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A NOTE ON MONOTONE BI-DERIVATIONS ON LATTICES

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ABSTRACT. The primary aim of this research is to delve deeply into the intriguing properties of symmetric bi-derivations on lattices, unveiling their complexities and potentials with unprecedented insight and clarity. Moreover, we characterize monotone bi-derivations and obtain some results in the context of the kernel of bi-derivations.

Key words and phrases: Lattice; distributive lattice; symmetric bi-derivation; monotone.

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1. INTRODUCTION

Since their introduction, binary operations have played a central role in the study of groups, monoids, semigroups, rings, and many other algebraic structures within abstract algebra [4]. Lattice theory, along with its numerous applications, now makes extensive use of binary operations [1, 5]. These operations can be employed to characterize a wide range of notions and properties, including the very notion of a lattice [2]. It is therefore natural that many theoretical and practical areas rely on binary operations with particular features. In particular, aggregation functions on bounded lattices (viewed as binary operations satisfying specific conditions) are widely used across various domains of applied science, including access control, information retrieval, cryptanalysis, and information theory. In recent years, significant work has focused on the properties of lattices [12, 11]. As generalizations of the fundamental relationships between fuzzy sets, these structures are also crucial in the development of fuzzy set and fuzzy logic theories [3].

An important topic of study in the theory of rings and other algebraic structures is the characteristics of derivations. The idea of derivation has several applications and was first presented mainly in connection with ring structures [6, 7]. The author in [14] has boldly expanded this intriguing concept to encompass lattice structures, ingeniously utilizing the fundamental operations of meeting and joining. Then, using a specious viewpoint, it is applied to lattices, where the operations $+$ and $\times$ are read as lattice operations $\vee$ and $\wedge$, respectively. Concurrently, the idea of ring theory derivation was used to BCI algebras, left (right) derivations, f-derivations, (f, g)-derivations, bi-derivations, and n-derivations on lattices. Check out [8, 9, 10, 12] and the references therein for some impressive results.

Definition 1.1. [5] Consider a scenario where L represents a robust, nonempty set thoroughly endowed with two operations, $\vee$ and $\wedge$. When we speak of a lattice $(L, \wedge, \vee)$, we are delving into the fascinating realm of set L, which unfolds its full potential by meticulously adhering to a suite of compelling conditions as follows:

(1) $\varsigma \wedge \varsigma = \varsigma, \varsigma \vee \varsigma = \varsigma$;
(2) $\varsigma \wedge \zeta = \zeta \wedge \varsigma, \varsigma \vee \zeta = \zeta \vee \varsigma$;
(3) $(\varsigma \wedge \zeta) \wedge z = \varsigma \wedge (\zeta \wedge z), (\varsigma \vee \zeta) \vee z = \varsigma \vee (\zeta \vee z)$,
(4) $(\varsigma \wedge \zeta) \vee \varsigma = \varsigma, (\varsigma \vee \zeta) \wedge \varsigma = \varsigma$ for all $\varsigma, \zeta, z \in L$.

Definition 1.2. [5] A distributive lattice L is characterized by the fulfillment of the powerful identities (i) or (ii):
(1) $\varsigma \wedge (\zeta \vee z) = (\varsigma \wedge \zeta) \vee (\varsigma \wedge z)$,
(2) $\varsigma \vee (\zeta \wedge z) = (\varsigma \vee \zeta) \wedge (\varsigma \vee z)$.
Remarkably, within any lattice structure, these pivotal conditions (i) and (ii) are not merely satisfied in isolation; they are intrinsically and fundamentally equivalent.

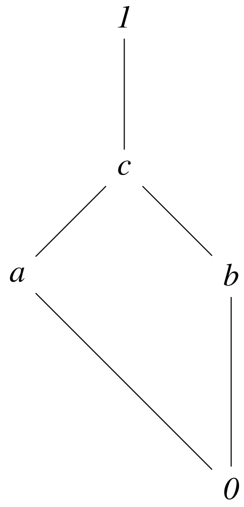
Definition 1.3. [1] A lattice L exhibits the distinguished property of being modular when it satisfies the pivotal identity as delineated in

$$\text{If } \varsigma \leq z,\ then \varsigma \vee (\zeta \wedge z) = (\varsigma \vee \zeta) \wedge z.$$

Definition 1.4. [11] A function $\mathfrak{d} : \mathfrak{L} \longrightarrow \mathfrak{L}$ is termed as a derivation on $\mathfrak{L}$ if it fulfills the ensuing condition for every $\varsigma, w, z \in \mathfrak{L}$, thereby inducing a transformative influence on

$$\mathfrak{d}(\varsigma \wedge w) = (\mathfrak{d}(\varsigma) \wedge w) \vee (\varsigma \wedge \mathfrak{d}(w)).$$

Example 1.1. *Let $\Lambda = \{0, a, b, c, 1\}$ be a lattice describe as below. It is easily seen that a mapping $d : \Lambda \longrightarrow \Lambda$ will be a derivation.*



Xin et al. [16] unveiled the concept of derivation within the framework of lattices and delved into an array of associated characteristics. They thoroughly charted the complexities of modular and distributive lattices via isotone derivations, offering a range of equivalent criteria under which a derivation becomes isotone for lattices with a greatest element, as well as for modular and distributive lattices. In addition, they clearly showed that, in the setting of distributive lattices, both Λ and $D(\Lambda)$ are strongly isomorphic to Λ, and they further proved that the fixed set $Fix_d(\Lambda)$ forms an ideal of $D(\Lambda)$, provided that d is an isotone derivation on a lattice Λ.

Researchers have previously examined several pertinent aspects related to the generalization of derivations on lattices. In particular, isotone derivations have been applied to characterize distributive and modular lattices. Our work is largely motivated by the investigations in [6, 8, 9, 10, 11]. Building on this literature, we establish a number of results on derivations to obtain further intriguing properties of derivations on lattices. Their important and distinctive characteristics are also addressed. This study investigates the properties of lattice derivations from a purely theoretical standpoint.

2. OUTCOMES ON SYMMETRIC BIDERIVATIONS

We commence by stating the following lemma:

Lemma 2.1. [5] *Consider $(L, \wedge, \vee)$, as a lattice imbued with a well-defined binary relation depicted as $\leq$. Consequently, $(L, \leq)$ emerges as a partially ordered set (poset), a structured framework where, for any elements $\varsigma, \zeta \in L$, $\varsigma \wedge \zeta$, $\{\varsigma, \zeta\}$ distinctly represents the greatest lower bound (g.l.b.), and remarkably, $\varsigma \vee \zeta$ stands as the least upper bound (l.u.b.) of $\{\varsigma, \zeta\}$.*

Theorem 2.2. *If ϖ is a bi-derivation, then*

(1) $\varpi(\varsigma, \varsigma) \leq \varsigma$.
(2) $\varpi(\varpi(\varsigma, \varsigma)) = \varpi(\varsigma, \varsigma)$.

Proof. (1)

$$\begin{aligned}\varpi(\varsigma,\varsigma) &= \varpi(\varsigma\wedge\varsigma,\varsigma)\\ &= (\varpi(\varsigma,\varsigma)\wedge\varsigma)\vee(\varsigma\wedge\varpi(\varsigma,\varsigma))\\ &= \varpi(\varsigma,\varsigma)\wedge\varsigma\end{aligned}$$

Thus $\varpi(\varsigma,\varsigma)\leq\varsigma$.

(2)

$$\begin{aligned}\varpi(\varpi(\varsigma,\varsigma)) =&\varpi(\varsigma\wedge\varpi(\varsigma,\varsigma),\varsigma\wedge\varpi(\varsigma,\varsigma))\\ =&(\varpi(\varsigma,\varsigma\wedge\varpi(\varsigma,\varsigma))\wedge\varpi(\varsigma,\varsigma))\vee(\varsigma\wedge\varpi(\varpi(\varsigma,\varsigma),\varsigma\wedge\varpi(\varsigma,\varsigma)))\\ =&(\varpi(\varsigma,\varsigma)\wedge\varpi(\varsigma,\varsigma))\vee(\varsigma\wedge\varpi(\varsigma,\varsigma)\wedge\varpi(\varsigma,\varpi(\varsigma,\varsigma)))\\ &\vee(\varsigma\wedge\varpi(\varpi(\varsigma,\varsigma),\varpi(\varsigma,\varsigma)))\\ =&\varpi(\varsigma,\varsigma)\vee\varpi(\varsigma,\varsigma)\vee\varpi(\varpi(\varsigma,\varsigma))\\ =&\varpi(\varsigma,\varsigma)\end{aligned}$$

Hence $\varpi(\varpi(\varsigma,\varsigma))=\varpi(\varsigma,\varsigma)$. ∎

Theorem 2.3. *If ϖ is a monotone, then*

(1) $\varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta)=\varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta)$.
(2) $\varpi(\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta))=\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta)$.

Proof. (1) Since $\varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta)\leq\varpi(\varsigma,\varsigma),\varpi(\zeta,\zeta)$, then we obtain

$$\begin{aligned}\varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta) &\leq \varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta)\\ &\leq \varsigma\wedge\zeta.\end{aligned}$$

Thus

$$\begin{aligned}\varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta) &= \varpi(\varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta))\\ &\leq \varpi(\varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta))\\ &\leq \varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta)\end{aligned}$$

This implies that

$$\begin{aligned}\varpi(\varsigma\wedge\zeta,\varsigma\wedge\zeta) &= \varpi(\varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta))\\ &= (\varpi(\varpi(\varsigma,\varsigma))\wedge\varpi(\zeta,\zeta))\vee(\varpi(\varsigma,\varsigma)\wedge\varpi(\varpi(\zeta,\zeta)))\\ &= (\varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta))\vee(\varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta))\\ &= \varpi(\varsigma,\varsigma)\wedge\varpi(\zeta,\zeta)\end{aligned}$$

(2) Since $\varpi(\varsigma,\varsigma),\varpi(\zeta,\zeta)\leq\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta)$, then

$$\varpi(\varpi(\varsigma,\varsigma),\varpi(\zeta,\zeta))\leq\varpi(\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta)).$$

That $\varpi(\varpi(\varsigma,\varsigma))=\varpi(\varsigma,\varsigma)$ and $\varpi(\varpi(\zeta,\zeta))=\varpi(\zeta,\zeta)$, implies

$$\varpi(\varsigma,\varsigma),\varpi(\zeta,\zeta)\leq\varpi(\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta)),$$

so that, $\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta)\leq\varpi(\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta))$. Note that

$$\varpi(\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta))\leq\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta)$$

Therefore,

$$\varpi(\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta))=\varpi(\varsigma,\varsigma)\vee\varpi(\zeta,\zeta) \quad ∎$$

Proposition 2.4. *If $\varpi(u,u)\leq\varpi(v,v)$ and $\varpi(w,w)\leq\varpi(s,s)$, then*

(1) $\varpi(u,u) \vee \varpi(w,w) \leq \varpi(v,v) \vee \varpi(s,s)$.
(2) $\varpi(u,u) \wedge \varpi(w,w) \leq \varpi(v,v) \wedge \varpi(s,s)$.

Proof. (1)

$$\begin{aligned}\varpi(u,u) \vee \varpi(w,w) &\leq \varpi(v,v) \vee \varpi(w,w)\\ &\leq \varpi(v,v) \vee \varpi(s,s)\end{aligned}$$

(2)

$$\begin{aligned}\varpi(u,u) \wedge \varpi(w,w) &\leq \varpi(v,v) \wedge \varpi(w,w)\\ &\leq \varpi(v,v) \wedge \varpi(s,s)\end{aligned}$$ ∎

Theorem 2.5. *If $\varpi(\varpi(\varsigma,\varsigma) \vee \zeta) = \varpi(\varsigma,\varsigma) \vee \varpi(\zeta,\zeta)$, then ϖ is monotone.*

Proof. Suppose that $\varsigma \leq \zeta$. Then $\varpi(\varsigma,\varsigma) \leq \varsigma \leq \zeta$. Thus

$$\begin{aligned}\varpi(\zeta,\zeta) &= \varpi(\varpi(\varsigma,\varsigma) \vee \zeta)\\ &= \varpi(\varsigma,\varsigma) \vee \varpi(\zeta,\zeta).\end{aligned}$$

This implies that $\varpi(\varsigma,\varsigma) \leq \varpi(\zeta,\zeta)$ and hence ϖ is monotone. ∎

Theorem 2.6. *If ϖ is a bi-derivation, then:*
(a) *If $\varpi(x,x) \vee \varpi(\zeta,\zeta) \leq \varpi(x \vee \zeta, x \vee \zeta)$, then ϖ is a monotone.*
(b) *If $\varpi(x,x) \wedge \varpi(\zeta,\zeta) = \varpi(x \wedge \zeta, x \wedge \zeta)$, then ϖ is a monotone.*

Proof. (a) Let $x \leq \zeta$. By assumption, $\varpi(x,x) \vee \varpi(\zeta,\zeta) \leq \varpi(x \vee \zeta, x \vee \zeta)$. Now, we have

$$\begin{aligned}\varpi(x,x) &\leq \varpi(x,x) \vee \varpi(\zeta,\zeta)\\ &\leq \varpi(x \vee \zeta, x \vee \zeta)\\ &= \varpi(\zeta,\zeta).\end{aligned}$$

Hence ϖ is a monotone.
(b) Let $x \leq \zeta$. By assumption, $\varpi(x,x) \wedge \varpi(\zeta,\zeta) = \varpi(x \wedge \zeta, x \wedge \zeta)$. Note that

$$\begin{aligned}\varpi(x,x) &= \varpi(x \wedge \zeta, x \wedge \zeta)\\ &= \varpi(x,x) \wedge \varpi(\zeta,\zeta)\end{aligned}$$

Therefore, $\varpi(x,x) \leq \varpi(\zeta,\zeta)$ and ϖ is a monotone. ∎

Finally, we examine the kernel of a symmetric biderivation D, which is given by the set

$$\mathfrak{K}_r(D) = \{k \in D \mid D(k,0) = 0\}.$$

Proposition 2.7. *Given a lattice $\mathfrak{L}$ having least element 0. Then $k \wedge w \in \mathfrak{K}_r(D)$ for every $k, w \in \mathfrak{K}_r(D)$.*

Proof. We are given that $k, w \in \mathfrak{L}$. By definition of $\mathfrak{K}_r(D)$, we obtain $D(k,0) = \{0\},\ D(w,0) = \{0\}$. Assume that

$$\begin{aligned}D(k \wedge w, 0) &= (D(k,0) \wedge w) \vee (k \wedge D(w,0))\\ &= (0 \wedge w) \vee (k \wedge 0)\\ &= 0 \vee 0\\ &= 0.\end{aligned}$$

Hence $k \wedge w \in \mathfrak{K}_r(D)$ for every $k, w \in \mathfrak{K}_r(D)$. ∎

Proposition 2.8. *Given a lattice $\mathfrak{L}$ having least element 0. If $k \leq w$ and $w \in \mathfrak{K}_r(D)$, then $k \in \mathfrak{K}_r(D)$.*

Proof. We are given that $w \in \mathfrak{L}$, by definition of $\mathfrak{K}_r(D)$, we obtain $D(w,0) = \{0\}$. Assume that $k \leq w$, it follows that

$$D(k,0) \leq D(w,0) = 0, \text{ that is, } D(k,0) = \{0\}.$$

Consider

$$\begin{aligned} D(k,0) &= D(k \wedge w, 0) \\ &= (D(k,0) \wedge w) \vee (k \wedge D(w,0)) \\ &= (0 \wedge w) \vee (k \wedge 0) \\ &= 0 \vee 0 \\ &= 0. \end{aligned}$$

We conclude $k \in \mathfrak{K}_r(D)$ as $D(k,0) = \{0\}$. ∎

3. CONCLUSION

The principal features and important properties addressed in this study are examined. From a purely theoretical standpoint, attention is given to a method for analyzing the characteristics of lattice derivations. At the same time, this notion opens up compelling questions for future investigation, particularly regarding its use in related disciplines such as computer science, combinatorics, discrete mathematics, and generalized algebraic structures.

Conflict of interest

The authors declare no conflicts of interest regarding this publication, ensuring the integrity and transparency of the findings.

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