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ALTERNATIVE PROOFS FOR THE POSITIVITY OF A FUNCTION INVOLVING THE PRODUCT OF TWO DIGAMMA FUNCTIONS

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ABSTRACT. Let $\psi(x)$ be the digamma function, that is, the logarithmic derivative $[\ln\Gamma(x)]' = \frac{\Gamma'(z)}{\Gamma(x)}$ of the classical Euler's gamma function $\Gamma(x)$. In this paper, the authors alternatively prove the positivity of the function

$$\psi(x) + x\psi'(x) - \psi(x)\psi\left(\frac{1}{x}\right), \quad x \in (0, \infty).$$

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1. INTRODUCTION

It is common knowledge [1, Chapter 6] that the functions

$$\Gamma(z)=\int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}\,t$$

and

$$\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$$

for $\Re(z)>0$ are respectively called the classical Euler gamma function and the digamma function. Furthermore, the derivatives $\psi'(z)$, $\psi''(z)$, $\psi^{(3)}(z)$, and $\psi^{(4)}(z)$ are called the trigamma, tetragamma, pentagamma, and hexagamma functions, respectively. All the derivatives $\psi^{(n)}(z)$ for $n\ge 0$ are known as the polygamma functions.

Let

$$f(x)=\psi(x)+x\psi'(x)-P_0(x),\quad x\in(0,\infty), \tag{1.1}$$

where

$$P_0(x)=\psi(x)\psi\biggl(\frac{1}{x}\biggr),\quad x\in(0,\infty). \tag{1.2}$$

In the paper [7], among other things, by establishing the monotonicity of the function $P_0(x)$, Qi and his coauthors confirmed the positivity of the function $f(x)$ on $(0,\infty)$.

Theorem 1 ([7, Theorem 2])**.** *The product* $P_0(x)$ *defined by* (1.2) *is increasing on* $(0,1)$ *and decreasing on* $(1,\infty)$*, with the limits*

$$\lim_{x\to0^+}P_0(x)=\lim_{x\to\infty}P_0(x)=-\infty.$$

Theorem 2 ([7, Theorem 3])**.** *The function* $f(x)$ *defined in* (1.1) *is positive on* $(0,\infty)$.

In this paper, we provide two alternative proofs of Theorem 2.

2. TWO ALTERNATIVE PROOFS OF THEOREM 2

In this section, we aim to provide two alternative proofs of Theorem 2.

2.1. **First alternative proof of Theorem 2.** From the monotonicity of the function $P_0(x)$, restated in Theorem 1, we deduce the inequality

$$P_0(x)=\psi(x)\psi\biggl(\frac{1}{x}\biggr)\le\gamma^2=P_0(1), \tag{2.1}$$

where $\gamma=0.57721566\dots$ stands for Euler–Mascheroni's constant. This inequality can also be found in [3, Proposition 4], in which the inequality (2.1) was proved using the power series for $\psi(1+x)$.

Since $\psi(x)$ is concave, with $\psi(1)=-\gamma$ and $\psi(2)=1-\gamma$, we acquire $\psi(y)\ge y-1-\gamma$ for $1<y<2$. Hence, for $0<x<1$,

$$\psi(x)+\frac{1}{x}=\psi(x+1)\ge x-\gamma.$$

For $x>0$, we have

$$\psi'(x)=\frac{1}{x^2}+\sum_{n=1}^\infty\frac{1}{(x+n)^2}>\frac{1}{x^2}+\int_1^\infty\frac{1}{(x+t)^2}\,\mathrm{d}\,t=\frac{1}{x^2}+\frac{1}{1+x}.$$

Accordingly,

$$x\psi'(x) \ge \frac{1}{x} + \frac{x}{x+1}. \tag{2.2}$$

Hence, for $0 < x < 1$,

$$\psi(x) + x\psi'(x) \ge x + \frac{x}{x+1} - \gamma. \tag{2.3}$$

When $0 < x \le \frac{1}{2}$, by virtue of (2.3), we obtain

$$\psi(x) + x\psi'(x) > -\gamma,$$

while

$$|P_0(x)| \ge |P_0(2)| = 0.8302\dots .$$

So, we find $P_0(x) \le -0.8302\dots$. Hence, we deduce $f(x) \ge 0.832\dots - \gamma > 0$.

Let $x_0 = 1.4616321\dots$ be the unique positive zero of the digamma function $\psi(x)$ on $(0,\infty)$. When $\frac{1}{2} \le x \le \frac{1}{x_0}$, it is easy to see that

$$\psi(x) + x\psi'(x) \ge g\left(\frac{1}{2}\right) = \frac{5}{6} - \gamma > 0$$

and $P_0(x) \le 0$.

When $\frac{1}{x_0} < x \le 1$, since $\frac{1}{x_0} > \frac{2}{3}$, it follows that $g(x) > g\left(\frac{2}{3}\right) > 1 - \gamma$. Making use of the inequality (2.1), we arrive at $f(x) > 1 - \gamma - \gamma^2 = 0.089\dots$.

When $1 \le x \le x_0$, employing $\psi(x) \ge -\gamma$ and $x\psi'(x) > 1$, which can be deduced from (2.2), we conclude $f(x) > 1 - \gamma - \gamma^2$ again. The first proof of Theorem 2 is thus complete.

2.2. **Second alternative proof of Theorem 2.** The positivity of the function $f(x)$ for $x > 0$ can be rewritten as $[x\psi(x)]' > P_0(x)$ for $x > 0$.

In [2, Lemma 1], Alzer proved that the first derivative $[x\psi(x)]'$ is increasing on $(0,\infty)$, with the limits

$$\lim_{x\to 0^+}[x\psi(x)]' = -\gamma \quad \text{and} \quad \lim_{x\to\infty}[x\psi(x)]' = \infty,$$

and that the only positive zero of $[x\psi(x)]'$ is $x_1 = 0.21609\dots$.

Since

$$\begin{aligned}
[x\psi(x)]'|_{x=1} &= [\psi(x) + x\psi'(x)]|_{x=1} = \psi(1) + \psi'(1) \\
&= -\gamma + \frac{\pi^2}{6} = 1.067718\dots > P_0(1) = [\psi(1)]^2 = \gamma^2 = 0.333\dots
\end{aligned}$$

and, by Theorem 1, the function $P_0(x)$ is decreasing on $(1,\infty)$, we are readily sure that the function $f(x)$ is positive on $[1,\infty)$.

For $x \in (0,1]$, the positivity of $f(x)$ can be rearranged as

$$x\psi'(x) + \psi(x)\left[1 - \psi\left(\frac{1}{x}\right)\right] > 0, \quad x \in (0,1]. \tag{2.4}$$

Let $x_2 = 0.312\dots$ be the zero of the equation $\psi\left(\frac{1}{x}\right) = 1$. When $x \in (0, x_2] \subset (0,1)$, the positivity of $f(x)$ is clearly true.

In [5, Lemma 2.4], there exists the double inequality

$$\ln\left(x + \frac{1}{2}\right) - \frac{1}{x} < \psi(x) < \ln(x+1) - \frac{1}{2x}, \quad x \in (0,\infty). \tag{2.5}$$

See also [4, Section 1.5]. In [4, p. 111, Theorem 1], the double inequality

$$\frac{(k-1)!}{\left(x+\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}\right)^k}+\frac{k!}{x^{k+1}}<\left|\psi^{(k)}(x)\right|<\frac{(k-1)!}{\left(x+\frac{1}{2}\right)^k}+\frac{k!}{x^{k+1}} \tag{2.6}$$

was proved for $x>0$ and $k\in\mathbb{N}$ and the constants $\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}$ and $\frac{1}{2}$ in (2.6) were proved to be the best possible. On the open interval $(x_2,1)$, making use of the left-hand inequalities in (2.5) and (2.6) for $k=1$, we can decrease the left-hand side in the inequality (2.4) as

$$\begin{aligned}
&x\psi'(x)+\psi(x)\left[1-\psi\left(\frac{1}{x}\right)\right]\\
&>x\left[\frac{1}{x+\frac{1}{|\psi'(1)|}}+\frac{1}{x^2}\right]+\left[\ln\left(x+\frac{1}{2}\right)-\frac{1}{x}\right]\left[1+x-\ln\left(\frac{1}{x}+\frac{1}{2}\right)\right]\\
&=\left[\frac{1}{x}-\ln\left(x+\frac{1}{2}\right)\right]\ln\left(\frac{1}{x}+\frac{1}{2}\right)+(x+1)\ln\left(x+\frac{1}{2}\right)-\frac{6}{\pi^2x+6}.
\end{aligned}$$

Therefore, it is sufficient to prove

$$\left[\frac{1}{x}-\ln\left(x+\frac{1}{2}\right)\right]\ln\left(\frac{1}{x}+\frac{1}{2}\right)>\frac{6}{\pi^2x+6}-(x+1)\ln\left(x+\frac{1}{2}\right)$$

for $x\in(x_2,1)$. Because

$$\ln(1+t)\ge\frac{2t}{2+t},\quad t\in(0,\infty),$$

see [6, p. 245, Remark 1, (3)], it suffices to show

$$\left[\frac{1}{x}-\ln\left(x+\frac{1}{2}\right)\right]\frac{2(2-x)}{3x+2}>\frac{6}{\pi^2x+6}-(x+1)\ln\left(x+\frac{1}{2}\right),$$

that is,

$$H(x)\triangleq\left(3x^2+7x-2\right)\ln\left(x+\frac{1}{2}\right)>\frac{2\left[\pi^2x^2-\left(2\pi^2-15\right)x-6\right]}{\pi^2x+6}\triangleq L(x), \tag{2.7}$$

for $x\in(x_2,1)$.

Since the quadratic equation $3x^2+7x-2=0$ has a unique positive root

$$\frac{\sqrt{73}-7}{6}=0.257\ldots<x_2=0.312\ldots$$

and the equation $\ln\left(x+\frac{1}{2}\right)=0$ has a unique root $\frac{1}{2}$, the function $H(x)$ is negative on $\left(\frac{\sqrt{73}-7}{6},\frac{1}{2}\right)$ and is positive on $\left(\frac{1}{2},1\right)$. It is easy to see that the quadratic equation

$$\pi^2x^2-\left(2\pi^2-15\right)x-6=0$$

has two roots

$$\frac{2\pi^2-15-\sqrt{225-36\pi^2+4\pi^4}}{2\pi^2}=-0.575\ldots$$

and

$$\frac{2\pi^2-15+\sqrt{225-36\pi^2+4\pi^4}}{2\pi^2}=1.055\ldots.$$

This means that the function $L(x)<0$ on $(0,1)$. Accordingly, the inequality (2.7) is trivially valid on $\left(\frac{1}{2},1\right)$.

It is standard that

$$\begin{aligned}
H'(x) &= \frac{2\big(3x^2+7x-2\big)}{2x+1} + (6x+7)\ln\left(x+\frac{1}{2}\right)\\
&\to \begin{cases} -4-7\ln 2, & x\to 0^+;\\ \dfrac{9}{4}, & x\to \dfrac{1}{2}, \end{cases}\\
H''(x) &= \frac{4\big(9x^2+13x+9\big)}{(2x+1)^2} + 6\ln\left(x+\frac{1}{2}\right)\\
&\to \begin{cases} 6(6-\ln 2), & x\to 0^+;\\ \dfrac{71}{4}, & x\to \dfrac{1}{2}, \end{cases}\\
H'''(x) &= \frac{16\big(3x^2+x-5\big)}{(2x+1)^3}\\
&< 0
\end{aligned}$$

on $\big(0,\frac{1}{2}\big]$. By Taylor's theorem, we obtain

$$\begin{aligned}
H(x) &= H\left(\frac{1}{2}\right) + H'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right) + \frac{H''\big(\frac{1}{2}\big)}{2!}\left(x-\frac{1}{2}\right)^2 + \frac{H'''(\xi(x))}{6}\left(x-\frac{1}{2}\right)^3\\
&> \frac{9}{4}\left(x-\frac{1}{2}\right) + \frac{71}{8}\left(x-\frac{1}{2}\right)^2 \qquad (2.8)\\
&\ge -\frac{81}{568}\\
&= -0.142\dots
\end{aligned}$$

for $x\in\big(0,\frac{1}{2}\big]$ and $\xi(x)\in\big(x,\frac{1}{2}\big)$.

It is straightforward that

$$L'(x) = \frac{2\big(\pi^4x^2+12\pi^2x+90-6\pi^2\big)}{(\pi^2x+6)^2}.$$

Since the quadratic equation

$$\pi^4x^2+12\pi^2x+90-6\pi^2=0$$

has two roots

$$-\frac{\sqrt{6(\pi^2-9)}+6}{\pi^2} = -0.839\dots$$

and

$$\frac{\sqrt{6(\pi^2-9)}-6}{\pi^2} = -0.376\dots,$$

the derivative $L'(x)$ is positive on $[0,\infty)$, and the function $L(x)$ is increasing on $[0,\infty)$. Hence, we acquire

$$L(x) \le L\left(\frac{1}{2}\right) = -\frac{3(\pi^2-2)}{12+\pi^2} = -1.079\dots, \quad x\in\left(0,\frac{1}{2}\right]. \tag{2.9}$$

Combining (2.8) with (2.9) concludes that the inequality (2.7) is valid for $x\in\big(x_2,\frac{1}{2}\big]$. The second proof of Theorem 2 is complete.

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] H. ALZER, A power mean inequality for the gamma function, *Monatsh. Math.*, **131** (2000), no. 3, 179–188. DOI: `https://doi.org/10.1007/s006050070007`.

[3] H. ALZER and G. JAMESON, A harmonic mean inequality for the digamma function and related results, *Rend. Semin. Mat. Univ. Padova*, **137** (2017), 203–209. DOI: `https://doi.org/10.4171/RSMUP/137-10`.

[4] B.-N. GUO, F. QI, J.-L. ZHAO, and Q.-M. LUO, Sharp inequalities for polygamma functions, *Math. Slovaca*, **65** (2015), no. 1, 103–120. DOI: `https://doi.org/10.1515/ms-2015-0010`.

[5] F. QI and B.-N. GUO, An inequality involving the gamma and digamma functions, *J. Appl. Anal.*, **22** (2016), no. 1, 49–54. DOI: `https://doi.org/10.1515/jaa-2016-0005`.

[6] F. QI, W. LI, and B.-N. GUO, Generalizations of a theorem of I. Schur, *Appl. Math. E-Notes*, **6** (2006), Art. 29, 244–250.

[7] F. QI, D. LIM, and K. NANTOMAH, Monotonicity and positivity of several functions involving ratios and products of polygamma functions, *J. Inequal. Appl.*, **2025**, Paper No. 5, 10 pages. DOI: `https://doi.org/10.1186/s13660-024-03245-8`.