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COMPACTLY SUPPORTED INTERPOLATORY ORTHOGONAL MULTIWAVELET PACKETS

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ABSTRACT. Compactly supported interpolatory orthogonal multiwavelet packets are introduced. Precisely, if both the multiscaling function and the corresponding multiwavelet have the same interpolatory property, then the multiwavelet packets are also interpolatory orthogonal. Thus, the coefficients of decomposition or synthesis of multiwavelet packets can be realized by sampling instead of inner products. This multiwavelet packets provide a finer decomposition of multiwavelet packets space and give a better localization.

Key words and phrases: Orthogonal, Interpolatory, Multiwavelet, Multiwavelet packets.

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1. INTRODUCTION

Wavelet analysis is a powerful tool for time-frequency localization. In order to have better localization for high frequency components in the wavelet decomposition, Coifman et al. (see [2]) introduced another kinds of bases called wavelet packets. In order to have symmetry, Cohen et al. (see [3]) constructed biorthogonal wavelets. The biorthogonal wavelet packets were considered by Chui and Li (see [4]). However, it was shown by Cohen and Daubechies (see [5]) that the biorthogonal wavelet packets are globally unstable. Orthogonal multiwavelet can have some features that scalar orthogonal wavelets cannot (see [6]). Another new feature of multiwavelet is that they can be made both orthogonal and interpolatory (see [9]-[1]), which provides nice sampling theorems for signal processing. The purpose of this paper is to study the construction of compactly supported interpolatory orthogonal multiwavelet packets. If multiwavelet is interpolatory orthogonal, then the corresponding multiwavelet packets is also interpolatory orthogonal.

Let $\Phi(x) = (\phi_1, \phi_2)^T, \phi_1, \phi_2 \in L^2(R)$, satisfy the following equation:

$$\Phi(x) = \sum_{k\in Z} P_k \Phi(2x-k), \tag{1.1}$$

where $\{P_k\}$ is a finitely supported sequence of 2×2 matrices called the two-scale matrix sequence. $\Phi(x)$ is called a multiscaling function with multiplicity two.

Define the two-scale matrix symbol of Φ as follows

$$P(z) = \frac{1}{2}\sum_{k\in Z} P_k z^k. \tag{1.2}$$

Define a subspace $V_j \subset L^2(R)$ by $V_j = \mathrm{clos}_{L^2(R)}\langle \phi_{\ell:j,k} : \ell = 1,2; k \in Z\rangle,\ j \in Z$, here and afterwards, for $f_\ell \in L^2$, we will use the notation $f_{\ell:j,k} = 2^{\frac{j}{2}} f_\ell(2^j x - k)$. The subspaces $\{V_j\}$ form a multiresolution analysis with multiplicity two. Let $W_j, j \in Z$, denote the complementary subspace of V_j in V_{j+1}. Suppose that the function $\Psi(x) = (\psi_1, \psi_2)^T, \psi_\ell \in L^2, \ell = 1,2$ is multiwavelet associated with multiscaling function Φ, and satisfy the following equation:

$$\Psi(x) = \sum_{k\in Z} Q_k \Phi(2x-k). \tag{1.3}$$

Define the two-scale matrix symbol of Ψ as follows

$$Q(z) = \frac{1}{2}\sum_{k\in Z} Q_k z^k. \tag{1.4}$$

We call $\Phi(x) = (\phi_1, \phi_2)^T$ orthogonal multiscaling function, if

$$\langle \Phi(\cdot), \Phi(\cdot - n)\rangle = \delta_{0,n} I_2, n \in Z. \tag{1.5}$$

$\Psi(x) = (\psi_1, \psi_2)^T$ will be said to be the orthogonal multiwavelet associated with multiscaling function Φ, if Φ, and Ψ satisfy the following equations

$$\langle \Phi(\cdot), \Psi(\cdot - n)\rangle = O, \quad \langle \Psi(\cdot), \Psi(\cdot - n)\rangle = \delta_{0,n} I_2, n \in Z, \tag{1.6}$$

where O and I_2 denote the zero matrix and unity matrix, respectively.

A multiscaling function Φ satisfying (1.1) is called interpolatory if ϕ_1, ϕ_2 are continuous, compactly supported and satisfy for $k \in Z, \ell = 0, 1$,

$$\phi_j(k + \frac{\ell}{2}) = \delta_{k,0}\delta_{j,\ell+1}, j = 1,2. \tag{1.7}$$

The condition (1.7) means that ϕ_1 is cardinal at integers and vanishes at half integers; ϕ_2 is cardinal at half integers and vanishes at integers.

Let $e_1 = (1,0)^T$, and $e_2 = (0,1)^T$. Then the interpolatory condition (1.7) is equivalent to the following equations:

$$\Phi(k) = \delta_{k,0}e_1, \quad \text{and} \quad \Phi(k+\frac{1}{2}) = \delta_{k,0}e_2. \tag{1.8}$$

In [8], Zhou forces the two-scale matrix symbol $P(z), Q(z)$ to take the forms,respectively

$$P(z) = \frac{1}{2}\begin{bmatrix} 1 & H(z) \\ z & G(z) \end{bmatrix}, \quad Q(z) = \frac{1}{2}\begin{bmatrix} 1 & -H(z) \\ z & -e^{i\theta}z^{2p+1}\overline{H(-z)} \end{bmatrix}, \tag{1.9}$$

where $H(z)$, $G(z)$ are both Laurent polynomials, satisfy $|H(z)|^2 + |H(-z)|^2 = 2, G(z) = e^{i\theta}z^{2p+1}\overline{H(-z)}, \theta \in R, p \in Z, \forall |z| = 1$. Then $\Phi(x)$ generated by $P(z)$ is interpolatory orthogonal multiscaling function, i.e., $\Phi(x)$ satisfies the interpolatory condition (1.7) or (1.8). $\Psi(x)$ generated by $Q(z)$ is orthogonal multiwavelet associated with multiscaling function Φ. Both the multiscaling function and the corresponding multiwavelet have the same interpolatory property.

2. INTERPOLATORY ORTHOGONAL MULTIWAVELET PACKETS

Suppose that $\mathbf{w}_0(x) = \Phi(x)$, $\mathbf{w}_1(x) = \Psi(x)$ and $P_k^{(0)} = P_k, P_k^{(1)} = Q_k$. Then (1.1) and (1.3) can be rewritten respectively as follows

$$\mathbf{w}_i(x) = \sum_{k\in Z} P_k^{(i)}\mathbf{w}_0(2x-k), i = 0,1. \tag{2.1}$$

The multiwavelet packets $\{\mathbf{w}_n(x)\}$ is defined by the following recursion:

$$\mathbf{w}_{2n+i}(x) = \sum_{k\in Z} P_k^{(i)}\mathbf{w}_n(2x-k), i = 0,1; n = 0,1,\cdots. \tag{2.2}$$

For any $n \in Z_+$, its 2-adic fractional expression is

$$n = \sum_{j=1}^{\infty} \varepsilon_j 2^{j-1}, \varepsilon_j \in \{0,1\}. \tag{2.3}$$

Theorem 2.1. *Let n be in $\in Z+$, and its 2-adic fractional expression given by (2.3). Then the Fourier transform $\overset{\wedge}{\mathbf{w}}_n\,(\omega)$ of orthogonal multiwavelet packets $\mathbf{w}_n(x)$ is given by*

$$\overset{\wedge}{\mathbf{w}}_n\,(\omega) = \prod_{j=0}^{\infty} P^{(\varepsilon_j)}(e^{-i\omega/2^j})\,\overset{\wedge}{\mathbf{w}}_0\,(0), \omega \in R. \tag{2.4}$$

Proof. The proof is by induction on n. By taking the Fourier transform for the both sides of (2.2), we have

$$\overset{\wedge}{\mathbf{w}}_{2n+i}\,(\omega) = P^{(i)}(e^{-i\omega/2})\,\overset{\wedge}{\mathbf{w}}_n\,(\frac{\omega}{2}). \tag{2.5}$$

The case $n = 0$ and $n = 1$ are trivial by (1.1) and (1.3), respectively. Let us assume that (2.4) holds for $0 \le n < 2^{s_0}$. Consider the setting of $2^{s_0} \le n < 2^{s_0+1}$. Since $n = 2[\frac{n}{2}]+\varepsilon_1 = 2n_1+\varepsilon_1$, by (2.5), we have

$$\overset{\wedge}{\mathbf{w}}_n\,(\omega) = P^{(\varepsilon_1)}(e^{-i\omega/2})\,\overset{\wedge}{\mathbf{w}}_{n_1}\,(\frac{\omega}{2}). \tag{2.6}$$

Since $n_1 = [\frac{n}{2}] = \sum_{j=1}^{s_0} \varepsilon_{j+1}2^{j-1} < 2^{s_0}$, hence, $\overset{\wedge}{\mathbf{w}}_{n_1}\,(\omega) = \prod_{j=1}^{\infty} P^{(\varepsilon_{j+1})}(e^{-i\omega/2^j})\,\overset{\wedge}{\mathbf{w}}_0\,(0)$. By (2.6), we prove Theorem 2.1 by induction on n. ∎

Lemma 2.2. *Let $\Phi(x)$ be an orthogonal multiscaling function defined in (1.1), and $P(z)$ be the two-scale matrix symbol defined in (1.2). Suppose that Ψ is an orthogonal multiwavelet associated with Φ, and $Q(z)$ is two-scale matrix symbol. Then*

$$P(\omega)P(\omega)^* + P(\omega+\pi)P(\omega+\pi)^* = I_2, \tag{2.7}$$

$$P(\omega)Q(\omega)^* + P(\omega+\pi)Q(\omega+\pi)^* = O, \tag{2.8}$$

$$Q(\omega)Q(\omega)^* + Q(\omega+\pi)Q(\omega+\pi)^* = I_2. \tag{2.9}$$

This result was obtained by Chui and Lian in [1]. The (2.7), (2.8) and (2.9) can be rewritten the following equation equivalently

$$P^{(i)}(\omega)P^{(j)}(\omega)^* + P^{(i)}(\omega+\pi)P^{(j)}(\omega+\pi)^* = \delta_{i,j}I_2, i,j = 0,1. \tag{2.10}$$

Theorem 2.3. *Let $\Phi(x)$ be orthogonal multiscaling function. Suppose $E = (1,1)^T$, and $\{\mathbf{w}_n(x), n \in Z_+\}$ defined in (2.2) is multiwavelet packets associated with multiscaling function Φ. Then $\forall n \in Z_+$,*

$$\langle \mathbf{w}_n(x-j), \mathbf{w}_n(x-k)\rangle = \delta_{j,k}E, j,k \in Z. \tag{2.11}$$

$$\langle \mathbf{w}_{2n}(x-j), \mathbf{w}_{2n+1}(x-k)\rangle = O, j,k \in Z. \tag{2.12}$$

Proof. It is clear that (2.11) holds for $n = 0$ and $n = 1$. Let us assume that (2.11) holds for $0 \le n < 2^{s_0}$. Now let us consider the setting of $2^{s_0} \le n < 2^{s_0+1}$. By Theorem 2.1 and (2.10), we have

$$\begin{aligned}
\langle \mathbf{w}_n(x-j), \mathbf{w}_n(x-k)\rangle &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |\overset{\wedge}{\mathbf{w}}_n(\omega)|^2 e^{-i(k-j)\omega}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |P^{(\varepsilon_1)}(e^{-i\omega/2})|^2 |\overset{\wedge}{\mathbf{w}}_{[\frac{n}{2}]}(\tfrac{\omega}{2})|^2 e^{-i(k-j)\omega}d\omega \\
&= \tfrac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}\int_{4\pi\ell}^{4\pi(\ell+1)} |P^{(\varepsilon_1)}(e^{-\frac{i\omega}{2}})|^2 |\overset{\wedge}{\mathbf{w}}_{[\frac{n}{2}]}(\tfrac{\omega}{2})|^2 e^{-i(k-j)\omega}d\omega \\
&= \tfrac{1}{2\pi}\int_0^{4\pi} |P^{(\varepsilon_1)}(e^{-i\omega/2})|^2 \sum_{\ell=-\infty}^{\infty} |\overset{\wedge}{\mathbf{w}}_{[\frac{n}{2}]}(\tfrac{\omega}{2}+2\pi\ell)|^2 e^{-i(k-j)\omega}d\omega \\
&= \tfrac{1}{2\pi}\int_0^{2\pi} e^{-i(k-j)\omega}[|P^{(\varepsilon_1)}(\omega)|^2 + |P^{(\varepsilon_1)}(\omega+\pi)|^2]Ed\omega = \delta_{j,k}E,
\end{aligned} \tag{2.13}$$

where $\varepsilon_1 = 0,1$. This completes the induction procedure. Hence (2.11) holds. Similarly, we can proved (2.12). ∎

Theorem 2.4. *The family $\{\mathbf{w}_n(x-k) : 0 \le n < 2^N; k \in Z\}$ is an orthogonal basis of V_N. Moreover, the collection $\{\mathbf{w}_n(x-k) : n \in Z_+; k \in Z\}$ is an orthogonal basis of $L^2(R)$.*

Theorem 2.5. *Let Φ be an orthogonal interpolatory multiscaling function, and satisfy the interpolatory condition (1.7) or (1.8). The corresponding multiwavelet $\Psi(x)$ has the same interpolatory property. Let $\{\mathbf{w}_n(x), n \in Z_+\}$ defined in (2.1) be multiwavelet packets associated with multiscaling function Φ. Then $\{\mathbf{w}_n(x), n \in Z_+\}$ is an orthogonal interpolatory multiwavelet packets.That is, $\mathbf{w}_n(x)$ also satisfies condition (1.7) or (1.8).*

Proof. The orthogonality of $\{\mathbf{w}_n(x), n \in Z_+\}$ has been given by Theorem 2.3-Theorem 2.4. Next, we only need to prove that $\{\mathbf{w}_n(x), n \in Z_+\}$ has interpolatory property. By applying the results in [8], we obtain that Φ and Ψ are both interpolatory, and satisfy the interpolatory condition: $\Phi(k) = \delta_{k,0}e_1$, and $\Phi(k+\frac{1}{2}) = \delta_{k,0}e_2$, and $\Psi(k) = \delta_{k,0}e_1$, and $\Psi(k+\frac{1}{2}) = \delta_{k,0}e_2$. It is clear that $\mathbf{w}_n(x)$ is interpolatory and satisfies the condition (1.8) for $n = 0,1$. Suppose that $\mathbf{w}_n(x)$ is interpolatory and satisfies the condition (1.8) for $0 \le n < 2^{s_0}$. i.e.,

$\mathbf{w}_n(k) = \delta_{k,0}e_1$ and $\mathbf{w}_n(k+\frac{1}{2}) = \delta_{k,0}e_2$, for $0 \le n < 2^{s_0}$. Now let us consider the setting of $2^{s_0} \le n < 2^{s_0+1}$. Since $n = 2[\frac{n}{2}] + \varepsilon_1 = 2n_1 + \varepsilon_1$, by (2.2), we have

$$\mathbf{w}_n(x) = \sum_{k\in Z} P_k^{(i)} \mathbf{w}_{[\frac{n}{2}]}(2x-k), i = 0,1; n = 2^{s_0}, 2^{s_0}+1, \cdots, 2^{s_0+1}-1. \tag{2.14}$$

Since $0 \le [\frac{n}{2}] < 2^{s_0}$, the induction hypothesis tells us that $\mathbf{w}_{[\frac{n}{2}]}(m) = \delta_{m,0}e_1, \mathbf{w}_{[\frac{n}{2}]}(m+\frac{1}{2}) = \delta_{m,0}e_2$. Therefore, by (1.1), (1.8) and (2.14), we have $\mathbf{w}_n(m) = \sum\limits_{k\in Z} P_k^{(i)} \mathbf{w}_{[\frac{n}{2}]}(2m-k) = \sum\limits_{k\in Z} P_{2m-k}^{(i)} \mathbf{w}_{[\frac{n}{2}]}(k) = \sum\limits_{k\in Z} P_{2m-k}^{(i)} \delta_{k,0} e_1 = P_{2m} e_1 = \delta_{m,0} e_1$. Similarly, applying (2.14), we obtain $\mathbf{w}_n(m+\frac{1}{2}) = \delta_{m,0}e_2$. This completes the induction procedure and the proof of Theorem 2.5. ∎

3. DECOMPOSITION OR RECONSTRUCTION ALGORITHMS

Define the linear spaces

$$U_j^n = close_{L^2}\langle 2^{j/2}\mathbf{w}_n(2^jx-k) : k \in Z\rangle, j \in Z, n \in Z_+. \tag{3.1}$$

Hence for all $j \in Z, U_j^0 = V_j, U_j^1 = W_j$. Applying $V_{j+1} = V_j + W_j$, we have

$$U_{j+1}^0 = U_j^0 \bigoplus U_j^1, j \in Z. \tag{3.2}$$

Since both the multiscaling function and the corresponding multiwavelet have the same interpolatory property. Therefore, we enable to compute the wavelets coefficients from the samples of the signal rather than the Mallat algorithm. That is, if a continuous signal $f(x)$ is in U_{j+1}^0 and has the orthogonal decomposition

$$f(x) = \sum_{\ell\in Z} [D_\ell^{j+1,0}]^T \mathbf{w}_{0:j+1,\ell}(x), \tag{3.3}$$

where $\mathbf{w}_{n:j,\ell}(x) = 2^{j/2}\mathbf{w}_n(2^jx-k), [D_\ell^{j+1,0}]^T = 2^{-\frac{j+1}{2}}[f(\frac{\ell}{2^{j+1}}, f(\frac{\ell}{2^{j+1}} + \frac{1}{2^{j+2}})]$. Applying (3.2), we have

$$f(x) = \sum_{\ell\in Z} [D_\ell^{j,0}]^T \mathbf{w}_{0:j,\ell}(x) + [D_\ell^{j,1}]^T \mathbf{w}_{1:j,\ell}(x). \tag{3.4}$$

By the Mallat algorithm, we can obtain

$$D_\ell^{j,0} = 2^{-\frac{1}{2}} \sum_{m\in Z} P_k D_{2\ell+m}^{j+1,0}. \tag{3.5}$$

In terms of interpolatory property of the multiscaling function and the corresponding multiwavelet, and (3.4), we can obtain

$$[D_\ell^{j,1}]^T = 2^{-j/2}[f(\frac{\ell}{2^j}), f(\frac{\ell}{2^j} + \frac{1}{2^{j+1}})] - [D_\ell^{j,0}]^T. \tag{3.6}$$

Furthermore, we have

$$U_{j+1}^n = U_j^{2n} \bigoplus U_j^{2n+1}, j \in Z. \tag{3.7}$$

Similarly, let $g_{j+1}^n(x)$ be in U_{j+1}^n. Then $g_{j+1}^n(x)$ can be expressed by

$$g_{j+1}^n(x) = \sum_{\ell\in Z} [D_\ell^{j+1,n}]^T \mathbf{w}_{n:j+1,\ell}(x), \tag{3.8}$$

By (3.7), proceed to the decomposition with an orthogonal interpolatory multiwavelet packets $\mathbf{w}_{n:j,\ell}(x)$

$$g^n_{j+1}(x) = \sum_{\ell \in Z} [D^{j,2n}_\ell]^T \mathbf{w}_{2n:j,\ell}(x) + [D^{j,2n+1}_\ell]^T \mathbf{w}_{2n+1:j,\ell}(x). \tag{3.9}$$

In light of interpolatory property of multiwavelet packets, we can establish the following relation:

$$2^{-j/2}[g^n_{j+1}(\frac{\ell}{2^j}), g^n_{j+1}(\frac{\ell}{2^j} + \frac{1}{2^{j+1}})] = [D^{j,2n}_\ell]^T + [D^{j,2n+1}_\ell]^T. \tag{3.10}$$

The above formula can play an important role in signal decomposition and reconstruction procedure instead of the Mallat algorithm.

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