



---

## GLOBAL EXISTENCE AND UNIQUENESS OF SOLUTION TO A NONLINEAR PARABOLIC EQUATION

ALEXANDER G. RAMM

*Received 17 November, 2025; accepted 20 March, 2026; published 22 April, 2026.*

MATHEMATICS DEPARTMENT, KANSAS STATE UNIVERSITY,, MANHATTAN, KS 66506-2602,, USA.  
ramm@ksu.edu

ABSTRACT. Consider the equation

$$u'(t) - \Delta u = |u|^\rho, \quad u(x, 0) = u_0(x), \quad (1),$$

where  $u' := \frac{du}{dt}$ ,  $\rho = \text{const} > 0$ ,  $x \in \mathbb{R}^3$ ,  $t > 0$ .

Assume that  $u_0$  is a smooth and rapidly decaying function belonging to a Banach space  $X$ .

It is proved that problem (1) has a global solution and this solution  $u \in X$ .

*Key words and phrases:* Nonlinear parabolic equations; Global solutions.

*2010 Mathematics Subject Classification.* 35K55.

## 1. INTRODUCTION

Let

$$(1.1) \quad u' - \Delta u = |u|^\rho, \quad u(x, 0) = u_0; \quad u' := \frac{du}{dt},$$

where  $\rho > 0$ ,  $t \in \mathbb{R}_+ = [0, \infty)$ ,  $x \in \mathbb{R}^3$ ,  $X$  is a Banach space of real-valued functions with the norm chosen so that  $Q$  is compact in  $X$ .

We prove the existence of a solution  $u \in X$  to (1.1):

$$(1.2) \quad \|u\| \leq c.$$

We say that  $u$  is a global solution to (1.1) if  $u$  exists  $\forall t \geq 0$ .

Our result is formulated in Theorem 1. Our method is simple and differs from the published results, see [2], [3] and references there.

The novel points in this work are:

a) There is no restriction on the upper bound of  $\rho$ .

In [2], (Section 1.1) a nonlinear hyperbolic equation with the same non-linearity is studied in a bounded domain, uniqueness of the solution is proved only for  $\rho \leq 2/(n-2)$ , and existence is proved by a different method. The contraction mapping theorem is not used.

In [3] the quasi-linear problems for parabolic equations are studied in Chapter 5 in a bounded domain and under the assumptions different from ours. There are many papers and books on non-linear problems for parabolic equations (see the bibliography in [2], [3]).

b) Existence of the global solution is proved.

c) Method of the proof differs from the methods in the cited literature.

Our result is formulated in Theorem 1:

**Theorem 1.** *Problem (1.1) has a global and uniformly bounded solution in  $X$  for smooth and rapidly decaying positive  $u_0 \in X$ , provided that  $\rho > 1$ .*

## 2. PROOFS

Let  $g(x, t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{3/2}}$ . If  $u$  solves (1.1) then

$$(2.1) \quad \begin{aligned} u(t) &= \int_0^t d\tau \int g(x-y, t-\tau) |u|^\rho dy + \\ &\int g(x-y, t) u_0(y) dy := A(u) + F := Q(u), \end{aligned}$$

where  $\int := \int_{\mathbb{R}^3}$ . Let  $X$  be the Banach space of continuous in  $\mathbb{R}^3 \times \mathbb{R}_+$  functions,  $\mathbb{R}_+ := [0, \infty)$ ,  $\|u\|$  is chosen so that  $Q$  is compact in  $X$ , for example,  $\|u\| = \max_{x \in \mathbb{R}^3, t \in [0, T]} |u(x, t)|$ .

If  $\|u\| \leq R$  then  $\|A(u)\| \leq TR^{\rho+1}$ , where the identity  $\int g(x-y, t-\tau) dy = 1$  was used. From (2.1) one gets

$$(2.2) \quad \|u\| \leq T\|u\|^\rho + \|F\|.$$

Thus,  $Q$  maps the ball  $B(R) = \{u : \|u\| \leq R\}$  into itself if  $T$  is such that

$$(2.3) \quad TR^\rho + \|F\| \leq R.$$

The  $Q$  is a contraction on  $B(R)$  if

$$\|Q(u) - Q(v)\| \leq T\rho R^{\rho-1} \|u - v\| \leq q \|u - v\|, \quad 0 < q < 1.$$

Thus, if

$$(2.4) \quad T\rho R^{\rho-1} \leq q < 1,$$

then  $Q$  is a contraction in  $B(R)$  in the Banach space  $X_T$  with the norm  $\|\cdot\|$ ,  $t \in [0, T]$ . We use the same notations for the norms in  $X_T$  and in  $X_\infty$ .

We have proved that

*For  $T$  satisfying (2.3)- (2.4) there exists and is unique the solution to (1.1), and this solution can be obtained from (2.1) by iterations.*

The problem now is:

*Does this solution exist and is unique on  $R_+$ ?*

From our proof it follows that if the solution exists and is unique in  $X_T$ , then the solution exists and is unique in  $X_{T_1}$  for some  $T_1 > T$  because the estimates (2.3) and (2.4) remain valid for  $T_1 > T$  if  $T_1 - T$  is sufficiently small.

To prove that the solution  $u(x, t)$  to (1.1) exists on  $\mathbb{R}_+$ , assume the contrary: this solution does not exist on any interval  $[0, T_1]$ ,  $T_1 > T$ , where  $T$  is the maximal interval of the existence of the continuous solution.

Then  $\lim_{t \rightarrow T-0} u(x, t) = \infty$ , because otherwise there is a sequence  $t_n \rightarrow T - 0$  such that  $u(x, t_n) \rightarrow u(x, T)$  and one may construct the solution defined on  $[T, T_1]$ ,  $T_1 > T$ , by using the local existence and uniqueness of the solution to (1.1) with the initial value  $u(x, T)$  for  $t \in [T, T_1]$ . This contradicts the assumption that  $T$  is the maximal interval of the existence of the continuous solution  $u$ . Of course, if  $T$  is such that the contraction mapping theorem is not valid, the solution to (1.1) may not exist or may be non-unique.

If  $T < \infty$  then one has  $\lim_{t \rightarrow T-0} u(x, t) = \infty$ . Let us prove that this also leads to a contradiction. Then we have to conclude that  $T = \infty$  and Theorem 1 is proved.

We need the following result (see [1], p. 61):

Lemma 1. (Leray-Schauder). Let  $A$  be a compact nonlinear operator in a Banach space  $X$ . If all solutions to the equation

$$(2.5) \quad u = \eta A(u)$$

for all  $\eta \in [0, 1]$  are uniformly bounded:  $\|u\| \leq R$ , then equation (2.5) has a solution in the ball of radius  $R$ .

If  $u_0 \geq 0$  then formula (2.1) implies

$$(2.6) \quad |u(x, t)| \geq \int_0^t \int_D g(x - y, t - s) |u(y, s)|^\rho dy ds$$

since the second term on the right side of formula (2.2) is nonnegative. Let  $\xi$  and  $\tau$  be points such that

$$(2.7) \quad |u(\xi, \tau)| > \max |u(x, t)| - \epsilon := M - \epsilon,$$

where  $\epsilon > 0$  is a small number. Let  $D_\epsilon \subset D$  be a subdomain of  $D$  where inequality (2.7) holds. Then (2.6) implies:

$$(2.8) \quad M > (M - \epsilon)^\rho \tau.$$

If  $M \rightarrow \infty$  and  $\rho > 1$  then (2.8) leads to a contradiction. This is so also if  $T \rightarrow \infty$ . Therefore,  $\max_{x \in \mathbb{R}^3, T \in \mathbb{R}_+} |u(x, t)| \leq C$  where  $C > 0$  is a constant.

Theorem 1 is proved. □

The ideas related to the ones used in this paper were developed and used in [4], [5], [7].

### REFERENCES

- [1] K. DEIMLING, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
- [2] J. LIONS, *Quelques methods de resolution des problemes aux limites non lineaires*, Dunod, Paris, 1969.

- [3] O. LADYZHENSKAYA, et al., Linear and quasilinear equations of parabolic type, Transl. of math. monogr., vol. 23, *Amer. Math. Soc.*, Providence RI, 1968.
- [4] A. G. RAMM, Stability of the solutions to evolution problems, *Mathematics*, 1, (2013), pp. 46-64. doi:10.3390/math1020046 <http://www.mdpi.com/journal/mathematics>
- [5] A. G. RAMM, *Large-time behavior of solutions to evolution equations*, *Handbook of Applications of Chaos Theory*, Chapman and Hall/CRC, 2016, pp. 183-200 (ed. C.Skiadas).
- [6] A. G. RAMM, N. S. HOANG, *Dynamical Systems Method and Applications. Theoretical Developments and Numerical Examples*. Wiley, Hoboken, 2012.
- [7] A. G. RAMM, Nonlinear elliptic problems, *Austr. J. Math. Anal.*, 22, N1, (2025) Art. 11.