain

$$(3.24)\qquad \|X\|_\infty\ \leq\ \kappa_2+\Big(T\int_0^T\|X'(s)\|^2ds\Big)^{\frac{1}{2}}\ \leq\ \kappa_2+T^{\frac{1}{2}}\|X'\|_{L^2}\,.$$

Substituting (3.24) into (3.21), with ϵ sufficiently small such that $\rho-\epsilon>0$, now yields some constants $\kappa_3>0$, $\kappa_4>0$ for which

$$(3.25)\qquad \|X'\|_{L^2}\ \leq\ \kappa_3$$

and

$$(3.26)\qquad \|X\|_\infty\ \leq\ \kappa_4\,.$$

Next, multiplying (3.17) scalarly by $X^{'}(t)$ and integrating over $[0,T]$ gives

$$-\tfrac{1}{T}\int_0^T\langle C^{-1}X'',X''\rangle\,dt+\tfrac{\lambda}{T}\int_0^T\langle C^{-1}\;\tfrac{d}{dt}\text{grad}\,g(X),X'\rangle\,dt + \tfrac{\lambda}{T}\int_0^T\langle\chi(t,X)-P(t),X'\rangle\,dt=0 \tag{3.27}$$

noting by definition that

$$\frac{1}{T}\int_0^T\langle\,(1-\lambda)C^{-1}N(t)X''+\lambda C^{-1}\widetilde{F}(t,X''),X'\,\rangle\,dt+\frac{\lambda}{T}\int_0^T\langle H(X),X'\rangle\,dt=0\,.$$

Thus, from (3.27) we obtain

$$0\geq\Delta_c^{-1}\|X''\|^2_{L^2}-\delta_c^{-1}\Delta_g\|X'\|^2_{L^2}-\delta_c^{-1}\big(\|\varphi\|_{L^1}+\|P\|_{L^1}\big)\|X'\|_\infty\,, \tag{3.28}$$

where Δ_c and δ_c are the greatest and least eigenvalues of C, and Δ_g is the greatest eigenvalue of the Hessian matrix corresponding to g. Again from the relation $\|X'\|_\infty\leq\sqrt{T}\|X''\|_{L^2}$, we deduce using (3.25) in (3.28) constants $\kappa_5>0$, $\kappa_6>0$ such that

$$\|X''\|_{L^2}\;\leq\;\kappa_5 \tag{3.29}$$

and

$$\|X'\|_\infty\;\leq\;\kappa_6\,. \tag{3.30}$$

Finally, integrating (3.17) and using the continuity of H and (3.26), we can find a constant κ_7, such that

$$\|X'''\|_{L^1}\leq\kappa_7 \tag{3.31}$$

so that

$$\|X''\|_\infty\leq T\|X'''\|_{L^1}=T\kappa_7. \tag{3.32}$$

Therefore, by (3.26), (3.30) and (3.32),

$$\|X\|_{C^2}=\|X\|_\infty+\|X'\|_\infty+\|X''\|_\infty\leq\kappa_8 \tag{3.33}$$

for some $\kappa_8>0$, and we are done. ∎

Finally, we give some uniqueness results for (1.1)-(1.2).

Theorem 3.3. *Assuming the hypotheses of Theorem 3.1 on $C,\ F,\ P,\ M$ and N hold, and suppose further that*
$(\mathcal{F}_5)$

$$(k+1)^{-2}\omega^{-2}\leq\frac{\langle C^{-1}M(t)(Z_1-Z_2),Z_1-Z_2\rangle}{\|Z_1-Z_2\|^2}\;\leq\;\frac{\langle\,C^{-1}\big(F(t,Z_1)-F(t,Z_2)\big),Z_1-Z_2\,\rangle}{\|Z_1-Z_2\|^2}\leq\frac{\langle C^{-1}N(t)(Z_1-Z_2),Z_1-Z_2\rangle}{\|Z_1-Z_2\|^2}\leq k^{-2}\omega^{-2}$$

or

$$\frac{\langle\,C^{-1}\big(F(t,Z_1)-F(t,Z_2)\big),Z_1-Z_2\,\rangle}{\|Z_1-Z_2\|^2}\;\geq\;\omega^{-2} \tag{$\mathcal{F}_6$}$$

holds uniformly for a.e. $t\in[0,T],\ k\in\boldsymbol{N},\ \omega=\frac{2\pi}{T}$ and all $Z_1,\ Z_2\in\boldsymbol{R}^n$ with $Z_1\neq Z_2$, with the strict inequalities holding on subsets $[0,T]$ of positive measure in $\mathcal{F}_5$, and of measure zero in $\mathcal{F}_6$.

Then, the system (1.1)-(1.2) has a unique T-periodic solution.

Proof. It follows from ($\mathcal{F}_5$) that ($\mathcal{F}_3$) holds, and from ($\mathcal{F}_6$) that ($\mathcal{F}_4$) holds (take $F(t,0) = 0$). Thus, the existence of at least one T-periodic solution for (1.1)-(1.2) is assured by Theorem 3.1 and Theorem 3.2.

Now, let $Z_1,\ Z_2 \in \mathbb{R}^n$ be any two T-periodic solutions of (1.1)-(1.2) and set $U(t) = Z_1(t) - Z_2(t)$, then $U(t)$ satisfies

$$U^{'''}(t) + D^\star\big(t, U^{''}(t)\big)U^{''}(t) + BU^{'}(t) + CU(t) = 0 \tag{3.34}$$

with the accompanying T-periodic boundary conditions on U, where $D^\star\big(\cdot, U^{''}(\cdot)\big)$ is a matrix defined as follows:

F *subject to ($\mathcal{F}_5$)*: Set

$$D^\star\big(t, U(t)\big)U(t) = \begin{cases} F(t, U + Z_2) - F(t, Z_2)\ , & \text{if } U \neq 0 \\ \\ M(t)\ , & \text{if } U = 0 \end{cases}$$

and satisfies $\lambda_i(C^{-1}M(t)) \le \lambda_i\big(\,C^{-1}D^\star(t, U(t))\,\big) \le \lambda_i(C^{-1}N(t))$ uniformly in $U \in \mathbb{R}^n$ for a.e. $t \in [0,T]$. It follows from Lemma 2.1 that (3.34) has only the trivial solution $U \equiv 0$, and we are done.

F *subject to ($\mathcal{F}_6$)*: Here

$$D^\star\big(t, U(t)\big)U(t) = \begin{cases} F(t, U + Z_2) - F(t, Z_2)\ , & \text{if } U \neq 0 \\ \\ \omega^{-2}\ , & \text{if } U = 0 \end{cases}$$

and satisfies $\langle\, C^{-1}D^\star(t, U(t))U, U\,\rangle \ge \langle\, C^{-1}N(t)U, U\,\rangle \ \ge\ \omega^{-2}$ uniformly in $U \in \mathbb{R}^n$ for a.e. $t \in [0,T]$. Multiply now (3.34) scalarly by $-U^{''}(t)$ and integrate over $[0,T]$, we get by Lemma 2.3

$$0 \ =\ \int_0^T \big(\langle\, C^{-1}D^\star(t, U(t))U^{'}(t), U^{'}(t)\,\rangle - \|U\|^2\big)\, dt \ \ge\ \rho\|U\|^2_{H^1} \tag{3.35}$$

from which we deduce that $U \equiv 0$, and the proof is complete. ∎

REFERENCES

[1] G. CONTI, R. IANNACCI and M.N. NKASHAMA, Periodic solutions of Liénard systems at resonance, *Ann. Math. Pura. Appl.* **141** (4), (1985), 313 - 327.

[2] J.O.C. EZEILO and M.N. NKASHAMA, Resonant and nonresonant oscillations for third-order nonlinear ordinary differential equations, *Nonlinear Analysis T.M.A.* **12** (10), (1988), 1029 - 1046.

[3] J. MAWHIN, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics **40**, American Math. Soc., Providence, R.I., (1979).

[4] J. MAWHIN and J.R. WARD, Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Liénard and Duffing equations, *Rocky Mount. J. Math.* **12** (4), (1982), 643 - 653.

[5] A.S UKPERA, Nonresonant oscillations for some third order differential systems with nonlinear acceleration term, *Nonlinear Studies* **9** (1), (2002), 27 - 40.

[6] A.S. UKPERA, Further results for some third order differential systems with nonlinear dissipation, *Acta Univ. Palacki. Olomuc. Mathematica* **43**, (2004), 153 - 167.

[7] A.S. UKPERA and A.U. AFUWAPE, Resonant oscillations for forced third order nonlinear periodic systems, *Nonlinear Studies* (to appear).