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$\mathbb{R}$-ORDER ANALYSIS OF e-OPEN CONTINUOUS MAPPING IN CUBIC PICTURE FUZZY TOPOLOGICAL SPACES

G. SARAVANAKUMAR AND K. SUGANTHI

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DEPARTMENT OF MATHEMATICS, VEL TECH RANGARAJAN DR.SAGUNTHALA R&D INSTITUTE OF SCIENCE AND TECHNOLOGY (DEEMED TO BE UNIVERSITY), AVADI, CHENNAI-600062, INDIA.
`saravananguru2612@gmail.com`

DEPARTMENT OF MATHEMATICS, VEL TECH RANGARAJAN DR.SAGUNTHALA R&D INSTITUTE OF SCIENCE AND TECHNOLOGY (DEEMED TO BE UNIVERSITY), AVADI, CHENNAI-600062, INDIA.
`suganthiselvam0980@gmail.com`

ABSTRACT. This paper investigates the structure and properties of *e-continuous mappings* within the framework of Cubic Picture Fuzzy Topological Spaces ($CPFTSs$) under $\mathbb{R}$-order. Building on the foundational role of e-open sets in generalized topological structures, we introduce several classes of continuity-namely e-continuous, $\delta\mathcal{P}$-continuous,$\delta\mathcal{S}$-continuous a-continuous, β-continuous, and e^*-continuous maps—and study their interrelationships through logical implications and set-theoretic operations. We establish necessary and sufficient conditions for each type of continuity, along with non-reversible inclusion results supported by counterexamples. Furthermore, we present preservation results involving closure and interior operations under e-continuous maps and identify conditions under which e-continuity implies classical $\mathbb{R}$-order continuity. The findings contribute to the refinement of topological structures in fuzzy environments and offer a foundational basis for functional modeling under uncertainty, with applications in decision theory, AI systems, and fuzzy information modeling.

Key words and phrases: Cubic Picture Fuzzy Topological Spaces; e-Open Sets, e-Continuous Mapping; $\mathbb{R}$-Order.

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1. INTRODUCTION

The increasing complexity and uncertainty of real-world systems have necessitated the development of advanced mathematical frameworks capable of modeling vagueness, indeterminacy, and imprecision. Among these, fuzzy set theory—pioneered by Zadeh [12]—has been greatly extended through the introduction of intuitionistic fuzzy sets [2], picture fuzzy sets [3], and more recently, cubic picture fuzzy sets ($CPFSs$). $CPFSs$ enrich the modeling capacity by combining interval-valued structure with the three-valued logic of picture fuzzy sets, enabling simultaneous representation of positive, neutral, and negative evaluations. Ashraf et al. [1] introduced $CPFSs$ to better capture such nuanced evaluations in fuzzy environments.

Subsequent developments by Jun et al. [9] provided a solid algebraic and operational foundation for $CPFSs$, which has since led to numerous applications in decision-making, medical diagnostics, and more recently, intelligent supply chain systems and blockchain-integrated metaverse platforms [10]. Within these domains, *Cubic Picture Fuzzy Topological Spaces ($CPFTSs$)* equipped with *P-order* and *R-order* structures have become important tools for modeling spatial uncertainty using dual-ordering mechanisms.

A parallel advancement in fuzzy topology has been the study of generalized open sets, such as *e-open*, *a-open*, and *β-open sets*, originally developed by Ekici [4, 5, 6, 7, 8]. These generalized open sets refine classical notions of openness and continuity, enabling deeper topological distinctions across fuzzy and intuitionistic fuzzy contexts. However, most of the prior literature has focused solely on the structure of these sets, with relatively limited work examining *mappings* between fuzzy topological spaces that preserve or reflect such generalized openness properties—particularly in the context of $CPFTSs$.

Research Gap. While $CPFSs$ have been utilized to capture complex uncertainty in topological modeling, there exists a notable lack of research extending *e-continuous mappings*—a natural generalization of continuity based on e-open sets into the cubic picture fuzzy setting. Previous studies largely limited themselves to defining and analyzing the structure of open sets, without exploring how *functions* behave with respect to these generalized topologies, especially under the dual ordering present in $CPFTSs$.

Motivation and Novelty. This paper seeks to fill that gap by introducing and analyzing the concept of *e-continuous mappings* and their nearby variants such as *$\delta\mathcal{P}$-continuous*, *$\delta\mathcal{S}$-continuous* *a-continuous*, *β-continuous*, and *e^*-continuous maps* within the rich framework of $CPFTSs$ *under $\mathbb{R}$-order*. Unlike previous efforts that treated open sets in isolation, this work systematically studies the *inverse image behavior* of these sets under fuzzy mappings. We establish logical implications, equivalence chains, and non-reversible inclusions among these mappings and validate the results with illustrative counterexamples.

Objectives.

- To formally define *$\mathbb{R}$-order e-continuous mappings* and their generalizations in $CPFTS$.
- To explore their structural and logical relationships through set-theoretic and order-theoretic lenses.
- To develop new results regarding preservation of topological constructs under such mappings.
- To provide examples and counterexamples that demonstrate non-reversibility and subtleties in these continuity concepts.

Significance. By shifting the focus from merely defining generalized open sets to studying *e-continuous mappings* and their implications, this work provides a foundational theory for understanding functional behavior in $CPFTSs$ under generalized topologies. The findings not only contribute to the theoretical refinement of fuzzy topological spaces but also open new directions for modeling *functional uncertainty* in AI, control systems, metaverse design, and decision support systems—particularly where *neighborhood sensitivity* and *continuity preservation* are critical under complex multi-valued data structures.

2. PRELIMINARIES

Definition 1 ([12])**.** *A fuzzy set* λ *in a set* X *is defined to be a function* $\lambda : \mathrm{X} \to \mathrm{I}$ *where* $\mathrm{I} = [0, 1]$.

Definition 2 ([13])**.** *A closed sub interval* $\tilde{a} = [a^-,\ a^+]$ *where* $0 \leq a^- \leq a^+ \leq 1$ *of* $\mathrm{I} = [0,\ 1]$ *is called Interval number. Where,* $[I]$ *denote the set of all interval numbers.*

Definition 3 ([13])**.** *Let* X *be a nonempty set. A function* $\mu : X \to [I]$ *is called an interval-valued fuzzy set(briefly,* $IVFS$*) in* X*. Let* $[I]$ *stand for the set of all* $IVFS$*'s in* X*. For every* $\mu \in [I]^X$ *and* $x \in \mathrm{X}, \mu(x) = [\mu^-(x), \mu^+(x)]$ *is called the degree of membership of an element* x *to* μ*, where* $\mu^- : X \to I$ *and* $\mu^+ : X \to I$ *are fuzzy sets in* X *which are called lower fuzzy set and an upper fuzzy set in* X*, respectively.*

Definition 4 ([9])**.** *Let* X *be a non-empty set. Then a structure* $A = \{\langle x, \mu(x), \lambda(x)\rangle \mid x \in X\}$ *is a cubic set in* X *in which* μ *is an* $IVFS$ *in* X *and* λ *is a fuzzy set in* X*.*

Definition 5 ([3])**.** *Consider a non-empty set* X*. We define a picture fuzzy set(*PFS*) as follows* $J(x) = \{\langle(s, \alpha(x), \beta(x), \gamma(x)) \mid x \in X\rangle\}$*, where* $\alpha : X \to [0, 1], \beta : X \to [0, 1], \gamma : X \to [0, 1]$ *represent the positive membership degree, neutral membership degree and negative membership degree of the element* x *with respect to the set* X*, respectively. Additionally,* α, β *and* γ *must satisfy the condition* $0 \leq \alpha(x) + \beta(x) + \gamma(x) \leq 1$ *for all* $x \in X$*.*

Definition 6 ([3])**.** *Consider a non-empty set* X*. Let* $S(x)$ *denote an interval valued picture fuzzy set(*$IVPFS$*) defined in* X *as follows* $S(x) = \{\langle x, \alpha(x), \beta(x), \gamma(x) \mid x \in X\}$*, where* $\alpha : X \to [0, 1], \beta : X \to [0, 1], \gamma : X \to [0, 1]$ *represent the positive membership degree, neutral membership degree and negative membership degree of the element* x *with respect to the set* X*. Moreover* $\alpha(x), \beta(x)$ *and* $\gamma(x)$ *must satisfy the condition* $0 \leq sup(\alpha(x)) + sup(\beta(x)) + sup(\gamma(x)) \leq 1,$ *for all* $x \in X$*.*

Definition 7 ([1])**.** *Consider a non-empty set* X*. The concept of a cubic picture fuzzy set(*$CPFS$*) can be represented as follows* $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$*, where* $\alpha : X \to [0, 1], \beta : X \to [0, 1], \gamma : X \to [0, 1]$ *represent the positive membership degree, neutral membership degree and negative membership degree of every element with respect to the set* X*.*

Definition 8 ([1])**.** *Let us consider a universal set* X*. Then the* $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ *is classified as follows*

(i) Positive-internal if $\alpha_i(x) \leq \alpha(x) \leq \alpha_j(x), \forall x \in X$*.*
(ii) Neutral-internal if $\beta_i(x) \leq \beta(x) \leq \beta_j(x), \forall x \in X$*.*
(iii) Negative-internal if $\gamma_i(x) \leq \gamma(x) \leq \gamma_j(x), \forall x \in X$*.*

If the $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ *in* X *is satisfied all the above properties then the* $CPFS$ *is said to be an internal* $CPFS$ *in* X*.*

Definition 9 ([1])**.** *Let us consider a universal set* X*. Then the* $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ *is classified as follows*

(i) Positive-external if $\alpha(x) \notin [\alpha_i(x), \alpha_j(x)], \forall x \in X$
(ii) Neutral-external if $\beta(x) \notin [\beta_i(x), \beta_j(x)], \forall x \in X$
(iii) Negative-external if $\gamma(x) \notin [\gamma_i(x), \gamma_j(x)], \forall x \in X$

Definition 10 ([1]). *Let X be a set. Consider a $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ in X. Then its complement is defined as $A^c = \{\langle[\gamma_i, \gamma_j], [\beta_i, \beta_j], [\alpha_i, \alpha_j]\rangle, \langle\gamma, \beta, \alpha\rangle\}$*

Definition 11. [10] Let us consider a universal set X. Then the cubic picture fuzzy set $\hat{0} = \{\langle[0, 0], [0, 0], [1, 1]\rangle, \langle 1, 0, 0\rangle\}$ is called a null R-cubic picture fuzzy set.

Definition 12. [10] Let us consider a universal set X. Then the cubic picture fuzzy set $\hat{1} = \{\langle[1, 1], [0, 0], [0, 0]\rangle, \langle 0, 0, 1\rangle\}$ is called a absolute R-cubic picture fuzzy set.

Definition 13. [1] Let $A = \{\langle[\alpha_{i_1}, \alpha_{j_1}], [\beta_{i_1}, \beta_{j_1}], [\gamma_{i_1}, \gamma_{j_1}]\rangle, \langle\alpha_1, \beta_1, \gamma_1\rangle\}$ and $B = \{\langle[\alpha_{i_2}, \alpha_{j_2}], [\beta_{i_2}, \beta_{j_2}], [\gamma_{i_2}, \gamma_{j_2}]\rangle, \langle\alpha_2, \beta_2, \gamma_2\rangle\}$ be two cubic picture fuzzy sets. Then the operations on cubic picture fuzzy sets are defined as follows for R-order

(i) $A \subseteq_{\mathbb{R}} B$ iff $\forall x \in X,\ [\alpha_{i_1}, \alpha_{j_1}] \leq [\alpha_{i_2}, \alpha_{j_2}], [\beta_{i_1}, \beta_{j_1}] \geq [\beta_{i_2}, \beta_{j_2}], [\gamma_{i_1}, \gamma_{j_1}] \geq [\gamma_{i_2}, \gamma_{j_2}], \alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2, \gamma_1 \leq \gamma_2$

(ii) Union of two cubic picture fuzzy sets

$$A \cup_{\mathbb{R}} B = \left\{ \begin{array}{l} \{max\{\alpha_{i_1}, \alpha_{i_2}\}, min\{\beta_{i_1}, \beta_{i_2}\}, min\{\gamma_{i_1}, \gamma_{i_2}\}\} \\ \{max\{\alpha_{j_1}, \alpha_{j_2}\}, min\{\beta_{j_1}, \beta_{j_2}\}, min\{\gamma_{j_1}, \gamma_{j_2}\}\} \\ \{min\{\alpha_1, \alpha_2\}, min\{\beta_1, \beta_2\}, max\{\gamma_1, \gamma_2\}\} \end{array} \right\}$$

(iii) Intersection of two cubic picture fuzzy sets

$$A \cap_{\mathbb{R}} B = \left\{ \begin{array}{l} \{min\{\alpha_{i_1}, \alpha_{i_2}\}, min\{\beta_{i_1}, \beta_{i_2}\}, max\{\gamma_{i_1}, \gamma_{i_2}\}\} \\ \{min\{\alpha_{j_1}, \alpha_{j_2}\}, min\{\beta_{j_1}, \beta_{j_2}\}, max\{\gamma_{j_1}, \gamma_{j_2}\}\} \\ \{max\{\alpha_1, \alpha_2\}, min\{\beta_1, \beta_2\}, min\{\gamma_1, \gamma_2\}\} \end{array} \right\}$$

Definition 14. [10] A $\mathbb{R}$-cubic picture fuzzy topology is the family $\mathcal{F}_{\mathbb{R}}$ of cubic picture fuzzy sets in X which satisfies the following conditions:

(i) $\hat{0}, \hat{1} \in \mathcal{F}_{\mathbb{R}}$
(ii) If $A_{\mathrm{i}} \in \mathcal{F}_{\mathbb{R}}$, then $\cup_{\mathbb{R}}^{i \in N} A_i \in \mathcal{F}_{\mathbb{R}}$
(iii) If $\mathrm{A}, \mathrm{B} \in \mathcal{F}_{\mathbb{R}}$, then $\mathrm{A} \cap_{\mathbb{R}} \mathrm{B} \in \mathcal{F}_{\mathbb{R}}$

The pair $(X, \mathcal{F}_{\mathbb{R}})$ is called the $\mathbb{R}$-cubic picture fuzzy topological space and any cubic picture fuzzy set in $\mathcal{F}_{\mathbb{R}}$ is known as $\mathbb{R}$-cubic picture fuzzy open set in X. The complement A^{c} of a $\mathbb{R}$-cubic picture fuzzy open set A in $\mathbb{R}$-cubic picture fuzzy topological space $(X, \mathcal{F}_{\mathbb{R}})$ is called a $\mathbb{R}$-cubic picture fuzzy closed set in X.

Definition 15. [10] Those cubic sets which are both $\mathbb{R}$-cubic picture fuzzy open and $\mathbb{R}$-cubic picture fuzzy close are called $\mathbb{R}$-cubic picture fuzzy clopen sets in $(X, \mathcal{F}_{\mathbb{R}})$.

Definition 16. [10] Let $(X, \mathcal{F}_{\mathbb{R}})$ be a $\mathbb{R}$-cubic picture fuzzy topological space and$A$ be a cubic picture fuzzy set in $(X, \mathcal{F}_{\mathbb{R}})$. Then the $\mathbb{R}$ -cubic picture fuzzy interior and $\mathbb{R}$ - cubic picture fuzzy closure are defined by

$CPFint_{\mathbb{R}}(A) = \cup_{\mathbb{R}}\{A_i \mid A_i$ is a $\mathbb{R}$- cubic picture fuzzy open set in X and $A_i \subseteq_{\mathbb{R}} A\}$
$CPFcl_{\mathbb{R}}(A) = \cap_{\mathbb{R}}\{A_i \mid A_i$ is a $\mathbb{R}$- cubic picture fuzzy closed set in X and$A \subseteq_{\mathbb{R}} A_i\}$.

Definition 17. [11] A cubic picture fuzzy set L in a $\mathbb{R}$ - cubic picture fuzzy topological space $(X, F_{\mathbb{R}})$ is said to be

(i) $\mathbb{R}$-cubic picture fuzzy regular open set(briefly,$CPF_{\mathbb{R}}ros$) if $L = CPF_{\mathbb{R}}int(CPF_{\mathbb{R}}cl(L))$
(ii) $\mathbb{R}$-cubic picture fuzzy regular closed set (briefly,$CPF_{\mathbb{R}}rcs$) if $L = CPF_{\mathbb{R}}cl(CPF_{\mathbb{R}}int(L))$

Definition 18. [11] A cubic picture fuzzy set L is said to be a $\mathbb{R}$-Cubic picture fuzzy

(i) δ interior of L (briefly, $CPF_{\mathbb{R}}\delta int$) is defined by $CPF_{\mathbb{R}}\delta int = \bigcup\{G : G \subseteq L$ and G is a $CPF_{\mathbb{R}}\delta ros$ in $X\}$.

(ii) δ closure of L (briefly, $CPF_{\mathbb{R}}\delta cl$) is defined by $CPF_{\mathbb{R}}\delta cl = \bigcap\{G : G \supseteq L$ and G is a $CPF_{\mathbb{R}}\delta rcs$ in $X\}$.

Definition 19. [11] A set L is said to be a $\mathbb{R}$-Cubic picture fuzzy

(i) δ-pre open set(briefly, $CPF_{\mathbb{R}}\delta\mathcal{P}os$) if $L \subseteq_{\mathbb{R}} CPF_{\mathbb{R}}int\big(CPF_{\mathbb{R}}\delta clL\big)$.
(ii) δ-pre closed set(briefly, $CPF_{\mathbb{R}}\delta\mathcal{R}cs$) if $L \supseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl\big(CPF_{\mathbb{R}}\delta intL\big)$.
(iii) δ-semi open set(briefly, $CPF_{\mathbb{R}}\delta\mathcal{S}os$) if $L \subseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl\big(CPF_{\mathbb{R}}\delta intL\big)$.
(iv) δ-semi closed set(briefly, $CPF_{\mathbb{R}}\delta\mathcal{S}cs$) if $L \supseteq_{\mathbb{R}} CPF_{\mathbb{R}}int\big(CPF_{\mathbb{R}}\delta clL\big)$.
(v) e-open set (briefly, $CPF_{\mathbb{R}}eos$) if $L \subseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl\big(CPF_{\mathbb{R}}\delta intL\big) \cup CPF_{\mathbb{R}}int\ \big(CPF_{\mathbb{R}}\delta clL\big)$.
(vi) e-closed set (briefly, $CPF_{\mathbb{R}}ecs$)$L \supseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl\big(CPF_{\mathbb{R}}\delta intL\big) \cap CPF_{\mathbb{R}}int\ \big(CPF_{\delta}clL\big)$.
(vii) e^*-open set(briefly, $CPF_{\mathbb{R}}e^*os$) if $L \subseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl\big(CPF_{\mathbb{R}}int(CPF_{\mathbb{R}}\delta clL)\big)$.
(viii) e^*-closed set(briefly, $CPF_{\mathbb{R}}e^*cs$) if $L \supseteq_{\mathbb{R}} CPF_{\mathbb{R}}int\big(CPF_{\mathbb{R}}cl(CPF_{\mathbb{R}}\delta intL)\big)$.
(ix) a-open set (briefly, $CPF_{\mathbb{R}}aos$) if $L \subseteq_{\mathbb{R}} CPF_{\mathbb{R}}int\big(CPF_{\mathbb{R}}cl(CPF_{\mathbb{R}}\delta intL)\big)$.
(x) a-closed set (briefly, $CPF_{\mathbb{R}}acs$) if $L \supseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl\big(CPF_{\mathbb{R}}int(CPF_{\mathbb{R}}\delta clL)\big)$.
(xi) β open set (briefly, $CPF_{\mathbb{R}}\beta os$) if $L \subseteq_{\mathbb{R}} CPF_{\mathbb{R}}cl(CPF_{\mathbb{R}}int(CPF_{\mathbb{R}}cl(L)))$
(xii) β closed set (briefly, $CPF_{\mathbb{R}}\beta cs$) if $L \supseteq_{\mathbb{R}} CPF_{\mathbb{R}}int(CPF_{\mathbb{R}}cl(CPF_{\mathbb{R}}int(L)))$

The family of all $CPF_{\mathbb{R}}\delta\mathcal{P}os$ (resp. $CPF_{\mathbb{R}}\delta\mathcal{P}cs$, $CPF_{\mathbb{R}}\delta\mathcal{S}os$, $CPF_{\mathbb{R}}\delta\mathcal{S}cs$, $CPF_{\mathbb{R}}eos$, $CPF_{\mathbb{R}}ecs$,$CPF_{\mathbb{R}}aos$,$CPF_{\mathbb{R}}acs$, $CPF_{\mathbb{R}}\beta os$,$CPF_{\mathbb{R}}\beta cs$, $CPF_{\mathbb{R}}e^*os$, & $CPF_{\mathbb{R}}e^*cs$) of X is denoted by $CPF_{\mathbb{R}}\delta\mathcal{P}OS(X)$ (resp. $CPF_{\mathbb{R}}\delta\mathcal{P}CS(X)$, $CPF_{\mathbb{R}}\delta\mathcal{S}OS(X)$, $CPF_{\mathbb{R}}\delta\mathcal{S}CS(X)$, $CPF_{\mathbb{R}} eOS(X)$, $CPF_eCS(X)$,$CPF_{\mathbb{R}}aOS(X)$, $CPF_aCS(X)$,$CPF_{\mathbb{R}}\beta OS(X)$, $CPF_{\mathbb{R}}\beta CS(X)$, $CPF_{\mathbb{R}}e^*OS(X)$ & $CPF_{\mathbb{R}}e^*CS(X)$).

Definition 20. [11] A set L is said to be a $\mathbb{R}$-cubic

(i) e-interior(resp. e^*-interior) of L is defined by $CPF_{\mathbb{R}}eintL$(resp. $CPF_{\mathbb{R}}e^*intL$) = $\bigcup_{\mathbb{R}}\{G : G \subseteq L$ & G is a $CPF_{\mathbb{R}}eos$(resp. $CPF_{\mathbb{R}}e^*os$)in$X\}$.
(ii) e-closure (resp. e^*-closure) of L is defined by $CPF_{\mathbb{R}}eclL$(resp. $CPF_{\mathbb{R}}e^*clL$) = $\bigcap_{\mathbb{R}}\{G : L \subseteq G$ & G is a $CPF_{\mathbb{R}}ecs$(resp. $CPF_{\mathbb{R}}e^*cs$)in$X\}$.
(iii) δ pre interior(resp. δ semi interior) of L (briefly, $CPF_{\mathbb{R}}\delta\mathcal{P}int$) (resp. $CPF_{\mathbb{R}}\delta\mathcal{S}int$) is defined by $CPF_{\mathbb{R}}\delta\mathcal{P}int$(resp. $CPF_{\mathbb{R}}\delta\mathcal{S}int$) = $\bigcup_{\mathbb{R}}\{G : G \subseteq L$ & G is a $CPF_{\mathbb{R}}\delta\mathcal{P}os$ (resp.$CPF_{\mathbb{R}}\delta\mathcal{S}os$) in $X\}$.
(iv) δ pre closure (resp. δ semi closure) of L (briefly, $CPF_{\mathbb{R}}\delta\mathcal{P}cl$ (resp.$CPF_{\mathbb{R}}\delta\mathcal{S}cl$)) is defined by $CPF_{\mathbb{R}}\delta\mathcal{P}cl$(resp. $CPF_{\mathbb{R}}\delta\mathcal{S}cl$) = $\bigcap_{\mathbb{R}}\{G : L \subseteq G$ & G is a $CPF_{\mathbb{R}}ecs$ (resp.$CPF_{\mathbb{R}}\delta\mathcal{P}cs$ & $CPF_{\mathbb{R}}\delta\mathcal{S}cs$) in $X\}$.
(v) a-interior(resp. β-interior) of L is defined by $CPF_{\mathbb{R}}aintL$(resp. $CPF_{\mathbb{R}}\beta intL$) = $\bigcup_{\mathbb{R}}\{G : G \subseteq L$ & G is a $CPF_{\mathbb{R}}aos$(resp. $CPF_{\mathbb{R}}\beta os$)in$X\}$.
(vi) a-closure (resp. β-closure) of L is defined by $CPF_{\mathbb{R}}aclL$(resp. $CPF_{\mathbb{R}}\beta clL$) = $\bigcap_{\mathbb{R}}\{G : L \subseteq G$ & G is a $CPF_{\mathbb{R}}acs$(resp. $CPF_{\mathbb{R}}\beta cs$)in$X\}$.

3. $\mathbb{R}$-ORDER e-OPEN CONTINUOUS MAPPING IN CUBIC PICTURE FUZZY TOPOLOGICAL SPACES

In this section we introduce $\mathbb{R}$ cubic e-continuous maps and study some of its properties.

Definition 21. Let $(X, \mathcal{F}_{\mathbb{R}})$ and $(Y, \mathcal{G}_{\mathbb{R}})$ be any two $\mathbb{R}$ cubic topological spaces (briefly $CPF_{\mathbb{R}}ts's$). A map $f : (X, \mathcal{F}_{\mathbb{R}}) \to (Y, \mathcal{G}_{\mathbb{R}})$ is said to be $\mathbb{R}$-Cubic

(i) e-continuous (briefly, $CPF_{\mathbb{R}}eCts$) if the inverse image of every $CPF_{\mathbb{R}}os$ in $(Y, \mathcal{G}_{\mathbb{R}})$ is a $CPF_{\mathbb{R}}eos$ in $(X, \mathcal{F}_{\mathbb{R}})$.

Definition 22. Let $(X, \mathcal{F}_{\mathbb{R}})$ and $(Y, \mathcal{G}_{\mathbb{R}})$ be any two $CPF_{\mathbb{R}}ts's$. A map $f : (X, \mathcal{F}_{\mathbb{R}}) \to (Y, \mathcal{G}_{\mathbb{R}})$ is said to be $\mathbb{R}$-Cubic

(i) $\delta\mathcal{S}$-continuous (briefly, $CPF_{\mathbb{R}}\delta\mathcal{S}Cts$) if the inverse image of every $CPF_{\mathbb{R}}os$ in $(Y, \mathcal{G}_{\mathbb{R}})$ is a $CPF_{\mathbb{R}}\delta\mathcal{S}os$ in $(X, \mathcal{F}_{\mathbb{R}})$.
(ii) $\delta\mathcal{P}$-continuous (briefly, $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$) if the inverse image of every $CPF_{\mathbb{R}}os$ in $(Y, \mathcal{G}_{\mathbb{R}})$ is a $CPF_{\mathbb{R}}\delta\mathcal{P}os$ in $(X, \mathcal{F}_{\mathbb{R}})$.
(iii) e^*-continuous (briefly, $CPF_{\mathbb{R}}e^*Cts$) if the inverse image of every $CPF_{\mathbb{R}}os$ in $(Y, \mathcal{G}_{\mathbb{R}})$ is a $CPF_{\mathbb{R}}e^*os$ in $(X, \mathcal{F}_{\mathbb{R}})$.
(iv) a-continuous (briefly, $CPF_{\mathbb{R}}aCts$) if the inverse image of every $CPF_{\mathbb{R}}os$ in $(Y, \mathcal{G}_{\mathbb{R}})$ is a $CPF_{\mathbb{R}}aos$ in $(X, \mathcal{F}_{\mathbb{R}})$.

Proposition 1. The statements are hold but the converse does not true. Every

(i) $CPF_{\mathbb{R}}Cts$ is a $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$.
(ii) $CPF_{\mathbb{R}}Cts$ is a $CPF_{\mathbb{R}}\delta\mathcal{S}Cts$.
(iii) $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$ is a $CPF_{\mathbb{R}}eCts$.
(iv) $CPF_{\mathbb{R}}\delta\mathcal{S}Cts$ is a $CPF_{\mathbb{R}}eCts$.
(v) $CPF_{\mathbb{R}}eCts$ is a $CPF_{\mathbb{R}}e^*Cts$.
(vi) $CPF_{\mathbb{R}}eCts$ is a $CPF_{\mathbb{R}}\beta Cts$.
(vii) $CPF_{\mathbb{R}}aCts$ is a $CPF_{\mathbb{R}}eCts$.
(viii) $CPF_{\mathbb{R}}aCts$ is a $CPF_{\mathbb{R}}\beta Cts$.
(ix) $CPF_{\mathbb{R}}\beta Cts$ is a $CPF_{\mathbb{R}}e^*Cts$.

Proof. (i) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}Cts$, $f^{-1}(\mathfrak{M})$ is $CPF_{\mathbb{R}}os$ in X. Since all $CPF_{\mathbb{R}}os$ are $CPF_{\mathbb{R}}\delta\mathcal{P}os$, $f^{-1}(\mathfrak{M})$ is $CPF_{\mathbb{R}}\delta\mathcal{P}os$ in X. Hence f is a $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$.
(ii) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}Cts$, $f^{-1}(\mathfrak{M})$ is $CPF_{\mathbb{R}}os$ in X. Since all $CPF_{\mathbb{R}}os$ are $CPF_{\mathbb{R}}\delta\mathcal{S}os$, $f^{-1}(\mathfrak{M})$ is $CPF_{\mathbb{R}}\delta\mathcal{S}os$ in X. Hence f is a $CPF_{\mathbb{R}}\delta\mathcal{S}Cts$.
(iii) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}\delta\mathcal{P}os$ in X. Since every $CPF_{\mathbb{R}}\delta os$ is a $CPF_{\mathbb{R}}eos$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X. Hence f is a $CPF_{\mathbb{R}}eCts$.
(iv) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}\delta\mathcal{S}Cts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}\delta\mathcal{S}os$ in X. Since every $CPF_{\mathbb{R}}\delta\mathcal{P}os$ is a $CPF_{\mathbb{R}}eos$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X. Hence f is a $CPF_{\mathbb{R}}eCts$.
(v) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}eCts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X. Since every $CPF_{\mathbb{R}}eos$ is a $CPF_{\mathbb{R}}e^*os$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}e^*os$ in X. Hence f is a $CPF_{\mathbb{R}}e^*Cts$.
(vi) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}eCts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X. Since every $CPF_{\mathbb{R}}eos$ is a $CPF_{\mathbb{R}}\beta os$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}\beta os$ in X. Hence f is a $CPF_{\mathbb{R}}\beta Cts$.
(vii) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}eCts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X. Since every $CPF_{\mathbb{R}}aos$ is a $CPF_{\mathbb{R}}eos$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X. Hence f is a $CPF_{\mathbb{R}}eCts$.
(viii) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}aCts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}\beta os$ in X. Since every $CPF_{\mathbb{R}}aos$ is a $CPF_{\mathbb{R}}\beta os$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}\beta os$ in X. Hence f is a $CPF_{\mathbb{R}}\beta Cts$.
(ix) Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}\beta Cts$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}e^*os$ in X. Since every $CPF_{\mathbb{R}}\beta os$ is a $CPF_{\mathbb{R}}e^*os$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}e^*os$ in X. Hence f is a $CPF_{\mathbb{R}}e^*Cts$.

∎

Remark 1. We obtain the following diagram from the results we discussed above and justified from the following examples.

$$\begin{tikzcd}
\boxed{CPF_{\mathbb{R}}\delta\mathcal{P}Cts} \arrow[dr] & & \boxed{CPF_{\mathbb{R}}e^{*}Cts} \\
\boxed{CPF_{\mathbb{R}}Cts} \arrow[u] \arrow[d] & \boxed{CPF_{\mathbb{R}}eCts} \arrow[ur] \arrow[r] & \boxed{CPF_{\mathbb{R}}\beta Cts} \arrow[u] \\
\boxed{CPF_{\mathbb{R}}\delta\mathcal{S}Cts} \arrow[ur] & & \boxed{CPF_{\mathbb{R}}aCts} \arrow[ul] \arrow[u]
\end{tikzcd}$$

Example 1. Let X be a non-empty set. Define an identity mapping $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}}=\{\hat{0},\hat{1},P_1,P_2,P_3,P_4\}$, $\mathcal{G}_R=\{\hat{0},\hat{1},P_5\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1&=\{\langle[0,0],[0.05,0.10],[0.85,0.90]\rangle,\langle 0.90,0.10,0\rangle\},\\
P_2&=\{\langle[0.85,0.90],[0.05,0.10],[0,0]\rangle,\langle 0,0.10,0.90\rangle\},\\
P_3&=\{\langle[0.25,0.35],[0.05,0.10],[0.45,0.47]\rangle,\langle 0.45,0.10,0.30\rangle\},\\
P_4&=\{\langle[0.35,0.42],[0.05,0.10],[0.43,0.46]\rangle,\langle 0.43,0.10,0.32\rangle\}\\
P_5&=\{\langle[0.29,0.35],[0.05,0.10],[0.45,0.46]\rangle,\langle 0.50,0.10,0.20\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$ but not a $CPF_{\mathbb{R}}Cts$, since P_5 is $CPF_{\mathbb{R}}\delta\mathcal{P}os$ but not $CPF_{\mathbb{R}}os$ in $(X,\mathcal{F}_{\mathbb{R}})$.

Example 2. Let X be a non-empty set. Define an identity mapping $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}}=\{\hat{0},\hat{1},P_1,P_2,P_3,P_4\}$, $\mathcal{G}_R=\{\hat{0},\hat{1},P_6\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1&=\{\langle[0,0],[0.05,0.10],[0.85,0.90]\rangle,\langle 0.90,0.10,0\rangle\},\\
P_2&=\{\langle[0.85,0.90],[0.05,0.10],[0,0]\rangle,\langle 0,0.10,0.90\rangle\},\\
P_3&=\{\langle[0.25,0.35],[0.05,0.10],[0.45,0.47]\rangle,\langle 0.45,0.10,0.30\rangle\},\\
P_4&=\{\langle[0.35,0.42],[0.05,0.10],[0.43,0.46]\rangle,\langle 0.43,0.10,0.32\rangle\}\\
P_6&=\{\langle[0.37,0.40],[0.05,0.10],[0.41,0.42]\rangle,\langle 0.41,0.10,0.36\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}\delta\mathcal{S}Cts$ but not a $CPF_{\mathbb{R}}Cts$, since P_6 is $CPF_{\mathbb{R}}\delta\mathcal{S}os$ but not $CPF_{\mathbb{R}}os$ in $(X,\mathcal{F}_{\mathbb{R}})$.

Example 3. Let X be a non-empty set. Define an identity mapping $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}}=\{\hat{0},P_1,P_2,P_3,P_4\}$, $\mathcal{G}_R=\{\hat{0},\hat{1},P_6\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1&=\{\langle[0,0],[0.05,0.10],[0.85,0.90]\rangle,\langle 0.90,0.10,0\rangle\},\\
P_2&=\{\langle[0.85,0.90],[0.05,0.10],[0,0]\rangle,\langle 0,0.10,0.90\rangle\},\\
P_3&=\{\langle[0.25,0.35],[0.05,0.10],[0.45,0.47]\rangle,\langle 0.45,0.10,0.30\rangle\},\\
P_4&=\{\langle[0.35,0.42],[0.05,0.10],[0.43,0.46]\rangle,\langle 0.43,0.10,0.32\rangle\}\\
P_6&=\{\langle[0.37,0.40],[0.05,0.10],[0.41,0.42]\rangle,\langle 0.41,0.10,0.36\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}eCts$ but not a $CPF_{\mathbb{R}}\delta\mathcal{P}Cts$, since P_6 is $CPF_{\mathbb{R}}eos$ but not $CPF_{\mathbb{R}}\delta\mathcal{P}os$ in $(X,\mathcal{F}_{\mathbb{R}})$.

Example 4. Let X be a non-empty set. Define an identity mapping $f : (X, \mathcal{F}_{\mathbb{R}}) \to (Y, \mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}} = \{\hat{0}, P_1, P_2, P_3, P_4\}$, $\mathcal{G}_R = \{\hat{0}, \hat{1}, P_5\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1 &= \{\langle [0,0], [0.05, 0.10], [0.85, 0.90]\rangle, \langle 0.90, 0.10, 0\rangle\},\\
P_2 &= \{\langle [0.85, 0.90], [0.05, 0.10], [0,0]\rangle, \langle 0, 0.10, 0.90\rangle\},\\
P_3 &= \{\langle [0.25, 0.35], [0.05, 0.10], [0.45, 0.47]\rangle, \langle 0.45, 0.10, 0.30\rangle\},\\
P_4 &= \{\langle [0.35, 0.42], [0.05, 0.10], [0.43, 0.46]\rangle, \langle 0.43, 0.10, 0.32\rangle\}\\
P_5 &= \{\langle [0.29, 0.35], [0.05, 0.10], [0.45, 0.46]\rangle, \langle 0.50, 0.10, 0.20\rangle\}
\end{aligned}$$

Here f is CPF_ReCts but not a $CPF_R\delta\mathcal{S}Cts$, since P_5 is CPF_Reos but not $CPF_R\delta\mathcal{S}os$ in $(X, \mathcal{F}_{\mathbb{R}})$.

Example 5. Let X be a non-empty set. Define an identity mapping $f : (X, \mathcal{F}_{\mathbb{R}}) \to (Y, \mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}} = \{\hat{0}, \hat{1}, P_1, P_2, P_3, P_4\}$, $\mathcal{G}_R = \{\hat{0}, \hat{1}, P_7\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1 &= \{\langle [0,0], [0.05, 0.10], [0.85, 0.90]\rangle, \langle 0.90, 0.10, 0\rangle\},\\
P_2 &= \{\langle [0.85, 0.90], [0.05, 0.10], [0,0]\rangle, \langle 0, 0.10, 0.90\rangle\},\\
P_3 &= \{\langle [0.25, 0.35], [0.05, 0.10], [0.45, 0.47]\rangle, \langle 0.45, 0.10, 0.30\rangle\},\\
P_4 &= \{\langle [0.35, 0.42], [0.05, 0.10], [0.43, 0.46]\rangle, \langle 0.43, 0.10, 0.32\rangle\}\\
P_7 &= \{\langle [0.15, 0.20], [0.05, 0.10], [0.42, 0.45]\rangle, \langle 0.42, 0.10, 0.30\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}e^*Cts$ but not a $CPF_{\mathbb{R}}eCts$, since μ_{13} is $CPF_{\mathbb{R}}e^*os$ but not $CPF_{\mathbb{R}}eos$ in $(X, \mathcal{F}_{\mathbb{R}})$.

Example 6. Let X be a non-empty set. Define an identity mapping $f : (X, \mathcal{F}_{\mathbb{R}}) \to (Y, \mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}} = \{\hat{0}, \hat{1}, P_1, P_2, P_3, P_4\}$, $\mathcal{G}_R = \{\hat{0}, \hat{1}, P_7\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1 &= \{\langle [0,0], [0.05, 0.10], [0.85, 0.90]\rangle, \langle 0.90, 0.10, 0\rangle\},\\
P_2 &= \{\langle [0.85, 0.90], [0.05, 0.10], [0,0]\rangle, \langle 0, 0.10, 0.90\rangle\},\\
P_3 &= \{\langle [0.25, 0.35], [0.05, 0.10], [0.45, 0.47]\rangle, \langle 0.45, 0.10, 0.30\rangle\},\\
P_4 &= \{\langle [0.35, 0.42], [0.05, 0.10], [0.43, 0.46]\rangle, \langle 0.43, 0.10, 0.32\rangle\}\\
P_7 &= \{\langle [0.15, 0.20], [0.05, 0.10], [0.42, 0.45]\rangle, \langle 0.42, 0.10, 0.30\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}e^*Cts$ but not a $CPF_{\mathbb{R}}\beta Cts$, since μ_{16} is $CPF_{\mathbb{R}}e^*Cts$ but not $CPF_{\mathbb{R}}\beta Cts$ in $(X, \mathcal{F}_{\mathbb{R}})$.

Example 7. Let X be a non-empty set. Define an identity mapping $f : (X, \mathcal{F}_{\mathbb{R}}) \to (Y, \mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}} = \{\hat{0}, \hat{1}, P_1, P_2, P_3, P_4\}$, $\mathcal{G}_R = \{\hat{0}, \hat{1}, P_7\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1 &= \{\langle [0,0], [0.05, 0.10], [0.85, 0.90]\rangle, \langle 0.90, 0.10, 0\rangle\},\\
P_2 &= \{\langle [0.85, 0.90], [0.05, 0.10], [0,0]\rangle, \langle 0, 0.10, 0.90\rangle\},\\
P_3 &= \{\langle [0.25, 0.35], [0.05, 0.10], [0.45, 0.47]\rangle, \langle 0.45, 0.10, 0.30\rangle\},\\
P_4 &= \{\langle [0.35, 0.42], [0.05, 0.10], [0.43, 0.46]\rangle, \langle 0.43, 0.10, 0.32\rangle\}\\
P_7 &= \{\langle [0.15, 0.20], [0.05, 0.10], [0.42, 0.45]\rangle, \langle 0.42, 0.10, 0.30\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}\beta Cts$ but not a $CPF_{\mathbb{R}}eCts$, since μ_{15} is $CPF_{\mathbb{R}}\beta Cts$ but not $CPF_{\mathbb{R}}eCts$ in $(X, \mathcal{F}_{\mathbb{R}})$.

Example 8. Let X be a non-empty set. Define an identity mapping $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}}=\{\hat{0},\hat{1},P_1,P_2,P_3,P_4\}$, $\mathcal{G}_R=\{\hat{0},\hat{1},P_7\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1&=\{\langle[0,0],[0.05,0.10],[0.85,0.90]\rangle,\langle 0.90,0.10,0\rangle\},\\
P_2&=\{\langle[0.85,0.90],[0.05,0.10],[0,0]\rangle,\langle 0,0.10,0.90\rangle\},\\
P_3&=\{\langle[0.25,0.35],[0.05,0.10],[0.45,0.47]\rangle,\langle 0.45,0.10,0.30\rangle\},\\
P_4&=\{\langle[0.35,0.42],[0.05,0.10],[0.43,0.46]\rangle,\langle 0.43,0.10,0.32\rangle\}\\
P_7&=\{\langle[0.15,0.20],[0.05,0.10],[0.42,0.45]\rangle,\langle 0.42,0.10,0.30\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}\beta Cts$ but not a $CPF_{\mathbb{R}}aCts$, since μ_{13} is $CPF_{\mathbb{R}}\beta Cts$ but not $CPF_{\mathbb{R}}aCts$ in $(X,\mathcal{F}_{\mathbb{R}})$.

Example 9. Let X be a non-empty set. Define an identity mapping $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_R)$. Let $\mathcal{F}_{\mathbb{R}}=\{\hat{0},\hat{1},P_1,P_2,P_3,P_4\}$, $\mathcal{G}_R=\{\hat{0},\hat{1},P_6\}$ be two $CPF_{\mathbb{R}}ts$'s. where

$$\begin{aligned}
P_1&=\{\langle[0,0],[0.05,0.10],[0.85,0.90]\rangle,\langle 0.90,0.10,0\rangle\},\\
P_2&=\{\langle[0.85,0.90],[0.05,0.10],[0,0]\rangle,\langle 0,0.10,0.90\rangle\},\\
P_3&=\{\langle[0.25,0.35],[0.05,0.10],[0.45,0.47]\rangle,\langle 0.45,0.10,0.30\rangle\},\\
P_4&=\{\langle[0.35,0.42],[0.05,0.10],[0.43,0.46]\rangle,\langle 0.43,0.10,0.32\rangle\}\\
P_6&=\{\langle[0.37,0.40],[0.05,0.10],[0.41,0.42]\rangle,\langle 0.41,0.10,0.36\rangle\}
\end{aligned}$$

Here f is $CPF_{\mathbb{R}}eCts$ but not a $CPF_{\mathbb{R}}aCts$, since P_6 is $CPF_{\mathbb{R}}eCts$ but not $CPF_{\mathbb{R}}aCts$ in $(X,\mathcal{F}_{\mathbb{R}})$.

Theorem 1. A map $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_{\mathbb{R}})$ is $CPF_{\mathbb{R}}eCts$ iff the inverse image of each $CPF_{\mathbb{R}}cs$ in Y is $CPF_{\mathbb{R}}ecs$ in X.

Proof. Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}cs$ in Y. This implies $\mathfrak{M}^c$ is $CPF_{\mathbb{R}}os$ in Y. Since f is $CPF_{\mathbb{R}}eCts$, $f^{-1}(\mathfrak{M}^c)$ is $CPF_{\mathbb{R}}eos$ in X. Since $f^{-1}(\mathfrak{M}^c)=((f^{-1}\mathfrak{M}))^c$, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}ecs$ in X.

Conversely, let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}cs$ in Y. Then $\mathfrak{M}^c$ is a $CPF_{\mathbb{R}}os$ in Y. By hypothesis $f^{-1}(\mathfrak{M}^c)$ is $CPF_{\mathbb{R}}eos$ in X. Since $f^{-1}(\mathfrak{M}^c)=((f^{-1}\mathfrak{M}))^c$, $(f^{-1}\mathfrak{M}))^c$ is a $CPF_{\mathbb{R}}eos$ in X. Therefore $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}ecs$ in X. Hence f is $CPF_{\mathbb{R}}eCts$. ∎

Definition 23. A $CPF_{\mathbb{R}}ts$ $(X,\mathcal{F}_{\mathbb{R}})$ is said to be $\mathbb{R}$-cubic $eU_{\frac{1}{2}}$ (briefly, $CPF_{\mathbb{R}}eU_{\frac{1}{2}}$)space, if every $CPF_{\mathbb{R}}eos$ in X is a $CPF_{\mathbb{R}}os$ in X.

Theorem 2. Let $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_{\mathbb{R}})$ be a $CPF_{\mathbb{R}}eCts$, then f is a $CPF_{\mathbb{R}}Cts$ if X is a $CPF_{\mathbb{R}}eU_{\frac{1}{2}}$-space.

Proof. Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Y. Then $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}eos$ in X, by hypothesis. Since X is a $CPF_{\mathbb{R}}eU_{\frac{1}{2}}$-space, $f^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}os$ in X. Hence f is a $CPF_{\mathbb{R}}eCts$. ∎

Theorem 3. Let $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_{\mathbb{R}})$ be a $CPF_{\mathbb{R}}eCts$ map and $g:(Y,\mathcal{G}_{\mathbb{R}})\to(Z,\mathcal{E}_p)$ be a $CPF_{\mathbb{R}}Cts$, then $g\circ f:(X,\mathcal{F}_{\mathbb{R}})\to(Z,\mathcal{E}_p)$ is a $CPF_{\mathbb{R}}eCts$.

Proof. Let $\mathfrak{M}$ be a $CPF_{\mathbb{R}}os$ in Z. Then $g^{-1}(\mathfrak{M})$ is a $CPF_{\mathbb{R}}os$ in Y, by hypothesis. Since f is a $CPF_{\mathbb{R}}eCts$ map, $f^{-1}\left(g^{-1}(\mathfrak{M})\right))$ is a $CPF_{\mathbb{R}}eos$ in X. Hence $g\circ f$ is a $CPF_{\mathbb{R}}eCts$ map. ∎

Theorem 4. Let $f:(X,\mathcal{F}_{\mathbb{R}})\to(Y,\mathcal{G}_{\mathbb{R}})$ be a $CPF_{\mathbb{R}}eCts$ map. Then the following conditions are hold.

(i) $f(CPF_{\mathbb{R}}ecl(\mathfrak{M}))\leq CPF_{\mathbb{R}}cl(f(\mathfrak{M}))$, for all $CPF_{\mathbb{R}}cs$ $\mathfrak{M}$ in X.

(ii) $CPF_{\mathbb{R}}ecl(f^{-1}\mathfrak{M}) \leq f^{-1}(CPF_{\mathbb{R}}cl\mathfrak{M})$, for all $CPF_{\mathbb{R}}cs$ $\mathfrak{M}$ in Y.

Proof. (i) Since $CPF_{\mathbb{R}}ecl(f(\mathfrak{M}))$ is a $CPF_{\mathbb{R}}ecs$ in Y and f is $CPF_{\mathbb{R}}eCts$, then $f^{-1}(CPF_{\mathbb{R}}ecl(f(\mathfrak{M})))$ is $CPF_{\mathbb{R}}ec$ in Y. Now, since $\mathfrak{M} \leq f^{-1}(CPF_{\mathbb{R}}cl(f(\mathfrak{M})))$, $CPF_{\mathbb{R}}ecl(\mathfrak{M}) \leq f^{-1}(CPF_{\mathbb{R}}ecl(f(\mathfrak{M})))$. Therefore, $f(CPF_{\mathbb{R}}ecl(\mathfrak{M})) \leq CPF_{\mathbb{R}}cl(f(\mathfrak{M}))$.
(ii) By replacing $\mathfrak{M}$ with $f^{-1}(\mathfrak{M})$ in (i), we obtain
$f(CPF_{\mathbb{R}}ecl(f^{-1}\mathfrak{M})) \leq CPF_{\mathbb{R}}cl(f(f^{-1}\mathfrak{M})) \leq CPF_{\mathbb{R}}cl\mathfrak{M}$.
Hence, $CPF_{\mathbb{R}}ecl(f^{-1}\mathfrak{M}) \leq f^{-1}(CPF_{\mathbb{R}}cl\mathfrak{M})$. ∎

Remark 2. If f is $CPF_{\mathbb{R}}eCts$, then

(i) $f(CPF_{\mathbb{R}}ecl(\mathfrak{M}))$ is not necessarily equal to $CPF_{\mathbb{R}}cl(f(\mathfrak{M}))$ where $(\mathfrak{M}) \in X$.
(ii) $CPF_{\mathbb{R}}ecl(f^{-1}\mathfrak{M})$ is not necessarily equal to $f^{-1}(CPF_{\mathbb{R}}cl\mathfrak{M})$ where $\mathfrak{M} \in Y$.

Theorem 5. f is $CPF_{\mathbb{R}}eCts$ iff $f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M})) \leq CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M}))$, for all $CPF_{\mathbb{R}}cs$ $\mathfrak{M}$ in Y.

Proof. If f is $CPF_{\mathbb{R}}eCts$ and $\mathfrak{M} \in Y$. $CPF_{\mathbb{R}}int(\mathfrak{M})$ is $CPF_{\mathbb{R}}os$ in Y and hence, $f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M}))$ is $CPF_{\mathbb{R}}eos$ in X. Therefore $CPF_{\mathbb{R}}eint(f^{-1}(CPF_{\mathbb{R}}eint(\mathfrak{M}))) = f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M}))$. Also, $CPF_{\mathbb{R}}int(\mathfrak{M}) \leq \mathfrak{M}$, implies that $f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M})) \leq f^{-1}(\mathfrak{M})$.
Therefore $CPF_{\mathbb{R}}eint(f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M}))) \leq CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M}))$.
That is $f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M})) \leq CPF_{\mathbb{R}}eint\ (f^{-1}(\mathfrak{M}))$.

Conversely, let $f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M})) \leq CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M}))$ for all subset $\mathfrak{M}$ of Y.
If $\mathfrak{M}$ is $CPF_{\mathbb{R}}os$ in Y, then $CPF_{\mathbb{R}}int(\mathfrak{M}) = \mathfrak{M}$. By assumption, $f^{-1}(CPF_{\mathbb{R}}int(\mathfrak{M})) \leq CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M}))$. Thus $f^{-1}(\mathfrak{M}) \leq CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M}))$.
But $CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M})) \leq f^{-1}(\mathfrak{M})$. Therefore $CPF_{\mathbb{R}}eint(f^{-1}(\mathfrak{M})) = f^{-1}(\mathfrak{M})$. That is, $f^{-1}(\mathfrak{M})$ is $CPF_{\mathbb{R}}eos$ in X, for all $CPF_{\mathbb{R}}os$ $\mathfrak{M}$ in Y. Therefore f is $CPF_{\mathbb{R}}eCts$ on X. ∎

Remark 3. If f is $CPF_{\mathbb{R}}ects$, then $CPF_{\mathbb{R}}eint(f^{-1}(A))$ is not necessarily equal to $f^{-1}(CPF_{\mathbb{R}}int(A))$ where $A \in Y$.

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