Aust. J. Math. Anal. Appl.
Vol. **23** (2026), No. 1, Art. 9, 10 pp.
AJMAA

HARNACK EXTENSION OF C[A, B]-VALUED FUNCTIONS

ANDREW FELIX IV SUAREZ CUNANAN

Received 3 March, 2026; accepted 27 May, 2026; published 30 June, 2026.

DAVAO DEL SUR STATE COLLEGE
andrewfelixiv.cunanan@dssc.edu.ph

ABSTRACT. In this paper, a version of Harnack extension of Henstock-Kurzweil integral of $C[a, b]$-valued functions was introduced. Moreover, some conditions were presented such that the said function is closed under Harnack extension.

Key words and phrases: $C[a, b]$ space-valued function; Henstock-Kurzweil Integrable Function; Harnack Extension.

2010 *Mathematics Subject Classification.* 26A39; 26A45; 35B53; 46E40.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The topic of integrating vector-valued functions remains a focus of ongoing research in functional analysis, particularly when the target space is the Banach space $C[a, b]$, which encompasses continuous real-valued functions defined on a compact interval. Functions that take values in $C[a, b]$ frequently emerge in various applications, such as the analysis of evolution equations, integral equations, and functional differential equations. However, conventional integration methods like the Riemann or Lebesgue integrals may not be adequate for addressing these functions, particularly in cases involving non-absolutely integrable or rapidly oscillating functions.

The Henstock-Kurzweil integral, also known as the generalized Riemann or gauge integral, offers a flexible and robust alternative. Unlike the Lebesgue integral, the Henstock-Kurzweil integral can integrate every derivative and allows for finer convergence theorems, such as the Dominated Convergence Theorem in broader contexts. Its ability to extend integration to functions with problematic behavior makes it a powerful tool for analyzing $C[a, b]$-valued mappings.

In parallel, the concept of Harnack extension plays a fundamental role in the theory of harmonic and subharmonic functions. Originating in potential theory, Harnack extensions provide a method to extend functions beyond their original domains while preserving harmonicity or related regularity properties. In the setting of vector-valued functions, particularly those valued in function spaces like $C[a, b]$, Harnack-type principles offer pathways to extend solutions to partial differential equations or variational problems in a controlled and analytically tractable manner.

This article aims to investigate analytic structure by Harnack extension for Henstock-Kurzweil integral of $C[a, b]$-valued functions.

2. $C[a, b]$-VALUED FUNCTIONS

Throughout, we consider the space $\mathcal{C}[a, b]$ of all continuous real-valued functions defined on $[a, b]$. For more details of the space $\mathcal{C}[a, b]$, see [2] or [5].

Let $[f, g]$ be a closed interval of $\mathcal{C}[a, b]$. A **partition** of $[f, g]$ is any finite set $\{h_0, h_1, \dots, h_n\} \subset [f, g]$ such that

$$h_0 = f,\ h_n = g \text{ and } h_{i-1} < h_i$$

for all $i = 1, 2, \dots, n$.

Denote the set of all real-valued continuous functions on $[a, b]$ by $\mathcal{C}[a, b]$; that is,

$$\mathcal{C}[a, b] = \{f | f : [a, b] \to \mathbb{R} \text{ is continuous on } [a, b]\}.$$

We denote the zero function in $\mathcal{C}[a, b]$ by θ. If $f, g \in \mathcal{C}[a, b]$ we define

$$f \le g \Leftrightarrow f(x) \le g(x),\ f < g \Leftrightarrow f(x) < g(x),\ \text{and } f = g \Leftrightarrow f(x) = g(x)$$

for every $x \in [a, b]$. Also $\mathcal{C}[a, b]$ is a Banach space, see [4], with norm defined by

$$|f| = \max_{x \in [a,b]} |f(x)|.$$

Definition 2.1. [5] Let $f, g \in \mathcal{C}[a, b]$ with $f \le g$. We define the following:

- $(f, g) = \{h \in \mathcal{C}[a, b] : f < h < g\}$, is called an **open interval**;
- $[f, g] = \{h \in \mathcal{C}[a, b] : f \le h \le g\}$, is called a **closed interval**;

- $[f, g) = \{h \in \mathcal{C}[a, b] : f \leq h < g\}$, is called $\mathbf{half-closed\, half-open\, interval}$; and
- $(f, g] = \{h \in \mathcal{C}[a, b] : f < h \leq g\}$, is called $\mathbf{half-open\, half-closed\, interval}$.

For $h, k \in \mathcal{C}[a, b]$, we define $\dfrac{h}{k}$, $h \vee k$, $h \wedge k$ and $|h|$ as follows :

$$\begin{aligned}
\left(\frac{h}{k}\right)(x) &= \frac{h(x)}{k(x)}, \quad \text{for all}\, x \in [a, b],\ k(x) \neq 0, \\
(h \vee k)(x) &= \sup\{h(x), k(x)\}, \quad \text{for all}\, x \in [a, b]. \\
(h \wedge k)(x) &= \inf\{h(x), k(x)\}, \quad \text{for all}\, x \in [a, b]. \\
|h|(x) &= |h(x)|, \quad \text{for all}\, x \in [a, b].
\end{aligned}$$

Definition 2.2. [2] A subset $S \subset \mathcal{C}[a, b]$ is said to be $\mathbf{bounded}$ if there exists $K > 0$ such that for all $h \in S$,

$$|h| \leq K \cdot e.$$

Definition 2.3. [2] A sequence $\{f_n\}$ of elements of $\mathcal{C}[a, b]$ is said to be $\mathbf{convergent}$ to $f \in \mathcal{C}[a, b]$ if for every $\epsilon > 0$ there is a positive integer K such that for every $n \geq K$, the terms f_n satisfy

$$|f_n - f| < \epsilon \cdot e.$$

A sequence $\{f_n\}$ which converges to f in $\mathcal{C}[a, b]$, will be written

$$f_n \to f \qquad\qquad n \to \infty.$$

Definition 2.4. [2] A subset $S \subset \mathcal{C}[a, b]$ is said to be $\mathbf{closed}$ if for all $h \in S$, there exists a sequence $\{h_n\}$ in S such that $h_n \to h$.

3. HENSTOCK-KURZWEIL INTEGRAL

Definition 3.1. [5] A function $F : [f, g] \to \mathcal{C}[a, b]$ is said to be Henstock-Kurzweil integrable, briefly $\mathcal{HK}$-integrable, on $[f, g]$ if there is $s \in \mathcal{C}[a, b]$ with the following property:
for every $\epsilon > 0$ there is a gauge δ on $[f, g]$ such that if $D = \{([h_{i-1}, h_i], ti) : i = 1, 2, \ldots n\}$ is any δ-fine tagged division of $[f, g]$, then

$$|S(F, D) - s| < \epsilon \cdot e.$$

The element $s \in \mathcal{C}[a, b]$ is called the Henstock-Kurweil integral, briefly $\mathcal{HK}$-integral, of F over $[f, g]$ and is written by

$$s = (\mathcal{HK}) \int_f^g F(h)\, dh = (\mathcal{HK}) \int_f^g F.$$

Definition 3.2. Let $F : [f, g] \to \mathcal{C}[a, b]$ be a function such that the product $F \cdot 1_X$ is Henstock-Kurzweil integrable on $[f, g]$ where $X \subset [f, g]$. A primitive of $F \cdot 1_X$ is a function $G : [f, g] \to \mathcal{C}[a, b]$ such that for every $h \in [f, g]$,

$$G(h) = (\mathcal{HK}) \int_f^g (F \cdot 1_X)(s)\, ds$$

.

4. HARNACK EXTENSION

Let $F : [f, g] \to \mathcal{C}[a, b]$ be Henstock-Kurzweil integrable on $[f, g]$. then $(\mathcal{HK}) \int_f^g F$ can be evualted as limit; that is,

$$\lim_{h \to f^+} (\mathcal{HK}) \int_h^g = (\mathcal{HK}) \int_f^g F.$$

Conversely, if F is Henstock-Kurzweil integrable on $(h, g]$ for all $f < h < g$ and $\lim_{h \to f^+} (\mathcal{HK}) \int_h^g F$ exists, then f is Henstock-Kurzweil integrable on $[f, g]$. This is known as the Cauchy extension for Henstock-Kurzweil Integral.

Definition 4.1. Let $\delta(h) : [f, g] \to \mathcal{C}[a, b]$ with $\delta(h) > \theta$ be a function and X be a closed subset of $[f, g]$. A finite collection of interval-point pair $Q = \{([u, v], t)\}$ is a Henstock δ-fine cover of X if each $([u, v], t)$ is a Henstock δ-fine with $t \in X$ and

$$X \subseteq \bigcup_{[u,v] \in Q} [u, v].$$

To formulate the Harnack extension, we need the notion of “nonabsolute” set.

Definition 4.2. A set $E \subset \mathcal{C}[a, b]$ is said to be an elementary set if E is an interval or a finite union of mutually non-overlapping intervals.

Definition 4.3. Let G be an open interval in $[f, g]$. An elementary set E is called a Henstock nonabsolute subset of G if there exists $\delta(t) > \theta$ on $[f, g]$ such that E is the complement of a Henstock δ-fine cover of $[f, g] \setminus G$; that is, there exists $\delta(t) > \theta$ on $[f, g]$ and a Henstock δ-fine cover $Q = \{([u, v], t)\}$ of $[f, g] \setminus G$ such that

$$E = \left(\bigcup_{[u,v] \in Q} [u, v] \right)^c.$$

We also say that E is a Henstock nonabsolute subset G involving δ.

We note that $F : [f, g] \to \mathcal{C}[a, b]$ is Henstock-Kurzweil integrable on $X \subseteq [f, g]$ if the function $F \cdot 1_X$ is Henstock-Kurzweil integrable on $[f, g]$ and

$$(\mathcal{HK}) \int_X F = (\mathcal{HK}) \int_f^g (F \cdot 1_X).$$

Definition 4.4. Let F_n be Henstock-Kurzweil integrable on $[f, g]$ for all $n \in \mathbb{N}$. We say that the sequence $\langle F_n \rangle$ is equi-integrable on $[f, g]$ if for every $\epsilon > 0$ there exists a $\delta(h) > \theta$ on $[f, g]$ such that for all δ-fine division D of $[f, g]$, we have

$$\left| (D) \sum_{i=1}^{n} F_n(t_i)[h_i - h_{i-1}] - (\mathcal{HK}) \int_f^g F \right| < \epsilon \cdot e.$$

5. MAIN RESULTS

Proposition 5.1. Let $F : [f, g] \to \mathcal{C}[a, b]$ and subsets $S_1, S_2 \in [f, g]$ be given. Then, whenever three of the integrals

$$(\mathcal{HK}) \int_{S_1} F, \quad (\mathcal{HK}) \int_{S_2} F, \quad (\mathcal{HK}) \int_{S_1 \cup S_2} F, \quad (\mathcal{HK}) \int_{S_1 \cap S_2} F$$

exist, then

$$(\mathcal{HK})\int_{S_1} F+(\mathcal{HK})\int_{S_2} F=(\mathcal{HK})\int_{S_1\cup S_2} F+(\mathcal{HK})\int_{S_1\cap S_2} F.$$

Proof: From the identity $\chi_{S_1\cup S_2}=\chi_{S_1}+\chi_{S_2}-\chi_{S_1\cap S_2}$, the Henstock-Kurzweil integral of $F\cdot\chi_{S_1\cup S_2}$ is given by

$$\begin{aligned}(\mathcal{HK})\int_f^g F\cdot\chi_{S_1\cup S_2}&=(\mathcal{HK})\int_f^g F\cdot\chi_{S_1}+(\mathcal{HK})\int_f^g F\cdot\chi_{S_2}\\&\quad-(\mathcal{HK})\int_f^g F\cdot\chi_{S_1\cap S_2}\\(\mathcal{HK})\int_{S_1\cup S_2} F&=(\mathcal{HK})\int_{S_1} F+(\mathcal{HK})\int_{S_2} F-(\mathcal{HK})\int_{S_1\cap S_2} F\\(\mathcal{HK})\int_{S_1\cup S_2} F+(\mathcal{HK})\int_{S_1\cap S_2} F&=(\mathcal{HK})\int_{S_1} F+(\mathcal{HK})\int_{S_2} F.\end{aligned}$$

■

In the case where $S_1\cap S_2=\emptyset$,

$$(\mathcal{HK})\int_{S_1\cup S_2} F=(\mathcal{HK})\int_{S_1} F+(\mathcal{HK})\int_{S_2} F.$$

Theorem 5.2. Let $F:[f,g]\to\mathcal{C}[a,b]$, X be a closed subset of $[f,g]$ and

$$S=[f,g]\setminus X=\bigcup_{i\in\mathbb{N}} E_i$$

where E_i are mutually disjoint elementary sets in $[f,g]$. Furthermore, put $S_n=\bigcup_{i=1}^n E_i$ for $n\in\mathbb{N}$ and assume that the integral $(\mathcal{HK})\int_X F$ exists and the sequence $\{F\chi_{S_n}\}$ is equi-integrable on $[f,g]$. Then the integrals $(\mathcal{HK})\int_S F$, $(\mathcal{HK})\int_{[f,g]} F$, and $(\mathcal{HK})\int_{E_i} F$ exist for all $i\in\mathbb{N}$. Moreover,

$$(\mathcal{HK})\int_{[f,g]} F=(\mathcal{HK})\int_X F+(\mathcal{HK})\int_S F$$

where

$$(\mathcal{HK})\int_S F=\sum_{i=1}^{\infty}(\mathcal{HK})\int_{E_i} F.$$

Proof: Since $E_i's$ are elementary sets, S_n is an elementary set in $[f,g]$ for every $n\in\mathbb{N}$, $E_i\subset S_n$ for $i=1,2,\cdots,n$ are mutually disjoint and the integrals $\int_{E_i} F$ for $i=1,2,\ldots,n$ exist. By induction of Proposition 5.1,

$$\begin{aligned}(\mathcal{HK})\int_{S_n} F&=(\mathcal{HK})\int_{\bigcup_{i=1}^n E_i} F\\&=(\mathcal{HK})\int_{E_1} F+(\mathcal{HK})\int_{E_2} F+\cdots+(\mathcal{HK})\int_{E_n} F\\&=\sum_{i=1}^{n}(\mathcal{HK})\int_{E_i} F\end{aligned}$$

Furthermore, as n gets large

$$\big(F \cdot \chi_{S_n}\big)(t) = \lim_{n\to\infty} \big(F \cdot \chi_{S_n}\big)(t) \qquad \text{for all } t \in [f,g],$$

we obtain

$$\begin{aligned}(\mathcal{HK})\int_S F &= (\mathcal{HK})\int_f^g F \cdot \chi_{S_n}\\ &= (\mathcal{HK})\int_f^g \lim_{n\to\infty} F \cdot \chi_{S_n}\\ &= \lim_{n\to\infty} (\mathcal{HK})\int_f^g F \cdot \chi_{S_n}\\ &= \lim_{n\to\infty} (\mathcal{HK})\int_{S_n} F\\ &= \lim_{n\to\infty} (\mathcal{HK})\int_{\bigcup_{i=1}^n E_i} F\\ &= \sum_{i=1}^{\infty} (\mathcal{HK})\int_{E_i} F.\end{aligned}$$

■

Theorem 5.3. Let $F : [f,g] \to \mathcal{C}[a,b]$ and X be a closed subset of $[f,g]$. Suppose the following conditions are satisfied:

(1) F is Henstock-Kurzweil integrable on X,
(2) F is Henstock-Kurzweil integrable on every elementary subset E of $[f,g] \setminus X$, and
(3) there exists A such that for every $\epsilon > 0$ there exists $\delta_0(h) > \theta$ such that for any Henstock nonabsolute subset E of $[f,g] \setminus X$ involving $\delta_0(h)$, we have

$$\left|(\mathcal{HK})\int_E F - A\right| < e \cdot \epsilon.$$

Then F is Henstock-Kurzweil integrable on $[f,g]$ and

$$(\mathcal{HK})\int_f^g F = (\mathcal{HK})\int_X F + A.$$

Proof Let $\epsilon > 0$ and

$$L = (\mathcal{HK})\int_X F = (\mathcal{HK})\int_f^g (F \cdot 1_X).$$

Then there exists $\delta_0(t) > \theta$ such that for any Henstock δ_0-fine division $D = \{[h_{i-1}, h_i], t_i\}$ of $[f,g]$, we have

$$\left|(D)\sum_{i=1}^{n}(F \cdot 1_X)(t_i)[h_i - h_{i-1}] - L\right| < \epsilon \cdot e. \tag{5.1}$$

Since $(f,g) \setminus X$ is open, choose a countable collection of pairwise disjoint open intervals $\{(p_i, q_i) : i = 1, 2, \ldots\}$ such that

$$(f,g) \setminus X = \bigcup_{i=1}^{\infty} (p_i, q_i).$$

Let $I_i = (p_i, q_i)$. By condition (2), F is Henstock-Kurzweil integrable on each I_i. Thus, for each i, there exists $\delta_i(t) > \theta$ on $[f, g]$ such that whenever $D = \{[h_{i-1}, h_i], t_i\}$ is a Henstock δ_i-fine partial division of $[f, g]$, we have

$$(D)\sum_{i=1}^{n} \left|(F \cdot 1_X)(t_i)[h_i - h_{i-1}] - \mathcal{F}_i[h_i - h_{i-1}]\right| < \frac{\epsilon}{2^i} \cdot e$$

wher $\mathcal{F}_i$ is the primitive of $F \cdot 1_{I_i}$ on $[f, g]$. Define $\mathcal{F}(h) = \mathcal{F}_i(h)$ if $h \in I_i$.

Note that if $t \in (f, g) \setminus X$, then there exists $i \in \mathbb{N}$ such that $t \in I_i$. Define $\delta(t) > \theta$ on $[f, g]$ by

$$\delta(t) = \min\{\delta_0(t), \delta_i(t)\}.$$

Since each I_i is open, we may assume that $(t - \delta(t), t + \delta(t)) \subset I_i$ whenever $t \in I_i$ for some i. Let $D = \{([h_{i-1}, h_i], t_i)\}$ be a Henstock δ-fine division of $[f, g]$ and

$$\mathcal{U} = \bigcup_{\substack{([h_{i-1},h_i],t_i)\in D \\ t_i \notin X}} [h_{i-1}, h_i].$$

Then $\mathcal{U}$ is a Henstock nonabsolute subset of $(f, g) \setminus X$ involving δ_0. Hence, by (3)

$$\left|(D)\sum_{t\notin X} \mathcal{F}(h_{i-1}, h_i) - A\right| = \left|(\mathcal{HK})\int_{\mathcal{U}} F - A\right| < \epsilon \cdot e. \tag{5.2}$$

Note that if $t \notin X$, then

$$[h_{i-1}, h_i] \subset (t - \delta(t), t + \delta(t)) \subseteq I_i \subseteq (f, g) \setminus X.$$

Then $\mathcal{U}$ is the union of some Henstock δ_i-fine division of I_i. Hence, we have

$$(D)\sum_{i\notin X} \left|F(t)[h_i - h_{i-1}] - \mathcal{F}(h_{i-1}, h_i)\right| < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \cdot e = \epsilon \cdot e. \tag{5.3}$$

Thus, by $(5.1), (5.3)$, and (5.2), we have

$$\begin{aligned}
&\left|(D)\sum_{i=1}^{n} F(t)[h_i - h_{i-1}] - (L+A)\right| \\
&= \left|(D)\sum_{t\in X} F(t)[h_i - h_{i-1}] + (D)\sum_{t\notin X} F(t)[h_i - h_{i-1}] - L - A\right| \\
&\leq \left|(D)\sum_{t\in X} F(t)[h_i - h_{i-1}] - L\right| + \left|(D)\sum_{t\notin X} F(t)[h_i - h_{i-1}] - A\right| \\
&= \left|(D)\sum_{t\in X} (F\cdot 1_X)(t)[h_i - h_{i-1}] - L\right| \\
&\qquad + \left|(D)\sum_{t\notin X} F(t)[h_i - h_{i-1}] - (D)\sum_{t\notin X} \mathcal{F}(t)(h_{i-1}, h_i) + (D)\sum_{t\notin X} \mathcal{F}(t)(h_{i-1}, h_i) - A\right| \\
&\leq \left|(D)\sum_{t\in X} (F\cdot 1_X)(t)[h_i - h_{i-1}] - L\right| + \left|(D)\sum_{t\notin X} F(t)[h_i - h_{i-1}] - (D)\sum_{t\notin X} \mathcal{F}(t)(h_{i-1}, h_i)\right| \\
&\qquad + \left|(D)\sum_{t\notin X} \mathcal{F}(t)(h_{i-1}, h_i) - A\right| \\
&< \epsilon\cdot e + \epsilon\cdot e + \epsilon\cdot e \\
&= 3\epsilon\cdot e.
\end{aligned}$$

This implies that F is Henstock integrable to $L + A$ on $[f, g]$, that is,

$$(\mathcal{HK})\int_f^g F = L + A = (\mathcal{HK})\int_X F + A.$$

■

Theorem 5.4. Let $F : [f, g] \to \mathcal{C}[a, b]$, X be a closed subset of $[f, g]$, and suppose that F is Henstock-Kurzweil integrable on every elementary subset E of $(f, g) \setminus X$. Then the following are equivalent:

(1) there exists A such that for every $\epsilon > 0$ there exists $\delta_0(t) > \theta$ such that for any Henstock nonabsolute subset E of $(f, g) \setminus X$ involving δ_0, we have

$$\left|(\mathcal{HK})\int_E F - A\right| < \epsilon\cdot e.$$

(2) for every $\epsilon > 0$, there exists $\delta(t) > \theta$ for any two Henstock nonabsolute subsets E and E' of $(f, g) \setminus X$ involving δ, we have

$$\left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E'} F\right| < \epsilon\cdot e.$$

Proof: $(i) \implies (ii)$: By (i), there exists A such that for every $\epsilon > 0$ there exists $\delta_0(t) > \theta$ such that for any Henstock nonabsolute subsets E of $(f, g) \setminus X$ involviing δ_0, we have

$$\left|(\mathcal{HK})\int_E F - A\right| < \frac{\epsilon}{2}\cdot e.$$

Thus, for any two Henstock nonabsolute subsets E and E' of $(f,g) \setminus X$ involving δ_0, we have

$$\begin{aligned}\left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E'} F\right| &= \left|(\mathcal{HK})\int_E F - A + A - (\mathcal{HK})\int_{E'} F\right| \\ &\leq \left|(\mathcal{HK})\int_E f - A\right| + \left|(\mathcal{HK})\int_{E'} F - A\right| \\ &< \frac{\epsilon}{2}\cdot e + \frac{\epsilon}{2}\cdot e \\ &= \epsilon\cdot e.\end{aligned}$$

Hence, (ii) holds.

$(ii) \implies (i)$: For each $n \in \mathbb{N}$, there exists $\delta_n(t) > \theta$ such that for any two Henstock nonabsolute subsets E and E' of $(f,g) \setminus X$ involving δ_n, we have

$$\left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E'} F\right| < \frac{1}{n}\cdot e.$$

We may assume that $\{\delta_n\}_{n=1}^{\infty}$ is decreasing. Now, for each $n \in \mathbb{N}$, fix a Henstock nonabsolute subset E_n of $(f,g)\setminus X$ involving δ_n and consider the sequence $\left\{(\mathcal{HK})\int_E F\right\}_{n=1}^{\infty}$ of real-valued functions.

Let $\epsilon > 0$. We claim that the sequence $\left\{(\mathcal{HK})\int_E F\right\}_{n=1}^{\infty}$ is Cauchy. By Archimedian property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Let $n, m \geq N$. Then both E_n and E_m are Henstock nonabsolute subsets of $(f,g) \setminus X$ involving δ_n. Hence,

$$\left|(\mathcal{HK})\int_{E_n} F - (\mathcal{HK})\int_{E_m} F\right| < \frac{1}{N}\cdot e < \epsilon\cdot e.$$

This shows that $\left\{(\mathcal{HK})\int_{E_n} F\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}[a,b]$. Let

$$A = \lim_{n\to\infty} (\mathcal{HK})\int_{E_n} F.$$

Then there exists $M \in \mathbb{N}$ such that for all $n \in M$, we have

$$\left|(\mathcal{HK})\int_{E_n} F - A\right| < \frac{\epsilon}{2}\cdot e.$$

By (ii), there exists $\delta(t) > \theta$ such that for any two Henstock nonabsolute subsets E and E' of $(f,g) \setminus X$ involving δ, we have

$$\left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E'} F\right| < \frac{\epsilon}{2}\cdot e.$$

Taking $\delta_0(t) = \min\{\delta_M(t), \delta(t)\}$ and E be any Henstock nonabsolute subset of $(f,g) \setminus X$ involving δ_0. Thus,

$$\begin{aligned}\left|(\mathcal{HK})\int_E F - A\right| &= \left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E_m} F + (\mathcal{HK})\int_{E_m} F - A\right| \\ &\leq \left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E_m} F\right| + \left|(\mathcal{HK})\int_{E_m} F - A\right| \\ &< \frac{\epsilon}{2}\cdot e + \frac{\epsilon}{2}\cdot e \\ &= \epsilon\cdot e.\end{aligned}$$

Therefore, (i) holds. ■

Corollary 5.5. Let $F : [f, g] \to \mathcal{C}[a, b]$ and X be a closed interval in $[f, g]$. Suppose the following conditions are satisfied:

1. F is Henstock-Kurzweil integrable on X,
2. F is Henstock-Kurzweil integrable on every elementary subset E of $(f, g) \setminus X$, and
3. for every $\epsilon > 0$, theexists $\delta(t) > \theta$ such that for any two Henstock nonabsolute subsets E and E' of $(f, g) \setminus X$ involving δ, we have
$$\left|(\mathcal{HK})\int_E F - (\mathcal{HK})\int_{E'} F\right| < \epsilon \cdot e.$$

Then F is Henstock-Kurzweil integrable on $[f, g]$.

6. CONCLUSION

The Harnack extension of the Henstock-Kurzweil integral to $\mathcal{C}[a, b]$-valued functions successfully generalizes the integrability framework to functions defined on subsets of $[f, g]$ with values in a Banach space. The central result establishes that if $F : [f, g] \to \mathcal{C}[a, b]$ is Henstock-Kurzweil integrable on a closed subset $X \subseteq [f, g]$, integrable on every elementary subset of $[f, g] \setminus X$ and satisfies a Harnack-type control condition with respect to an additive correction term A, then F is Henstock-Kurzweil integrable on the whole interval $[f, g]$, and the integral decomposes as
$$(\mathcal{HK})\int_f^g F = (\mathcal{HK})\int_X F + A.$$
Moreover, this formulation is supported by an equivalence criterion: the existence of a limiting value A of the Henstock-Kurzweil integral over arbitrarily fine nonabsolute subsets of $(f, g) \setminus X$ is equivalent to the Cauchy-type condition that integrals over such subsets converge uniformly in difference. This characterization not only generalizes the scalar-valued Harnack extension but also provides a robust convergence structure for function-valued Henstock-Kurzweil integrals. As such, it deepens the theoretical foundation for integrating in Banach spaces and paves the way for further development in vector integration theory.

REFERENCES

[1] T.M. APOSTOL, *Mathematical Analysis, 2nd edition*. Narosa Publishing House, New Delhi, (2002).

[2] A.S. CUNANAN and J.V. BENITEZ, Simple Properties and Existence Theorem for the Henstock-Kurzweil-Stieltjes Integral of Functions Taking Values on $\mathcal{C}[a, b]$ Space-valued Functions. *European Journal of Pure and Applied Mathematics*, New York Business Global, 2020. `https://doi.org/10.29020/nybg.ejpam.v13i1.3626`

[3] U.M. HANUNG, Role of the Harnack Extension Principle in the Kurzweil-Stieltjes Integral, *Math. CA*, 2022. `https://doi.org/10.48550/arXiv.2207.04426`

[4] E. KREYSZIG, *Introductory Functional Analysis with Application*, John Wiley and Sons. Inc., New York, (1978)

[5] F. UBAIDILLAH , S. DARMAWIJAYA and CH.R. INDRATI, On the Henstock-Kurzweil Integral of $\mathcal{C}[a, b]$ Space-valued Functions, *IJMA*, vol. 37 (9)(2015), pp. 1831-1846.