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NEW COINCIDENCE AND FIXED POINT THEOREMS FOR STRICTLY CONTRACTIVE HYBRID MAPS

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ABSTRACT. The purpose of this paper is to study the (EA)-property and noncompatible maps of a hybrid pair of single-valued and multivalued maps in fixed point considerations. Such maps have the remarkable property that they need not be continuous at their common fixed points. We use this property to obtain some coincidence and fixed point theorems for strictly contractive hybrid maps without using their continuity and completeness or compactness of the space.

Key words and phrases: Coincidence point, fixed point, hybrid maps, non-compatible maps, (EA)-property, IT-commuting maps, R-weakly commuting maps.

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1. INTRODUCTION

The notion of compatible maps, due to Jungck [6, 7], has proven fruitful in fixed point considerations. On the other hand, noncompatible maps appear to play a vital role in metric fixed point theory for contractive type maps. Pant [14], Pant et al. [15], [16], Aamri and El Moutawakil [1] and others have initiated work along these lines. However, the concept of the (EA)-property [1] generalizes both compatible and noncompatible maps. It is remarkable that maps having (EA)-property need not be continuous at their common fixed points (see [14] and [19]). For an excellent discussion on the continuity of maps on their fixed points, one may refer to Rhoades [17] and Hicks and Rhoades [4].

The purpose of this paper is to extend the concept of (EA)-property to a hybrid pair of single-valued and multivalued maps on an arbitrary nonempty set with values in a metric space, and use the same to study coincidence and fixed points of contractive type hybrid maps without appealing to the continuity of the maps involved. Our results extend, improve and generalize several results for single-valued and multivalued maps on metric spaces.

2. PRELIMINARIES

We generally follow the definitions and notations used in [2], [7], [10], [11] and [21]. Given a metric space (X, d), let $(CL(X), H)$ and $(CB(X), H)$ denote respectively the hyper spaces of nonempty closed and nonempty closed bounded subsets of X, where H is the Hausdrorff metric induced by d. Notice that the hyper space $CL(X)$ contains the space $CB(X)$. Throughout, $d(A, B)$ will denote the ordinary distance between subsets A and B of X and $d(x, B)$ will stand for $d(A, B)$ when A is the singleton $\{x\}$. Further, let Y be an arbitrary nonempty set and $C(S, A) = \{u : Su \in Au\}$, the collection of coincidence points of the maps $S : X \to X$ and $A : X \to CL(X)$.

The following definition is due to Ito and Takahashi [5] (see also [21], page 488) when $Y = X$ and S and A both are self-maps of X.

Definition 2.1. Let Y be a nonempty set, $S : Y \to Y$ and $A : Y \to 2^Y$, the collection of nonempty subsets of Y. Then the maps of the hybrid pair (S, A) are (IT)-commuting at $x \in Y$ if $SAx \subset ASx$. They are (IT)-commuting on Y if $SAx \subset ASx$ for each $x \in Y$. (This formulation in [22], p. 625 is correct with the interchange of symbols for single-valued and multivalued maps).

Maps A and S are commuting at $x \in Y$ when $ASx = SAx$. Clearly a commuting hybrid pair of maps is IT-commuting and the reverse implication is not true. For example, if $Y = [0, \infty), Sx = 4x$ and $Ax = [3 + x, \infty), x \in Y$, then the pair (S, A) is not commuting but (IT)-commuting.

The following definition is essentially due to Kaneko and Sessa [9] and Beg and Azam [2] when $S : X \to CB(X)$.

Definition 2.2. [22]. Maps $A : X \to CL(X)$ and $S : X \to X$ are compatible if $SAx \in CL(X)$ for each $x \in X$ and $lim_{n\to\infty} H(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Ax_n = M \in CL(X)$ and $lim_{n\to\infty} Sx_n = t \in M$. They are weakly compatible if $SAx = ASx$ whenever $Sx \in Ax$ [8].

For fundamental discussions and applications of compatible self-maps of metric spaces, one may refer to Jungck [6]-[7]. For a good comparison of various weaker forms of commuting maps (such as weakly/R-weakly commuting maps, compatible maps etc.), one may refer to Singh and Tomar [23]. We remark that commutativity, compatibility, R-weak commutativity and weak compatibility of $A : X \to CL(X)$ and $S : X \to X$ are equivalent at their coincidence points (cf. [21] and [22]). Further, the maps are noncompatible if there exists at least one sequence $\{x_n\}$ in X, such that $lim_{n\to\infty}Ax_n = M$ and $lim_{n\to\infty}Sx_n = t \in M$ for some $t \in X$ but $lim_{n\to\infty}H(SAx_n, ASx_n)$ is either nonzero or nonexistent.

Definition 2.3. [20], [21]. Maps $A : X \to CL(X)$ and $S : X \to X$ are R-weakly commuting if $SAx \in CL(X)$ for all $x \in X$, and there exists a real number $R > 0$ such that $H(ASx, SAx) \leq Rd(Sx, Ax)$. Further, A and S may be called pointwise R-weakly commuting on X if given x in X there exists R such that $H(ASx, SAx) \leq Rd(Sx, Ax)$.

If R = 1, we get a similar concept due to Hȧdžić and Gajić [3]. If the map A is also single-valued then we get the definition of R-weak commutativity of single-valued maps due to Pant [12, 13].

Definition 2.4. Let $A : Y \to CL(X)$ and $S : Y \to X$. Then A and S will be called to satisfy the (EA)-property if there exists a sequence $\{x_n\}$ in Y such that

$$lim_{n\to\infty}Ax_n = M \in CL(X) \; and \; lim_{n\to\infty}Sx_n = t \in M$$

.

In this definition, if A is a map on Y with values in X then we get the definition of (EA)-property due to Singh and Kumar [19]. If we take $Y = X$ then we get the definition of (EA)-property (also called tangential maps by Sastry and Murthy [18]) for two self-maps on X essentially due to Aamri and El Moutawakil [1].

Example 2.1. *Let $X = [0, \infty)$ with the usual metric, $Ax = [0, 3x/2]$ and $Sx = x/2$. We consider the sequence $\{x_n = 1 + 1/n : n \geq 1\}$ to see that A and S satisfy the(EA)-property.*

Example 2.2. *Let $X = [2, \infty)$ with the usual metric and $Ax = \{1 + x\}$ and $Sx = 2x + 1$. We see that there does not exist a sequence $\{x_n\}$ in X for which $\{Ax_n\}$ and $\{Sx_n\}$ both converge to the same element. So A and S lack the (EA)-property.*

3. MAIN RESULTS

First we present a basic result for a hybrid pair of maps.

Theorem 3.1. *Let $A : Y \to CL(X)$ and $S : Y \to X$ be such that*
(i) $AY \subset SY$;
(ii) the pair (S, A) satisfies the (EA)-property;
(iii) $H(Ax, Ay) < m(x, y)$ when $m(x, y) > 0$, where

$$m(x, y) = max\left\{d(Sx, Sy), \alpha[d(Sx, Ax) + d(Sy, Ay)], \alpha[d(Sx, Ay) + d(Sy, Ax)]\right\},$$

$0 \leq \alpha < 1$.
If $A(Y)$ or $S(Y)$ is a complete subspace of X then $C(S, A)$ is nonempty. Further, A and S have a common fixed point provided that $SSz = Sz$ and A and S are (IT)-commuting at $z \in C(S, A)$.

Proof. Since the pair (A, S) satisfies the (EA)-property, there exists a sequence $\{x_n\}$ in Y such that $lim_{n\to\infty}Ax_n = M \in CL(X)$ and $lim_{n\to\infty}Sx_n = t \in M$. If SY is a complete subspace of X, there exists a point $z \in Y$ such that $t = Sz$. Suppose $Sz \notin Az$. Then by (iii),
$d(Az, Ax_n) \leq H(Az, Ax_n) < max\{d(Sz, Sx_n), \alpha[d(Sz, Az)+d(Sx_n, Ax_n)], \alpha[d(Sz, Ax_n)+ d(Sx_n, Az)]\}$.

Making $n \to \infty$ yields $H(Az, M) \leq \alpha d(Az, Sz)$, a contradiction. So $Sz \in Az$, and $C(S, A)$ is nonempty. Further, if $SSz = Sz$ and A, S are (IT)-commuting at $z \in C(S, A)$ then $Sz \in SAz \subset ASz$, and Sz is a common fixed point of A and S. If $A(Y)$ is a complete subspace of X, then in view of (i), $C(S, A)$ is evidently nonempty. ∎

We remark that the Proof of Theorem 3.1 may also follow from Theorem 3.2 (below). Notice that Theorem 3.1 (without the commutativity requirement) guarantees that A and S have a coincidence, while A, B, S and T of Theorem 3.2 (without the commutativity requirements) need not have a common coincidence. Indeed, the conclusion "... then $C(A,S)$ and $C(B,T)$ are nonempty" in Theorem 3.2 clearly means that there exist points u, v in Y such that $Su \in Au$ and $Tv \in Bv$, and notice the important feature that S and T have the same coincidence value, i.e., $Su = Tv$.

Example 3.1. *Let $X = [1,\infty)$ be endowed with the usual metric, $Ax = [1, 2x-1]$ and $Sx = x^2$. We consider a sequence $\{x_n = 1 + 1/n, n \geq 1\}$ to see that the maps A and S satisfy the (EA)-property. Also A and S are (IT)-commuting at $x = 1$. Indeed, it is easy to verify that A and S satisfy all the hypotheses of Theorem 3.1. Evidently $S1 = 1 \in A1$.*

The following is our main result for a hybrid quadruple of maps on an arbitrary nonempty set.

Theorem 3.2. *Let (X,d) be a metric space and $A, B : Y \to CL(X)$ and $S, T : Y \to X$ such that*
(iv) $AY \subset TY$ and $BY \subset SY$;
(v) one of the pairs (S,A) or (T,B) satisfies the (EA)-property;
(vi) $H(Ax, By) < M(x,y)$ when $M(x,y) > 0$, where

$$M(x,y) = max\{d(Sx,Ty), \alpha[d(Sx,Ax) + d(Ty,By)], \\ \alpha[d(Ty,Ax) + d(Sx,By)]\}, 0 \leq \alpha < 1.$$

If $A(Y)$ or $B(Y)$ or $S(Y)$ or $T(Y)$ is a complete subspace of X then $C(S,A)$ and $C(B,T)$ are nonempty. Further,
(I) A and S have a common fixed point Su provided that $SSu = Su$ and A, S are (IT)-commuting at $u \in C(S,A)$;
(II) B and T have a common fixed point Tv provided that $TTv = Tv$ and B, T are (IT)-commuting at $v \in C(T,B)$;
(III) A, B, S and T have a common fixed point provided that (I) and (II) are true.

Proof. If the pair (B,T) satisfies the (EA)-property then there exists a sequence $\{x_n\}$ in Y such that $lim_{n\to\infty} Bx_n = M \in CL(X)$ and $lim_{n\to\infty} Tx_n = t \in M$.

Since $BY \subset SY$ for each x_n, there exist a sequence $\{y_n\}$ in Y such that $Sy_n \in BX_n$ and $lim_{n\to\infty} Sy_n = t \in M = lim_{n\to\infty} Bx_n$. We show that $lim_{n\to\infty} Ay_n = M$. If not, there exists a subsequence $\{Ay_k\}$ of $\{Ay_n\}$, a positive integer N, and a real number $\epsilon > 0$ such that for some $k \geq N$ we have $H(Ay_k, M) \geq \epsilon$. From (vi),

$$H(Ay_k, Bx_k) \leq H(Ay_k, Bx_k) + H(Bx_k, M)$$

$$< max\{d(Sy_k, Tx_k), \alpha[d(Sy_k, Ay_k) + d(Bx_k, Tx_k)], \alpha[d(Tx_k, Ay_k) + d(Sy_k, Bx_k)]\} + H(Bx_k, M).$$

$$\leq max\{d(Sy_k, Tx_k), \alpha[d(Sy_k, M) + H(M, Bx_k) + d(Bx_k, Tx_k)],$$

$$\alpha[d(Tx_k, M) + H(M, Ay_k) + d(Sy_k, Bx_k)]\} + H(Bx_k, M).$$

Making $k \to \infty$ gives

$$lim_{n\to\infty} H(Ay_n, M) \leq \alpha H(M, Ay_k),$$

and so

$$lim_{n\to\infty}Ay_n = M.$$

Suppose SY or BY is a complete subspace of X, then there exists a point $u \in Y$ such that $t = Su$. To show that $Su \in Au$, we suppose otherwise and use the condition (vi) to have
$d(Au, Bx_n) \leq H(Au, Bx_n) < max\{d(Su, Tx_n), \alpha[d(Su, Au)+d(Tx_n, Bx_n)], \alpha[d(Tx_n, Au)+ d(Su, Bx_n)]\}$.
Making $n \to \infty$ implies
$H(Au, M) \leq \alpha d(Au, Su) \leq \alpha H(Au, M)$, a contradiction. Consequently $C(S, A)$ is nonempty.

Since $AY \subset TY$, there exists a point $v \in Y$ such that $Su = Tv \in Au$. So by (vi),

$$\begin{aligned} d(Tv, Bv) = d(Su, Bv) \leq H(Au, Bv) \\ < max\{d(Su, Tv), \alpha[d(Su, Au) + d(Tv, Bv)], \alpha[d(Tv, Au) + d(Su, Bv)]\}. \end{aligned}$$

So $d(Tv, Bv) < d(Tv, Bv)$, and $C(T, B)$ is nonempty.

Further, $Su = SSu$ and the (IT)-commutativity of A and S at $u \in C(A, S)$ imply that $Su \in SAu \subset ASu$. So Su is a common fixed point of A and S. The proof of (II) is similar. Now (III) is immediate. Analogous argument establishes the theorem when AY or TY is a complete subspace of X. ∎

In view of the above proof, we have other versions of Theorem 3.2.

Theorem 3.3. *Theorems 3.2 with M(x, y) replaced by* $M^1(x, y)$, *where*
(vii) $M^1(x, y) = max\{d(Sx, Ty), \alpha[d(Sx, Ax)+d(Ty, By)]/2, [d(Ty, Ax)+d(Sx, By)]/2\}$, $1 \leq \alpha < 2$.

Theorem 3.4. *Let* $A, B : X \to CL(X)$ *and* $S, T : X \to X$ *be such that (iv) holds with* $Y = X$. *Further, assume*
(viii) one of the pairs (S, A) *or* (T, B) *is noncompatible;*
(ix) pairs (S, A) *and* (T, B) *are R-weakly commuting;*
Then $C(S, A)$ *and* $C(T, B)$ *are nonempty. Further*
(Ia) A *and* S *have a common fixed point* Su *provided that* $SSu = Su$.
(IIa) B *and* T *have a common fixed point* Tv *provided that* $TTv = Tv$.
(IIIa) A, B, S *and* T *have a common fixed point provided that (Ia) and (IIa) are true.*

Now we derive some corollaries with slightly different versions. The following result is an improvement of Pant and Pant [15], Th. 2.3 (see also [16]).

Corollary 3.5. *Let* $A, B, S, T : X \to X$ *such that* (S, A) *and* (T, B) *are pointwise R-weakly commuting selfmaps of a metric space* (X, d) *satisfying the conditions (iv), (viii), (ix) and the following:*
(x) $d(Ax, By) < L(x, y)$ *when* $L(x, y) > 0$, *where* $L(x, y) = max\{d(Sx, Ty), \alpha[d(Sx, Ax) + d(Ty, By)]/2, [d(Ty, Ax) + d(Sx, By)]/2\}, 1 \leq \alpha \leq 2$. *If the range of one of the maps is a complete subspace of* X *then* $C(S, A)$ *and* $C(T, B)$ *are nonempty. Further,*
(Ib) A *and* S *have a common fixed point provided that* A *and* S *commute at* $u \in C(S, A)$;
(IIb) B *and* T *have a common fixed point provided that* B *and* T *commute at* $v \in C(T, B)$.
(IIIb) A, B, S *and* T *have a common fixed point provided that (Ib) and (IIb) are true.*

The following result is a considerable improvement and extension of Aamri and El Moutawakil's result [1], Theorem 1.

Corollary 3.6. *Let $A, S : Y \to X$ be self-maps of a metric space (X, d) such that*
(xi) $AY \subset SY$;
(xii) the pair (S, A) satisfies the (EA)-property;
(xiii) $d(Ax, Ay) < f(x, y)$ when $f(x, y) > 0$, where
$f(x, y) = \{d(Sx, Sy), [d(Sx, Ax) + d(Sy, Ay)]/2, [d(Sy, Ax) + d(Sx, Ay)]/2\}$. If the range of AY or SY is a complete subspace of X then $C(S, A)$ is nonempty. Further, A and S have a common fixed point provided that they are weakly compatible.

Remark 3.1. If $S = T$ in Theorems 3.2 - 3.4 and Corollary 3.5, then the conclusion regarding the coincidence part is a slightly improved and consequently A, B and $S(= T)$ have a common coincidence. Further, our results are good variants and generalizations of several results from Jungck [7], Nadler, Jr. [10], Smithson [24], Tan and Minh [25] and others. Finally, we may conclude that our results are obtained effectively under tight minimal conditions and are not subject to further simplification.

REFERENCES

[1] M. AAMRI and D. EL MOUTAWAKIL, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*, **270** (2002), 181-188.

[2] ISMAT BEG and AKBAR AZAM, Fixed point of asymptotically regular multivalued mappings, *J. Austral. Math. Soc. Ser. A*, **53** (1992), 313-326.

[3] OLGA HȦDŽIĆ and LJ. GAJIĆ, Coincidence points for set-valued mappings in convex metric spaces, *Univ. Novom. Sadu. Zb. Rad. Prirod. Mat. Fak. Ser. Mat.*, **16**(1) (1986), 13-25.

[4] T. L. HICKS and B. E. RHOADES, Fixed points and continuity for multivalued mappings, *Internat. J. Math. Math. Sci.*, **15** (1992), 15-30.

[5] S. ITOH and W. TAKAHASHI, Single-valued mappings, multivalued mappings and fixed point theorems, *J. Math. Anal. Appl.*, **59** (1977), 514-521.

[6] G. JUNGCK, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.*, **9** (4) (1986), 771-779.

[7] G. JUNGCK, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, **103** (1988), 977-983.

[8] G. JUNGCK and B.E. RHOADES, Fixed points for set-valued functions without continuity, *Indian J. Pure Appl. Math.*, **29** (3) (1998), 227-238.

[9] H. KANEKO and S. SESSA, Fixed point theorems for compatible multivalued and single-valued mappings, *Internat. J. Math. Math. Sci.*, **12** (1989), 257-262.

[10] S. B. NADLER, Jr., Multivalued contraction mappings, *Pacific J. Math.*, **30** (1969), 475-488.

[11] S. A. NAIMPALLY, S. L. SINGH and J. H. M. WHITFIELD, Coincidence theorems for hybrid contractions, *Math. Nachr.*, **127** (1986), 177-180.

[12] PANT, R.P., Common fixed points of non-commuting mappings, *J. Math. Anal. Appl.*, **188** (1994), 436-440.

[13] R. P. PANT, Common fixed point theorems for contractive maps, *J. Math. Anal. Appl.*, **226** (1998), 251-258.

[14] R. P. PANT, Discontinuity and fixed points, *J. Math. Anal. Appl.*, **240** (1999), 284-289.

[15] R. P. PANT and V. PANT, Common fixed points under strict contractive conditions, *J. Math. Anal. Appl.*, **248** (2000), 327-332.

[16] R. P. PANT, V. PANT and K. JHA, Note on common fixed points under strict contractive conditions, *J. Math. Anal. Appl.*, **274** (2002), 879-880.

[17] B. E. RHOADES, Contractive definitions and continuity, fixed point theory and its applications (Berkeley, CA, 1986), 233-245, *Contemp. Math., 72, Amer. Math. Soc.*, Providence, RI, 1988.

[18] K. P. R. SASTRY and I. S. R. KRISHNA MURTHY, Common fixed points of two partially commuting tangential self-maps on a metric space, *J. Math. Anal. Appl.*, **250** (2000), 731-734.

[19] S. L. SINGH and ASHISH KUMAR, A fixed point theorems for Lipschitz type maps, *Rivista. Mat. Univ. Parma* (2004).

[20] S. L. SINGH and S. N. MISHRA, On general hybrid contractions, *J. Austral. Math. Soc. Ser. A*, **66** (1999), 244-254.

[21] S. L. SINGH and S. N. MISHRA, Coincidences and fixed points of nonself hybrid contractions, *J. Math. Anal. Appl.*, **256** (2001), 486-497.

[22] S. L. SINGH and S. N. MISHRA, Coincidences and fixed points of reciprocally continuous and compatible hybrid maps, *Internat. J. Math. Math. Sci.*, **30** (2002), 627-635.

[23] S. L. SINGH and ANITA TOMAR, Weaker forms of commuting maps and existence of fixed points, *J. Korea Soc. Math. Edu. Ser. B: Pure Appl. Math.*, **10** (2003), 145-161.

[24] R. E. SMITHSON, Fixed points for contractive multifunctions, *Proc. Amer. Math. Soc.*, **27** (1971), 192-194.

[25] D. H. TAN and N. A. MINH, Some fixed point theorems for mappings of contractive type, *Acta Math. Vietnam.*, **3** (1978), 24-42.