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RANDOM FIXED POINT RESULT IN BANACH SPACE

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ABSTRACT. The present paper deals with an extension of classical fixed point theory using random variable for advancement because it has far-reaching applications in calculus, optimization, and stochastic processes. However, many nonlinear operators arise in applied mathematics which do not fulfill the strict contraction conditions of the classical Banach principle. This limitation has inspired the development of generalized non-contractive mappings. New generalized contractive-type conditions for random operators in a complete probability measure space as well as Banach space are established. The proposed conditions incorporate nonlinear, fractional, and minimum-type terms. Random fixed point theorem are proved for rational inequalities. These results not only generalize existing theorems, but also contribute to further development of functional analysis and its applications in mathematical modeling.

Key words and phrases: Contraction and Non-contraction mapping, Banach Fixed Point (BFT), Fixed Point (F_P), Random Operator (R_O), Random Fixed Point(R_{F_P}), Common Fixed Point.

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1. INTRODUCTION

Fixed point theory holds a significant position in functional analysis, and its origins can be traced back to Poincare [4] in 1886 and Brower [5] gave the Brower F_P theory in 1912. After a decade, Banach introduced a concept using the contraction principle, named Banach's Contraction Principle [13]. This leads to algorithmic applications in numerical analysis and differential equations, integral equations [34, 35, 12], optimization, and stochastic processes. Moreover, this principle has been generalized in various directions, including the works of Kannan [7], Chatterjea [8], Kirk [9], Reich, and others. However, many researchers have observed that some operators that arise in functional analysis, differential equations, and applied mathematics do not satisfy the strict contractive condition. This observation has driven the study of non contraction or non expansive mappings, which extend the scope of fixed point theory beyond the classical framework. Recently several mathematicians Datson [14], Goebel [15], kirk and Simi [16], Iseki [17], Singh and Chatterjee [18], Sharma and Rajput [19], Pathak and Maity [20], Sharma and Bhagwan [21], Prajapati [36], Bhardwaj [37], Shahzad and Udomene [22] introduced the generalization on non expansive mappings, as well as explored non contraction mappings concerning fixed point theorem in different types of topological spaces [10].

The *random fixed points* is an emerging topic for research in non-linear analysis in present time [1, 2, 38, 40] due to interaction with an uncertain and ambiguous real-life problem. Moreover, random non-linear analysis is the most efficient branch to deal with the solution of random operator related problem such as developing models or equation which demonstrate certain physical phenomena in engineering mathematics, physics and different biological systems. In 1950s this theory has been originated from the work of the Prague school of probabilities. It is viewed as a natural generalization of classical (deterministic) fixed point theorems and plays an essential role in the study of stochastic dynamic programming, random equations, random matrices, random partial differential equations, and other classes of random operators arising in physical systems. Over the years, this field has attracted considerable attention, leading to the development of powerful techniques for solving non-linear random systems.

In this direction, Hans [23] proved a stochastic analogue of the BFT in separable complete metric spaces. Later, Itoh [24] (1979) extended these results to the setting of multivalued contraction random operators. More recently, Zhang et al. (2011) [25] investigated φ-weakly contractive random operators and established stability. Many mathematicians worked in the said field. Furthermore, Khan et al. developed a class of generalized random operators and established convergence and stability results that improved and extended many existing random fixed point theorems, including those of Okeke [27]. In addition, In 2012, Chugh et al. [26] introduced a new three-step iterative process known as the CR iterative scheme. They demonstrated that this scheme is equivalent to, and converges more rapidly than the Picard, Mann [29], Ishikawa [28], Agarwal et al. [39], Noor [30], and SP [32] iteration methods for certain classes of contractive operators, in the sense of Berinde [31]

The developments have provided the motivation for undertaking my research in the area of random fixed point [3]. The objective of this paper is to advance the theory of R_{Fp}, where the underlying operator depends not only on a Banach space but also on an associated probability space. In this study, we focus on R_O, $\mathcal{T} : \Theta \times \mathcal{E} \to \mathcal{E}$ defined on a complete probability measure space, where $\mathcal{E}$ is a subset of a separable Banach space. Our study is particularly concerned with operators that satisfy the involution condition $\mathcal{T}^2 = \mathcal{I}$, where $\mathcal{I}$ is the identity operator, that $\mathcal{T}$ satisfies a generalized contractive inequality involving the parameters $\xi, \zeta, \sigma, \psi, \eta, \tau_1, \tau_2$, and the central contribution of this work is the introduction of a generalized contractive-type condition using these random operators. Unlike classical contractions, which are typically linear or involve a single control parameter, our formulation incorporates multiple parameters, namely $\xi, \zeta, \sigma, \psi, \eta, \tau_1, \tau_2$, along with new contractive-type conditions that incorporate combinations of nonlinear contractive distance terms, fractional expressions, and minimum functions. In addition, the presence of minimum-type terms provides greater flexibility and allows us to address a wider class of random operators. This generalized inequality allows us to capture a wide range of operator behaviors, including those that are nonlinear or exhibit variable contraction strength depending on the points in the Banach space or the outcomes in the probability space. which is a versatile and powerful framework. Under these generalized contractive conditions, the R_{Fp} [11] also opens avenues for further research using two and more than two random operators and iterative methods to approximate random fixed point, which is called common F_P.

2. **BASIC DEFINITIONS**

Some fundamental definitions and findings that will be used throughout the article are reviewed in this section.
One can see the details about contraction mapping and Banach fixed point theorem for Banach spaces in the given literatures. [13], [33]

Definition 2.1 **(R_{Fp}).** Let $\Phi = (\Theta, \Sigma, \mu)$ define a complete probability measure space, $\mathcal{X}$ be a Banach space, and $\mathcal{E} \subset \mathcal{X}$ i.e. nonempty. A mapping $\mathcal{T} : \Theta \times \mathcal{E} \to \mathcal{E}$ is said to be a R_O if for each $x \in \mathcal{E}$, the mapping

$$\partial \mapsto \mathcal{T}(\partial, x)$$

is measurable.

A *random fixed point* of $\mathcal{T}$ is a measurable function $x : \Theta \to \mathcal{E}$ such that

$$\mathcal{T}(\partial, x(\partial)) = x(\partial), \quad \text{for } \mu\text{-almost every } \partial \in \Theta.$$

Here,Φ represents complete probability measure space and (Θ, Σ) is measure where, Θ is a non empty sample space, Σ denotes sigma algebra and μ demonstrate probability measure on Σ. A random fixed point, $x(\partial)$ therefore represents a random variable in $\mathcal{E}$ which maps $x : \Theta \to \mathcal{E}$. $\mathcal{T} : \Theta \times \mathcal{E} \to \mathcal{E}$ is known R_O if, for all particular $x \in \mathcal{E}$ the map $x : \Theta \to \mathcal{E}$ is measurable. It not only depend on the variable $x \in \mathcal{E}$ but also depends on a random element ∂ drawn from Φ [24].

3. **ESTABLISHED RESULTS**

Theorem 3.1. *Let $\Phi = (\Theta, \Sigma, \mu)$ be a complete probability measure space and $\mathcal{X}$ be Banach space, let $\mathcal{E} \subset \mathcal{X}$ i.e. nonempty. Let*

$$\mathcal{T} : \Theta \times \mathcal{E} \to \mathcal{E}$$

be a R_O satisfying

$$\mathcal{T}^2 = \mathcal{I},$$

for every $\partial \in \Theta$ *and every* $X \in \mathcal{E}$ *the following inequality holds:*

$$\begin{aligned}
\|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\| \le \xi\, & \frac{\big(\|X(\partial) - \mathcal{T}(\partial, X(\partial))\| + \|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|\big)\|X(\partial) - Y(\partial)\|}{1 + \|Y(\partial) - \mathcal{T}(\partial, X(\partial))\|^2 + \|\mathcal{T}(\partial, Y(\partial)) - Y(\partial)\|^3 + \|X(\partial) - Y(\partial)\|^4} \\
& + \zeta\, \frac{\big(\|X(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|Y(\partial) - \mathcal{T}(\partial, X(\partial))\|\big)\|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\|}{1 + \|X(\partial) - \mathcal{T}(\partial, Y(\partial))\|^2 + \|\mathcal{T}(\partial, Y(\partial)) - Y(\partial)\|^3 + \|\mathcal{T}(\partial, X(\partial)) - Y(\partial)\|^4} \\
& + \sigma\big(\|X(\partial) - \mathcal{T}(\partial, X(\partial))\| + \|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|\big) \\
& + \psi\big(\|X(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|Y(\partial) - \mathcal{T}(\partial, X(\partial))\|\big) \\
& + \eta\, \|X(\partial) - Y(\partial)\| \\
& + \tau_1 \min\{\|X(\partial) - \mathcal{T}(\partial, X(\partial))\|,\ \|X(\partial) - \mathcal{T}(\partial, Y(\partial))\|\} \\
& + \tau_2 \min\{\|Y(\partial) - \mathcal{T}(\partial, X(\partial))\|,\ \|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|\}.
\end{aligned}$$

where $\xi, \zeta, \sigma, \psi, \eta, \tau_1, \tau_2 \ge 0$.
If

$$2\xi\, + \zeta\, + 4\sigma + 2\psi + \eta + \tau_1 + \tau_2 < 2$$

then $\mathcal{T}$ *has at least one* R_{Fp}.
Moreover, if

$$\zeta + 2\psi + \eta < 1,$$

then $\mathcal{T}$ *has a unique* R_{Fp}.

Proof. **existence:**
Suppose $X \in \mathcal{X}$
Let

$$Y(\partial) = \tfrac{1}{2}\big(\mathcal{T}(\partial, X(\partial)) + \mathcal{I}(\partial, X(\partial))\big), \quad Z(\partial) = \mathcal{T}(\partial, Y(\partial)), \quad U(\partial) = 2Y(\partial) - Z(\partial).$$

Then

$$\begin{aligned}
\|Z(\partial) - X(\partial)\| \le \xi\, & \frac{\big(\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\big)\|Y(\partial) - \mathcal{T}(\partial, X(\partial))\|}{1 + \|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\|^2 + \|X(\partial) - \mathcal{T}(\partial, X(\partial))\|^3 + \|Y(\partial) - \mathcal{T}(\partial, X(\partial))\|^4} \\
& + \zeta\, \frac{\big(\|Y(\partial) - X(\partial)\| + \|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\|\big)\|\mathcal{T}(\partial, Y(\partial)) - X(\partial)\|}{1 + \|Y(\partial) - X(\partial)\|^2 + \|X(\partial) - \mathcal{T}(\partial, X(\partial))\|^3 + \|\mathcal{T}(\partial, Y(\partial)) - \mathcal{T}(\partial, X(\partial))\|^4} \\
& + \sigma\Big(\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big) \\
& + \psi\Big(\|Y(\partial) - X(\partial)\| + \|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\|\Big) \\
& + \eta\, \|Y(\partial) - \mathcal{T}(\partial, X(\partial))\| \\
& + \tau_1 \min\Big\{\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|,\ \|Y(\partial) - X(\partial)\|\Big\} \\
& + \tau_2 \min\Big\{\|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\|,\ \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big\}.
\end{aligned}$$

Using the inequality $\frac{1}{1+a^4+B^3+c^2} \le \frac{1}{a+b+c}$ on the coefficient of ξ and ζ in right hand side

$$\begin{aligned}
\|Z(\partial)-X(\partial)\| \leq \xi\, & \frac{\big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\big)\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|}{\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|+\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|} \\
& + \zeta\, \frac{\big(\|Y(\partial)-X(\partial)\|+\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|\big)\|\mathcal{T}(\partial,Y(\partial))-X(\partial)\|}{\|Y(\partial)-X(\partial)\|+\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|\mathcal{T}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|} \\
& + \sigma\Big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big) \\
& + \psi\Big(\|Y(\partial)-X(\partial)\|+\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|\Big) \\
& + \eta\,\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\| \\
& + \tau_1 \min\Big\{\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|,\ \|Y(\partial)-X(\partial)\|\Big\} \\
& + \tau_2 \min\Big\{\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|,\ \|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}.
\end{aligned}$$

Equating the denominator in the first term and the second term in the right-hand side, we get the next step as follows:

$$\begin{aligned}
\|Z(\partial)-X(\partial)\| \leq \xi\, & \frac{\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|}{\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|}\,\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\| \\
& + \zeta\, \frac{\|Y(\partial)-X(\partial)\|+\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|}{\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|\mathcal{T}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|}\,\|\mathcal{T}(\partial,Y(\partial))-X(\partial)\| \\
& + \sigma\Big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big) \\
& + \psi\Big(\|Y(\partial)-X(\partial)\|+\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|\Big) \\
& + \eta\,\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\| \\
& + \tau_1 \min\Big\{\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|,\ \|Y(\partial)-X(\partial)\|\Big\} \\
& + \tau_2 \min\Big\{\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|,\ \|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}.
\end{aligned}$$

Substituting $Y(\partial)=\frac{1}{2}\big(\mathcal{T}(\partial,X(\partial))+\mathcal{I}(\partial,X(\partial))\big)$ in the term $\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|$, $\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|$, $\|\mathcal{T}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|$,$\|Y(\partial)-X(\partial)\|$ and using the condition $\mathcal{T}^2=\mathcal{I}$ we obtain the following

$$\begin{aligned}
\|Z(\partial) - X(\partial)\| \le \xi\, & \frac{[\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|]\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|}{\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| + \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|} \\
& + \zeta\, \frac{[\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| + \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|]\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|}{\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| + \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|} \\
& + \sigma\Big(\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big) \\
& + \psi\Big(\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| + \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big) \\
& + \eta\, \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| \\
& + \tau_1 \min\Big\{\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|,\ \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big\} \\
& + \tau_2 \min\Big\{\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|,\ \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\|Z(\partial) - X(\partial)\| \le \xi\, & \frac{\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| + \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|^2}{\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|} \\
& + \zeta\, \frac{\frac{1}{4}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|^2 + \frac{1}{4}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|^2}{\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|} \\
& + \sigma\Big(\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\| + \|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big) \\
& + \psi\Big(\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big) \\
& + \eta\, \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| \\
& + \tau_1 \min\Big\{\|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|,\ \frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big\} \\
& + \tau_2 \Big\{\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\|\Big\}.
\end{aligned}$$

Case 1. If $\frac{1}{2}\|\mathcal{T}(\partial, X(\partial)) - X(\partial)\| < \|Y(\partial) - \mathcal{T}(\partial, Y(\partial))\|$ then the inequality becomes as follows:

$$\begin{aligned}\|Z(\partial)-X(\partial)\| \le \xi\,\frac{1}{2}\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\| + \frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\zeta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\sigma\Big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\ +\psi\Big(\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\ +\eta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\tau_1\Big\{\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}\\ +\tau_2\Big\{\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}.\end{aligned}$$

Now,

$$\begin{aligned}\|Z(\partial)-X(\partial)\| \le \big(\xi\,\frac{1}{2}+\zeta\,\frac{1}{2}+\sigma+\psi+\eta\,\frac{1}{2}+\tau_1\frac{1}{2}+\tau_2\frac{1}{2}\big)\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\Big(\xi\frac{1}{2}+\sigma\Big)\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\end{aligned}$$

Case 2. if $\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\| < \frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|$ then using this relation we obtain

$$\begin{aligned}\|Z(\partial)-X(\partial)\| \le \xi\,\frac{1}{2}\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\| + \frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\zeta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\sigma\Big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\ +\psi\Big(\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\ +\eta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\tau_1\Big\{\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\Big\}\\ +\tau_2\Big\{\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}.\end{aligned}$$

Then the inequality should be

$$\begin{aligned}\|Z(\partial)-X(\partial)\| \le \big(\xi\,\frac{1}{2}+\zeta\,\frac{1}{2}+\sigma+\psi+\eta\,\frac{1}{2}+\tau_2\frac{1}{2}\big)\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\Big(\xi\frac{1}{2}+\sigma+\tau_1\Big)\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\end{aligned}$$

Combining both cases, we get the result

$$\|Z(\partial)-X(\partial)\|\le \min\big(case1, case2\big)$$

Again, solving for $\|U(\partial)-X(\partial)\|$,

$$\begin{aligned}
\|U(\partial)-X(\partial)\| \le{}& \xi\,\frac{\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\big)\|X(\partial)-Y(\partial)\|}{1+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|^2+\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|^3+\|X(\partial)-Y(\partial)\|^4}\\
&+\zeta\,\frac{\big(\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|\big)\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|}{1+\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|^2+\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|^3+\|\mathcal{T}(\partial,X(\partial))-Y(\partial)\|^4}\\
&+\sigma\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\big)\\
&+\psi\big(\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|\big)\\
&+\eta\,\|X(\partial)-Y(\partial)\|\\
&+\tau_1\min\{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|\}\\
&+\tau_2\min\{\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\}.
\end{aligned}$$

implementing this inequality $\dfrac{1}{1+a^4+B^3+c^2}\le\dfrac{1}{a+b+c}$ in the first and second terms on the right-hand side

$$\begin{aligned}
\|U(\partial)-X(\partial)\| \le{}& \xi\,\frac{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|}{\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|+\|X(\partial)-Y(\partial)\|}\,\|X(\partial)-Y(\partial)\|\\
&+\zeta\,\frac{\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|}{\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|+\|\mathcal{T}(\partial,X(\partial))-Y(\partial)\|}\\
&\quad\times\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|\\
&+\sigma\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\big)\\
&+\psi\big(\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|\big)\\
&+\eta\,\|X(\partial)-Y(\partial)\|\\
&+\tau_1\min\{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|\}\\
&+\tau_2\min\{\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\}.
\end{aligned}$$

We calculate this by equating the denominator of the first two terms, as we have done above in a previous calculation

$$
\begin{aligned}
\|U(\partial)-X(\partial)\| \leq{}& \xi\,\frac{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|}{\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|+\|X(\partial)-Y(\partial)\|}\,\|X(\partial)-Y(\partial)\| \\
&+\zeta\,\frac{\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|}{\|X(\partial)-Y(\partial)\|+\|\mathcal{T}(\partial,X(\partial))-Y(\partial)\|}\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\| \\
&+\sigma\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\big) \\
&+\psi\big(\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|\big) \\
&+\eta\,\|X(\partial)-Y(\partial)\| \\
&+\tau_1\min\{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|\} \\
&+\tau_2\min\{\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\}.
\end{aligned}
$$

Now we replace these term $\|Y(\partial)-\mathcal{T}(\partial,X(\partial))\|$, $\|X(\partial)-\mathcal{T}(\partial,Y(\partial))\|$, $\|\mathcal{T}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|$, $\|Y(\partial)-X(\partial)\|$ by using $Y(\partial)=\frac{1}{2}\big(\mathcal{T}(\partial,X(\partial))+\mathcal{I}(\partial,X(\partial))\big)$ and utilize the condition $\mathcal{T}^2=\mathcal{I}$

$$
\begin{aligned}
\|U(\partial)-X(\partial)\| \leq{}& \xi\,\frac{[\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\|]\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|}{\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|} \\
&+\zeta\,\frac{[\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\|]\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|}{\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|} \\
&+\sigma\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\big) \\
&+\psi\big(\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\big) \\
&+\eta\,\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\| \\
&+\tau_1\min\{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\} \\
&+\tau_2\min\{\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\}.
\end{aligned}
$$

Then we have

$$\begin{aligned}\|U(\partial)-X(\partial)\| \leq{}& \xi\,\frac{\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|^2+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|}{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|}\\
&+\zeta\,\frac{\frac{1}{4}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|^2+\frac{1}{4}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|^2}{\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|}\\
&+\sigma\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|+\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\big)\\
&+\psi\big(\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\big)\\
&+\eta\,\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\\
&+\tau_1\{\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\}\\
&+\tau_2\min\{\frac{1}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|,\ \|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\}.\end{aligned}$$

Case 3. if $\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\| < \|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|$ then the inequality of the form of

$$\begin{aligned}\|U(\partial)-X(\partial)\| \leq{}& \xi\,\frac{1}{2}\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\
&+\zeta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\
&+\sigma\Big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\
&+\psi\Big(\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\
&+\eta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\
&+\tau_1\Big\{\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}\\
&+\tau_2\Big\{\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}.\end{aligned}$$

Computing all these we get

$$\begin{aligned}\|U(\partial)-X(\partial)\| \leq{}& \big(\xi\,\frac{1}{2}+\zeta\,\frac{1}{2}+\sigma+\psi+\eta\,\frac{1}{2}+\tau_1\frac{1}{2}+\tau_2\frac{1}{2}\big)\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\
&+\Big(\xi\frac{1}{2}+\sigma\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\end{aligned}$$

Case 4. if $\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\| < \frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|$ then we have the inequality in the form of

$$\begin{aligned}\|U(\partial)-X(\partial)\| \leq \xi\,\frac{1}{2}\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\zeta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\sigma\Big(\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\ +\psi\Big(\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big)\\ +\eta\,\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\tau_1\Big\{\frac{1}{2}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\Big\}\\ +\tau_2\Big\{\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\Big\}.\end{aligned}$$

Continuing similarly, we obtain

$$\begin{aligned}\|U(\partial)-X(\partial)\| \leq (\xi\,\frac{1}{2}+\zeta\,\frac{1}{2}+\sigma+\psi+\eta\,\frac{1}{2}+\tau_1\frac{1}{2})\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\Big(\xi\frac{1}{2}+\sigma+\tau_2\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\end{aligned}$$

$$\|U(\partial)-X(\partial)\| \leq \min\Big(case3, case4\Big)$$

$$\begin{aligned}\|Z(\partial)-U(\partial)\| &= \|(Z(\partial)-X(\partial))-(U(\partial)-X(\partial))\|\\ &\leq \|(Z(\partial)-X(\partial))\|-\|(U(\partial)-X(\partial))\|\end{aligned}$$

$$\begin{aligned}\|Z(\partial)-U(\partial)\| \leq (\xi\,\frac{1}{2}+\zeta\,\frac{1}{2}+\sigma+\psi+\eta\,\frac{1}{2}+\tau_1\frac{1}{2}+\tau_2\frac{1}{2})\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\Big(\xi\frac{1}{2}+\sigma\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|+\\ (\xi\,\frac{1}{2}+\zeta\,\frac{1}{2}+\sigma+\psi+\eta\,\frac{1}{2}+\tau_1\frac{1}{2}+\tau_2\frac{1}{2})\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|+\\ \Big(\xi\frac{1}{2}+\sigma\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\end{aligned}$$

$$\begin{aligned}\|Z(\partial)-U(\partial)\| \leq (\xi\,+\zeta\,+2\sigma+2\psi+\eta+\tau_1+\tau_2)\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\Big(\xi+2\sigma\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\end{aligned}$$

$$\begin{aligned}\|Z(\partial)-U(\partial)\| &= \|\mathcal{T}(\partial,Y(\partial))-(2y(\partial)-Z(\partial))\|\\ &= \|2T(\partial,Y(\partial))-2y(\partial)\|\\ &= 2\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\end{aligned}$$

$$\begin{aligned}2\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\| \leq \big(\xi\ +\zeta\ +2\sigma+2\psi+\eta+\tau_1+\tau_2\big)\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|\\ +\Big(\xi+2\sigma\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\|\end{aligned}$$

$$\Big(2-\xi+2\sigma\Big)\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\| \leq \big(\xi\ +\zeta\ +2\sigma+2\psi+\eta+\tau_1+\tau_2\big)\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|$$

$$\|\mathcal{T}(\partial,Y(\partial))-Y(\partial)\| \leq \frac{\big(\xi\ +\zeta\ +2\sigma+2\psi+\eta+\tau_1+\tau_2\big)}{\Big(2-\xi+2\sigma\Big)}\|\mathcal{T}(\partial,X(\partial))-X(\partial)\|$$

$$p=\frac{\big(\xi\ +\zeta\ +2\sigma+2\psi+\eta+\tau_1+\tau_2\big)}{\Big(2-\xi+2\sigma\Big)}<1$$

$$\big(\xi\ +\zeta\ +2\sigma+2\psi+\eta+\tau_1+\tau_2\big)<\Big(2-\xi+2\sigma\Big)$$

Since

$$\big(2\xi\ +\zeta\ +4\sigma+2\psi+\eta+\tau_1+\tau_2\big)<2$$

Let, $\mathcal{S}(\partial,X(\partial))=\frac{1}{2}\big(\mathcal{T}(\partial,X(\partial))+\mathcal{I}(\partial,X(\partial))\big)$ for every $\partial\in\Theta$ and $X\in X$

$$\begin{aligned}\|\mathcal{S}^2(\partial,X(\partial))-\mathcal{S}(\partial,X(\partial))\| &= \|\mathcal{S}(\partial,Y(\partial))-Y(\partial)\|\\ &= \|\tfrac{1}{2}\big(\mathcal{T}(\partial,Y(\partial))+\mathcal{I}(\partial,X(\partial))\big)-Y(\partial)\|\\ &= \frac{1}{2}\|Y(\partial)-\mathcal{T}(\partial,Y(\partial))\|\\ &\leq \frac{p}{2}\|X(\partial)-\mathcal{T}(\partial,X(\partial))\|\end{aligned}$$

For p, $\mathcal{S}^n$ is a Cauchy seq. in $\mathcal{X}$ and using completeness, $\mathcal{S}^n\to X_0(\partial)$

$$\lim_{n\to\infty} \mathcal{S}^n(\partial, X(\partial)) = X_0(\partial)$$

Which implies $\mathcal{S}(\partial, X_0(\partial)) = X_0(\partial)$
Therefore, $\mathcal{T}(\partial, X_0(\partial)) = X_0(\partial)$ is fixed point of $\mathcal{T}$.

uniqueness: If possible, let $Y_0(\neq X_0)$ be another R_{Fp} of $\mathcal{T}$, then

$$\begin{aligned}\|X_0(\partial) - Y_0(\partial)\| \leq \xi \, & \frac{\big(\|X_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\| + \|Y_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\|\big)\|X_0(\partial) - Y_0(\partial)\|}{1 + \|Y_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\|^2 + \|\mathcal{T}(\partial, Y_0(\partial)) - Y_0(\partial)\|^3 + \|X_0(\partial) - Y_0(\partial)\|^4} \\ & + \zeta \, \frac{\big(\|X_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\| + \|Y_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\|\big)\|\mathcal{T}(\partial, X_0(\partial)) - \mathcal{T}(\partial, Y_0(\partial))\|}{1 + \|X_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\|^2 + \|\mathcal{T}(\partial, Y_0(\partial)) - Y_0(\partial)\|^3 + \|\mathcal{T}(\partial, X_0(\partial)) - Y_0(\partial)\|^4} \\ & + \sigma\big(\|X_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\| + \|Y_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\|\big) \\ & + \psi\big(\|X_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\| + \|Y_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\|\big) \\ & + \eta \, \|X_0(\partial) - Y_0(\partial)\| \\ & + \tau_1 \min\{\|X_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\|, \ \|X_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\|\} \\ & + \tau_2 \min\{\|Y_0(\partial) - \mathcal{T}(\partial, X_0(\partial))\|, \ \|Y_0(\partial) - \mathcal{T}(\partial, Y_0(\partial))\|\}.\end{aligned}$$

$$\|X_0(\partial) - Y_0(\partial)\| \leq \zeta \frac{2\|X_0(\partial) - Y_0(\partial)\|^2}{2\|X_0(\partial) - Y_0(\partial)\|} + 2\psi\|X_0(\partial) - Y_0(\partial)\| + \eta\|X_0(\partial) - Y_0(\partial)\|$$

$$\|X_0(\partial) - Y_0(\partial)\| \leq \big(\zeta + 2\psi + \eta\big)\|X_0(\partial) - Y_0(\partial)\|$$

$$\zeta + 2\psi + \eta < 1$$

$$\|X_0(\partial) - Y_0(\partial)\| = 0 \text{ implies } X_0(\partial) = Y_0(\partial)$$

∎

Corollary 3.2. *Let Φ be a complete probability measure space and $\mathcal{X}$ be a Banach space let $\mathcal{E} \subset \mathcal{X}$ where $\mathcal{E} \neq \phi$. If two R_O, $\mathcal{T}, \mathcal{S} : \Theta \times \mathcal{E} \to \mathcal{E}$ commute and both satisfy the condition $\mathcal{T}^2 = \mathcal{I}, \mathcal{S}^2 = \mathcal{I}$ and this mapping holds*

$$\|\mathcal{T}(\partial, X(\partial)) - \mathcal{T}(\partial, Y(\partial))\| \leq$$

$$
\begin{aligned}
&\xi\,\frac{\big(\|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,X(\partial))\|+\|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,Y(\partial))\|\big)\|\mathcal{S}(\partial,X(\partial))-\mathcal{S}(\partial,Y(\partial))\|}{1+\|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|^2+\|\mathcal{T}(\partial,Y(\partial))-\mathcal{S}(\partial,Y(\partial))\|^3+\|\mathcal{S}(\partial,X(\partial))-\mathcal{S}(\partial,Y(\partial))\|^4}\\
&+\zeta\,\frac{\big(\|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|\big)\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|}{1+\|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|^2+\|\mathcal{T}(\partial,Y(\partial))-\mathcal{S}(\partial,Y(\partial))\|^3+\|\mathcal{T}(\partial,X(\partial))-\mathcal{S}(\partial,Y(\partial))\|^4}\\
&+\sigma\big(\|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,X(\partial))\|+\|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,Y(\partial))\|\big)\\
&+\psi\big(\|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|+\|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|\big)\\
&+\eta\,\|\mathcal{S}(\partial,X(\partial))-\mathcal{S}(\partial,Y(\partial))\|\\
&+\tau_1\min\{\|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,X(\partial))\|,\ \|\mathcal{S}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|\}\\
&+\tau_2\min\{\|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|,\ \|\mathcal{S}(\partial,Y(\partial))-\mathcal{T}(\partial,Y(\partial))\|\}.
\end{aligned}
$$

For every $\partial \in \Theta$,
$\xi, \zeta, \sigma, \psi, \eta, \tau_1, \tau_2 \geq 0$ *and*

$$2\xi + \zeta + 4\sigma + 2\psi + \eta + \tau_1 + \tau_2 < 2$$

then $\mathcal{T}$ and $\mathcal{S}$ has at least one R_{Fp}.
Furthermore, if

$$\zeta + 2\psi + \eta < 1,$$

then $\mathcal{T}$ and $\mathcal{S}$ has a unique R_{Fp}.

Corollary 3.3. *Let Φ be a complete probability measure space and $\mathcal{X}$ is a Banach space let $\mathcal{E} \subset \mathcal{X}$ where $\mathcal{E} \neq \phi$. If three R_O, $\mathcal{T}, S, U : \Theta \times \mathcal{E} \to \mathcal{E}$ commute and satisfy the condition $\mathcal{T}^2 = \mathcal{I}, S^2 = \mathcal{I}, U^2 = \mathcal{I}$ and this mapping holds*

$$\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\| \leq$$

$$
\begin{aligned}
&\xi\frac{\big(\|SU(\partial,X(\partial))-\mathcal{T}(\partial,X(\partial))\|+\|SU(\partial,Y(\partial))-\mathcal{T}(\partial,Y(\partial))\|\big)\|SU(\partial,X(\partial))-SU(\partial,Y(\partial))\|}{1+\|SU(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|^2+\|\mathcal{T}(\partial,Y(\partial))-SU(\partial,Y(\partial))\|^3+\|SU(\partial,X(\partial))-SU(\partial,Y(\partial))\|^4}\\
&+\zeta\,\frac{\big(\|SU(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|+\|SU(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|\big)\|\mathcal{T}(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|}{1+\|SU(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|^2+\|\mathcal{T}(\partial,Y(\partial))-SU(\partial,Y(\partial))\|^3+\|\mathcal{T}(\partial,X(\partial))-SU(\partial,Y(\partial))\|^4}\\
&+\sigma\big(\|SU(\partial,X(\partial))-\mathcal{T}(\partial,X(\partial))\|+\|SU(\partial,Y(\partial))-\mathcal{T}(\partial,Y(\partial))\|\big)\\
&+\psi\big(\|SU(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|+\|SU(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|\big)\\
&+\eta\,\|SU(\partial,X(\partial))-SU(\partial,Y(\partial))\|\\
&+\tau_1\min\{\|SU(\partial,X(\partial))-\mathcal{T}(\partial,X(\partial))\|,\ \|SU(\partial,X(\partial))-\mathcal{T}(\partial,Y(\partial))\|\}\\
&+\tau_2\min\{\|SU(\partial,Y(\partial))-\mathcal{T}(\partial,X(\partial))\|,\ \|SU(\partial,Y(\partial))-\mathcal{T}(\partial,Y(\partial))\|\}.
\end{aligned}
$$

For every $\partial \in \Theta$,
$\xi, \zeta, \sigma, \psi, \eta, \tau_1, \tau_2 \geq 0$ *and*

$$2\xi + \zeta + 4\sigma + 2\psi + \eta + \tau_1 + \tau_2 < 2$$

then $\mathcal{T}$, $\mathcal{S}$ and U has at least one R_{Fp}.
Furthermore, if

$$\zeta + 2\psi + \eta < 1,$$

then $\mathcal{T}$, $\mathcal{S}$ and U has a unique R_{Fp}.

Example 3.1. *Let $\mathcal{X} \subset \mathbb{R}$ be a Banach space and let Φ be a probability space. Define a R_O $\mathcal{T} : \Theta \times \mathcal{X} \to \mathcal{X}$ by*

$$\mathcal{T}(\theta, X) = k(\theta)X,$$

where $|k(\theta)| \leq c < 1$ for all $\theta \in \Theta$.

Then, for any $X, Y \in \mathcal{X}$,
we have

$$\|\mathcal{T}(\theta, X) - \mathcal{T}(\theta, Y)\| = \|k(\theta)X - k(\theta)Y\| = |k(\theta)| \, \|X - Y\|.$$

Since $|k(\theta)| \leq c$, it follows that

$$\|\mathcal{T}(\theta, X) - \mathcal{T}(\theta, Y)\| \leq c \, \|X - Y\|.$$

Now, comparing this with the inequality given in Theorem 3.1, we choose the constants as follows:

$$\eta = c, \quad \xi = \zeta = \sigma = \psi = \tau_1 = \tau_2 = 0.$$

Substituting these values into the existence condition

$$2\xi + \zeta + 4\sigma + 2\psi + \eta + \tau_1 + \tau_2 < 2,$$

we obtain

$$c < 2,$$

which holds since $c < 1$.

For the uniqueness condition,

$$\zeta + 2\psi + \eta < 1,$$

we get

$$c < 1,$$

which is also satisfied since $c < 1$.

As a result, every requirement of Theorem 3.1 is met. This means that the operator $\mathcal{T}$ has a distinct R_{Fp}.

CONCLUSION

The Present manuscript has explored new developments in R_{Fp} theorem by extending the classical deterministic result. Building on the involution property and generalized contractive inequalities with multiple control constant parameters, we have proved existence and uniqueness theorems for R_{Fp} using R_O on Banach spaces associated with complete probability spaces. By introducing nonlinear distance terms, fractional expressions, and minimum-type conditions into the contractive framework, we have broadened the scope of F_P theory to include classes of random operators that cannot be treated by standard contraction principles.

Furthermore, the results can be extended to commuting families of operators, demonstrating the existence of unique common random fixed points for pairs and triples of operators under the proposed conditions. These findings enhance the theoretical foundation of stochastic functional analysis stochastic differential equations, and optimization problems under uncertainty.

The work presented here unifies and generalizes existing results in both deterministic and random fixed point theory. Its significance lies not only in enriching the mathematical theory but also in its potential applications to problems in engineering, economics, biological systems, and physical sciences where randomness plays a central role. Future research can build upon these results by considering Hilbert spaces, uniformly convex Banach spaces, or neutrosophic fuzzy metric spaces, thus incorporating additional layers of uncertainty and indeterminacy.

In conclusion, this thesis has contributed to advancing random fixed point theory through generalized contractive conditions and extensions to commuting two or more then two operators, offering a versatile and robust framework for addressing nonlinear problems in functional analysis.

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Conflicts of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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