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ORTHOGONALITY AND ε-ORTHOGONALITY IN BANACH SPACES

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ABSTRACT. A concept of orthogonality on normed linear space was introduced by Brickhoff, also the concept of ε-orthogonality was introduced by Vaezpour. In this note, we will consider the relation between these concepts and the dual of X. Also some results on best coapproximation will be obtained.

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1. Introduction

We recall that if X is a normed linear space and $x, y \in X$, x is said to be *orthogonal* to y and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. If G_1 and G_2 are subsets of X, it is defined $G_1 \perp G_2$ if and only if for all $g_1 \in G_1, g_2 \in G_2$, $g_1 \perp g_2$ (see [3], [5], [8]). Also if X is a normed linear space, $\varepsilon > 0$ and $x, y \in X$, we say x is ε*-orthogonal* to y and is denoted by $x \perp_\varepsilon y$ if and only if $\|x\| \leq \|x + \alpha y\| + \varepsilon$ for all scalar α. If G_1 and G_2 are subsets of X, define $G_1 \perp_\varepsilon G_2$ if and only if for all $g_1 \in G_1, g_2 \in G_2$ we have, $g_1 \perp_\varepsilon g_2$ (see [8]).

Let X be a normed linear space and G be a subspace of X, it is defined,

$$\hat{G} = \{x \in X : \ x \perp G\},$$

and

$$\breve{G} = \{x \in X : \ G \perp x\}.$$

Similarly for $\varepsilon > 0$

$$\hat{G}_\varepsilon = \{x \in X : x \perp_\varepsilon G\},$$

and

$$\breve{G}_\varepsilon = \{x \in X : G \perp_\varepsilon x\}.$$

Let X be a normed linear space and G be a subset of X. A point $g_0 \in G$ is said to be a best approximation (best coapproximation) for $x \in X$ if and only if $\|x - g_0\| \leq \|x - g\| \ for \ all \ g \in G$ $(\|g_0 - g\| \leq \|x - g\| \ \forall \ g \in G)$. It can be easily proved that g_0 is a best approximation (best coapproximation) for $x \in X$ if and only if $x - g_0 \in \hat{G}$ $(x - g_0 \in \breve{G})$. The set of all best approximations (best coapproximations) of $x \in X$ in G is shown by $P_G(x)$ $(R_G(x))$. In other words,

$$P_G(x) = \{g_0 \in G \ : \ x - g_0 \in \hat{G}\}$$

and

$$R_G(x) = \{g_0 \in G \ : \ x - g_0 \in \breve{G}\}.$$

(For more details see [1], [3], [4], [5]).

For $\epsilon > 0$, a point $g_0 \in G$ is said to be an ϵ-approximation (ϵ-coapproximation) for $x \in X$ if $x - g_0 \in \hat{G}_\epsilon$ $(x - g_0 \in \breve{G}_\epsilon)$. The set of all ϵ-approximation (ϵ-coapproximation) for $x \in X$ will be denoted by $P_{G,\varepsilon}(x)$ $(R_{G,\varepsilon}(x))$. It can be easily proved that,

$$P_{G,\epsilon}(x) = \{g_0 \in G : \|x - g_0\| \leq \|x - g\| + \epsilon \ for \ all \ g \in G\}.$$

It is clear that the set $P_{G,\epsilon}(x)$ is a nonempty set. (see [9])

Definition 1.1. If $R_{G,\epsilon}(x)$ is non-empty for every $x \in X$, then G is called an ϵ-coproximinal set. The set G is ϵ-cochebyshev if $R_{G,\epsilon}(x)$ is a singleton set for every $x \in X$.

Let X^* be the dual of the normed space X. For $x \in X$ and $\epsilon > 0$, put

$$M_x = \{f \in X^* \ : \ \|f\| = 1, \ f(x) = \|x\|\}$$

and

$$M_{x,\epsilon} = \{f \in X^* \ : \ \|f\| = 1, \ f(x) \geq \|x\| + \epsilon\}.$$

Buck in 1965 introduced the elements ε-approximation and Singer in [8] gave another characterization of these elements which is more concrete for application in convenient spaces. Franchetti and Furi in [2] introduced the concept of coapproximation. Vaezpour in [9] introduced the concept of ε-best coapproximation and ε-orthogonality. We shall obtain a necessary and sufficient condition for orthogonality and ε-orthogonality. Also see [6] and [7].

At first we state lemma which is needed in the proof of the main results.

Lemma 1.1. ([H. Hahn]) *Suppose that X is a normed space. Let A be a nonempty subset of X and let $\{c_x : x \in A\}$ be a corresponding collection of scalars. Then the following are equivalent:*

a. *There is a bounded linear functional f on X such that $f(x) = c_x$ for each x in A.*
b. *There is a nonnegative real number M such that:*

$$|\alpha_1 c_{x_1} + \cdots + \alpha_n c_{x_n}| \leq M \, \|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$$

for each linear combination $\alpha_1 x_1 + \cdots + \alpha_n x_n$ of elements of A.

2. MAIN RESULTS

In this section we state and prove our main results.

Theorem 2.1. *Let X be a normed linear space, and $x, y \in X$. Then the following statements are equivalent:*

1. $x \perp y$.
2. *There exists $f \in M_x$ such that $f(y) = 0$.*

Proof. $1. \longrightarrow 2.$ Suppose $x \perp y$, consider the set $A = \{x, y\}$, put $c_y = 0$ and $c_x = \|x\|$. Since $x \perp y$ we have,

$$|\alpha_1 c_x + \alpha_2 c_y| = |\alpha_1| \|x\| = \|\alpha_1 x\| \leq \|\alpha_1 x + \alpha_2 y\|.$$

So from Lemma 1.1, there is a bounded linear functional f on X such that $f(x) = c_x = \|x\|$, and $f(y) = c_y = 0$. Also we have $\|f\| = 1$.

$2. \longrightarrow 1.$ Suppose there exists $f \in M_x$ such that $f(y) = 0$. Then,

$$\|x\| = f(x) = f(x + \alpha y) \leq \|f\| \|x + \alpha y\| = \|x + \alpha y\|,$$

for all scalar α. ∎

Corollary 2.2. *Let X be a normed linear space. Let G be a linear subspace of X, and $u \in X \backslash G$. Then the following statements are equivalent:*

1. $u \in \hat{G}$.
2. *There exists $f \in M_u$ such that $f|_G = 0$.*

Proof. $1. \longrightarrow 2.$ It is enough we put $A = G \cup \{u\}$, $c_u = \|u\|$, $c_g = 0$ for all $g \in G$ in the Lemma 1.1
$2. \longrightarrow 1.$ Suppose there exists $f \in M_u$ such that $f|_G = 0$ that is $f(g) = 0$ for all $g \in G$. From Theorem 2.1, it follows that $u \perp g$ for all $g \in G$ and so $u \perp G$. ∎

Corollary 2.3. *Let X be a normed linear space. Let G be a linear subspace of X, and $u \in X \backslash G$. Then the following statements are equivalent:*

1. $u \in \breve{G}$
2. *For all $g \in G$, there exists $f \in M_g$ such that $f(u) = 0$.*

Let E be a nonempty subset of a linear space X, the subspace spanned by E is denoted by $< E >$.

Theorem 2.4. *Let X be a normed linear space. Let G be a linear subspace of X, and E be a nonempty subset of $X \backslash G$. Then the following statements are equivalent:*

1. $< E > \subseteq \breve{G}$
2. *For all $g \in G$ there exists $f \in M_g$ such that $f|_E = 0$.*

Proof. $1. \longrightarrow 2.$ Suppose $g \in G$, consider the set $A = E \cup \{g\}$. Put $c_g = \|g\|$ and $c_u = 0$ for all $u \in E$. If $u \in <A>$, then $u = \alpha_1 u_1 + \cdots + \alpha_n u_n + \alpha g$ for some scalars $\alpha_1, \ldots \alpha_n, \alpha$ and $u_1, \ldots, u_n \in E$. If we choose $M = 1$, then we have,

$$\begin{aligned} |\alpha_1 c_{u_1} + \cdots + \alpha_n c_{u_n} + \alpha c_g| &= |\alpha|\|g\| = \|\alpha g\| \\ &\leq M\|\alpha_1 u_1 + \cdots + \alpha_n u_n + \alpha g\|. \end{aligned}$$

Thus from Lemma 1.1, there exits $f \in X^*$ such that $f(u) = 0$ for all $u \in E$, and $f(g) = \|g\|$. Therefore $\|f\| = 1$, and $f \in M_g$.
$2. \longrightarrow 1.$ Suppose $u \in <E>$, then $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$ for some scalars $\alpha_1, \ldots \alpha_n$ and $u_1, \ldots, u_n \in E$. If $g \in G$, then there exits $f^g \in M_g$ such that $f^g|_E = 0$. Now we have,

$$\begin{aligned} \|g\| &= f^g(g) = f^g(-u+g) \\ &\leq \|f^g\|\|g-u\| = \|g-u\| \end{aligned}$$

and it follows that $u \in \breve{G}$. ∎

Theorem 2.5. *Let X be a normed linear space, and $x, y \in X$. Then the following statements are equivalent:*

1. $x \perp_\varepsilon y$.
2. *There exists $f \in M_{x,\epsilon}$ such that $f(y) = 0$.*

Proof. $1. \longrightarrow 2.$ Suppose $x \perp_\varepsilon y$ and consider $H = <y>$. Since H is a finite subspace of X, therefore there is a scalar α_0 such that,

$$d = d(x, H) = \|x - \alpha_0 y\|.$$

Now by virtue of a well known corollary of the Hahn-Banach Theorem, there is an $f_0 \in X^*$ such that,

$$\|f_0\| = \frac{1}{d},\ f_0|_H = 0,\ f_0(x) = 1.$$

Put $f = df_0$, then,

$$\begin{aligned} \|x\| &\leq \|x - \alpha_0 y\| + \varepsilon \\ &= d + \varepsilon = f(x) + \varepsilon. \end{aligned}$$

$2. \longrightarrow 1.$ Suppose there exists $f \in M_{x,\epsilon}$ such that $f(y) = 0$, then,

$$\begin{aligned} \|x\| &\leq f(x) + \varepsilon = f(x + \alpha y) + \varepsilon \\ &\leq \|f\|\|x + \alpha y\| + \varepsilon \\ &= \|x + \alpha y\| + \varepsilon \end{aligned}$$

for all scalar α. ∎

Corollary 2.6. *Let X be a normed linear space. Let G be a linear subspace of X, and $u \in X \backslash G$. Then the following statements are equivalent:*

1. $u \in \hat{G}_\varepsilon$.
2. *There exists $f \in M_{u,\epsilon}$ such that $f|_G = 0$.*

Corollary 2.7. *Let X be a normed linear space. Let G be a linear subspace of X, and $u \in X \backslash G$. Then the following statements are equivalent:*

1. $u \in \breve{G}_\varepsilon$
2. *For all $g \in G$, there exists $f \in M_{g,\epsilon}$ such that $f(u) = 0$.*

Theorem 2.8. *Let X be a normed linear space and G be a linear subspace of X. Then,*

1. *G is ε- coproximinal if and only if $X = G + \breve{G}_\varepsilon$.*

2. *G is ε- cochebyshev if and only if $X = G \oplus \breve{G}_\varepsilon$.*

Where $\oplus$ means that the sum decomposition of each element $x \in X$ is unique.

Proof. 1. Let G be ε- coproximinal and $x \in X$. Suppose $g_0 \in R_{G,\varepsilon}(x)$ and so $x - g_0 \in \breve{G}_\varepsilon$. If we put $g_\varepsilon = x - g_0$, then $x = g_0 + g_\varepsilon$.

Now suppose $X = G + \breve{G}_\varepsilon$ and $x \in X$. Then $x = g_0 + g_\varepsilon$ for some $g_0 \in G$ and $g_\varepsilon \in \breve{G}_\varepsilon$. Since $x - g_0 = g_\varepsilon \in \breve{G}_\varepsilon$, therefore $g_0 \in R_{G,\varepsilon}(x)$.

2. If G is ε-cochebyshev and $x \in X$. Then from (1) there exits $g_0 \in R_{G,\varepsilon}(x)$, and $g_\varepsilon \in \breve{G}_\varepsilon$ such that $x = g_0 + g_\varepsilon$.

Also if $x = h_0 + h_\varepsilon$ for some $h_0 \in G$ and $h_\varepsilon \in \breve{G}_\varepsilon$, then $h_0 \in R_{G,\varepsilon}(x)$, and so $g_0 = h_0$. It follows that $g_\varepsilon = h_\varepsilon$.

Conversely, suppose $g_0, h_0 \in R_{G,\varepsilon}(x)$, then $x - g_0 \ and \ x - h_0 \ are \ in \ \breve{G}_\varepsilon$. It follows that $x = h_0 + h_\varepsilon = g_0 + g_\varepsilon$ for some $g_\varepsilon, h_\varepsilon \in \breve{G}_\varepsilon$. Since $X = G \oplus \breve{G}_\varepsilon$, then $g_0 = h_0$. ∎

Example 2.1. *Let X be a normed linear space, $x_0 \in X$ and,*

$$G = \{x \in X : \|x - x_0\| = 1\}.$$

Then $R_{G,\varepsilon}(x_0) = R_G(x_0) = \emptyset$ and $P_G(x_0) = P_{G,\varepsilon}(x_0) = G$

REFERENCES

[1] F. DEUTSCH, *Best Approximation In Inner Product Spaces,* NewYork, Springer, 2000.

[2] C. FRANCHETTI and M. FURI, Some characteristic properties of real Hilbert spaces, *Rev. Roumaine Math. Pures Appl*, **17** (1972), 10945-1048.

[3] C. LI, R. NI and G. A. WATSON, On nonlinear coapproximatin in Banach spaces, *Approx. Theory Appl.* (N. S) **17** (2001), no. 2, 54-63.

[4] T. D. NARANG, Best coapproximation in metric spaces, *Rocky Mountain J. Math,* **51** (1992) 71-76.

[5] C. S. RAO and R. SARAVANAN, Some result concerning best uniform coapproximation, *Journal of Inequal. in Pure and Appl. Math,* **3** (2002), no 2, Article 24, 1-13.

[6] J. M. RASSIAS, Two new criteria on characterizations of inner products, *Discuss. Math.* **9** (1988), 255-267.

[7] J. M. RASSIAS, Four new criteria on characterizations of inner products, *Discuss. Math.* **10** (1990), 139-146.

[8] I. SINGER, *Best Approximation in Normal Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, 1970.

[9] S. M. VAEZPOUR, R. HASANI and H. MAZAHERI, On the ε-best coapproximation, *to appear*.