DYNAMICAL PROPERTIES FOR MULTIPLICATION OPERATORS ON THE BLOCH SPACE

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ABSTRACT. We study some dynamical properties such as uniformly mean ergodicity, mean ergodicity and power boundedness of multiplication operators on the Bloch spaces defined on the unit ball of a Banach or a Hilbert space. Necessary and sufficient conditions are given for dynamical properties of multiplication operators defined on the Bloch spaces endowed with two different topologies: the one of uniform convergence on compact sets and the one defined by the usual norm.

1. INTRODUCTION, NOTATION AND PRELIMINARIES

1.1. **Introduction.** The classical Bloch space $\mathcal{B}(\mathbb{D})$ of analytic functions on the open unit disk $\mathbb{D}$ of $\mathbb{C}$ plays an important role in geometric function theory and it has been studied by many authors. In the last decades, many authors have considered higher dimensions, providing appropriate definitions of Bloch spaces on the unit ball of finite and infinite dimensional Hilbert or Banach spaces. O. Blasco, P. Galindo and A. Miralles extended the notion to the infinite dimensional setting by considering Bloch functions on the unit ball of an infinite dimensional Hilbert space (see [9], [10]). A. Miralles also dealt with a finite or infinite dimensional complex Banach space and considered two possible extensions of the classical Bloch space (see [1]). To have a more complete insight on the theory of the Bloch space in the finite dimensional space, see the book by Zhu [7].

Let X be a Banach space with open unit ball B_X. We define $H(B_X)$ as the space of all holomorphic functions $f : B_X \to \mathbb{C}$, endowed with the topology τ_c of uniform convergence on compact sets. This space is a locally convex Hausdorff space. The space $H^\infty(B_X)$ is defined as the set of functions $f \in H(B_X)$ satisfying $\|f\|_\infty := \sup\{|f(x)| : x \in B_X\} < \infty$. It is well known that $(H^\infty(B_X), \|\cdot\|_\infty)$ is a Banach space.

For each $f \in H(B_X)$, we define the natural Bloch semi-norm of f by

$$\|f\|_{nat} = \sup\{(1 - \|x\|^2)\|f'(x)\| : x \in B_X\},$$

where $f'(x) \in X^*$ denotes the Fréchet derivative of f at the point x. The Bloch type space $\mathcal{B}(B_X)$ is given by

$$\mathcal{B}(B_X) = \{f \in H(B_X) : \|f\|_{nat} < \infty\}.$$

It is clear that $\|\cdot\|_{nat}$ is a seminorm on $\mathcal{B}(B_X)$ and that with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{nat}, \tag{1.1}$$

the space $(\mathcal{B}(B_X), \|\cdot\|_{\mathcal{B}})$ is a Banach space. One of the interesting topics concerning Bloch functions is that of multiplication operators. Suppose ψ is a fixed analytic function on $\mathbb{D}$. The multiplication operator on the Bloch space is defined as

$$M_\psi(f) = \psi f.$$

Multiplication operators on the Bloch space of the open unit disk $\mathbb{D}$ have been studied in [4, 8, 13, 3]. In this sense, criteria for the boundedness of multiplications operators were obtained by Arazy in [4], and independently by Brown and Shields. In [8], the authors proved that M_ψ is a bounded operator on the classical Bloch space $\mathcal{B}(\mathbb{D})$ if and only if $\psi \in H^\infty(\mathbb{D})$ and

$$|\psi'(z)| = O\left(\frac{1}{(1 - |z|)\log(1/(1 - |z|))}\right).$$

In [13], Allen and Colonna defined

$$\sigma_\psi^\infty := \sup_{z \in \mathbb{D}} \frac{1}{2}(1 - |z|^2)|\psi'(z)| \log \frac{1 + |z|}{1 - |z|}$$

to determine an upper bound on $\|M_\psi\|$ and gave estimates on the norm of the multiplication operator in [13, Corollary 2.4], they deduced that if $\psi \in H(\mathbb{D})$ induces a bounded multiplication operator M_ψ on the Bloch space, then

$$\max\{\|\psi\|_{\mathcal{B}}, \|\psi\|_\infty\} \le \|M_\psi\| \le \max\{\|\psi\|_{\mathcal{B}}, \|\psi\|_\infty + \sigma_\psi^\infty\}.$$

Also mean ergodicity, power boundedness and related properties of multiplication operators acting on spaces of holomorphic functions on the unit disc $\mathbb{D}$ have been investigated and characterized in [3].

In this paper we firstly present some basic definitions for maps defined on locally convex space and fix the notation used throughout the paper and later study some dynamical properties, such as uniformly mean ergodicity, mean ergodicity and power boundedness, of multiplication operators on the Bloch spaces when these spaces are endowed with two different topologies on the unit ball of a Banach or a Hilbert space. For standard theory and notation of functional analysis we refer to [5].

1.2. **Notation and Preliminaries.** Let E be a locally convex Hausdorff space. We denote the set of all continuous seminorms on E by $cs(E)$, and by E' the dual space of E that is, the collection of all functions $u : E \to \mathbb{C}$ that are linear and continuous and by $\sigma(E, E')$ the topology of weak convergence on E, where each $u \in E'$ defines a seminorm $p_{\{u\}} : E \to \mathbb{R}$ given by $p_{\{u\}}(x) = |u(x)|$ for every $x \in E$. $(E, cs(E))$ is a Banach space whenever $cs(E)$ is a singleton.

Let $T \in L(E)$, where as usual we denote by $L(E)$ the space of continuous linear operators on E. We consider the topology of uniform convergence on bounded sets is given by the family of seminorms defined for $T \in L(E)$ as

$$p_{B,q}(T) = \sup_{x \in B} q(Tx),$$

where $B \subseteq E$ is bounded, and q is a continuous seminorm on E. T is called *power bounded* if the set of iterates of T, i.e. $\{T^n : n \in \mathbb{N}\}$, is an equicontinuous subset of $L(E)$. Thus T is power bounded when for every $q \in cs(E)$ there is $p \in cs(E)$ such that $q(T^n x) \leq p(x)$ for all $x \in E$ and all $n \in \mathbb{N}$. We denote by

$$T_{[n]} := \frac{1}{n} \sum_{k=1}^{n} T^k, \ n \in \mathbb{N},$$

the Cesàro means of T. Thus, T is *Cesàro bounded* if the family $\{T_{[n]} : n \in \mathbb{N}\}$ is an equicontinuous subset of $L(E)$, T is called *mean ergodic* if $(T_{[n]})_{n \in \mathbb{N}}$ converges in the strong operator topology, i.e. if for each $x \in E$ the sequence $(T_{[n]}(x))_{n \in \mathbb{N}}$ converges in E. In case $(T_{[n]})_{n \in \mathbb{N}}$ converges uniformly on bounded subsets of E we call T *uniformly mean ergodic*. For $T \in L(E)$ and $n \in \mathbb{N}$ we have the following identities

$$(1.2) \qquad \frac{1}{n} T^n = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$$

so that $\lim_{n \to \infty} n^{-1} T^n x$ for every $x \in E$ whenever T is mean ergodic. If E is normed space, then for any operator $T \in L(E)$ we denote by $\|T\| = \sup\{\|Tx\|_E : \|x\|_E \leq 1\}$ the operator norm. Recall that by $\sigma(T)$ (spectrum of T) we mean the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

Proposition 1.1. *Let $\{T_n\}$ be a sequence in $L(E)$ where E is locally convex Hausdorff space and $T_{(n)} = \frac{1}{n} \sum_{k=1}^{n} T_k$ be Cesàro means of T_n for $n \in \mathbb{N}$. If $T_n \to T$ in the topology of uniform convergence on bounded sets, then we have that $T_{(n)} \to T$ in the same topology.*

Proof. We observe that being $\{T_n\}$ convergent implies that it is bounded, so for all $q \in cs(E)$ and bounded set $B \subseteq E$, we can find a constant $M > 0$ such that

$$p_{B,q}(T_n - T) = \sup_{x \in B} q(T_n x - Tx) \leq M$$

for all $n \in \mathbb{N}$. Also since sequence $\{T_n\}$ in $L(E)$ converges to T, then for all $q \in cs(E)$ and bounded set $B \subseteq E$ and $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$p_{B,q}(T_n - T) = \sup_{x \in B} q(T_n x - Tx) < \epsilon$$

for every $n > n_0$. Joining these two facts together, we finally have

$$
\begin{aligned}
p_{B,q}(T_{(n)} - T) &= \sup_{x \in B} q(T_{(n)}x - Tx) \\
&= \sup_{x \in B} q\left[\frac{(T_1 x - Tx) + \cdots + (T_{n_0}x - Tx) + \cdots + (T_n x - Tx)}{n}\right] \\
&\leq \frac{1}{n}\sup_{x \in B} q(T_1 x - Tx) + \cdots + \frac{1}{n}\sup_{x \in B} q(T_{n_0}x - Tx) \\
&+ \frac{1}{n}\sup_{x \in B} q(T_{n_0+1}x - Tx) + \cdots + \frac{1}{n}\sup_{x \in B} q(T_n x - Tx) \\
&< \frac{n_0 M}{n} + \frac{(n - n_0)\epsilon}{n} \\
&< \frac{n_0 M}{n} + \epsilon
\end{aligned}
$$

for every $n > n_0$. This implies that $p_{B,q}(T_{(n)} - T) < \epsilon$ as $n \to \infty$. Since ϵ is arbitrary, as a result we obtain that $T_{(n)} \to T$ in the topology of uniform convergence on bounded sets. ∎

Before stating our results obtained, we present following two preparatory lemmas used quite a few in the article and proved by Miralles in [1, Proposition 3.3] and by X. Liu and Y. Liu in [16, Lemma 1.3], respectively.

Lemma 1.2. *Let $f \in H^\infty(B_X)$ such that $\|f\|_\infty \leq 1$. Then, we have that*

$$(1 - \|x\|)\|f'(x)\| \leq 1 - |f(x)|^2, \quad (x \in B_X).$$

Lemma 1.3. *If $f \in \mathcal{B}(B_X)$, then*

$$|f(x)| \leq C \log \frac{2}{1 - \|x\|^2}\|f\|_{\mathcal{B}}, \quad (x \in B_X).$$

2. DYNAMICS WITH THE COMPACT-OPEN TOPOLOGY

In addition to the Banach space topology on $\mathcal{B}(B_X)$, we also consider the compact-open topology, denoted by τ_c, which is the topology of uniform convergence on compact subsets of B_X. Throughout this section, we equip the Bloch space with the compact-open topology τ_c and study the multiplication operator

$$M_\psi : (\mathcal{B}(B_X), \tau_c) \to (\mathcal{B}(B_X), \tau_c), \quad M_\psi(f) = \psi f,$$

whenever this operator is *well-defined*. In this topology, every well-defined multiplication operator M_ψ is automatically τ_c-continuous also note that $M_\psi(1) = \psi \in \mathcal{B}(B_X)$ for $f \equiv 1$.

Lemma 2.1. *Let $\psi \in H(B_X)$ and $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ be multiplication operator. Then M_ψ is power bounded if and only if $\{\psi^n\}_{n \in \mathbb{N}}$ is bounded on $\mathcal{B}(B_X)$.*

Proof. Assume that M_ψ is power bounded, then the set $\{M_\psi^n(f) : n \in \mathbb{N}\}$ is bounded on $\mathcal{B}(B_X)$ for every $f \in \mathcal{B}(B_X)$. In particular, for $f \equiv 1$ we have that $\{\psi^n : n \in \mathbb{N}\}$ is bounded on $\mathcal{B}(B_X)$. Conversely, let K be a compact subset of B_X and consider

$$C := \sup_{n \in \mathbb{N}} \sup_{x \in K} |\psi^n(x)| < \infty.$$

Then

$$\sup_{x \in K} |M_\psi^n f(x)| \leq C \sup_{x \in K} |f(x)|, \quad n \in \mathbb{N}.$$

Hence, $\{M_\psi^n(f)\}_{n \in \mathbb{N}}$ is bounded on $\mathcal{B}(B_X)$ for every $f \in \mathcal{B}(B_X)$, and so M_ψ is power bounded. ∎

By combining Lemma 2.1 and [6, Proposition 9.15] we obtain at once the following result.

Corollary 2.2. *Let $\psi \in H(B_X)$ and $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ be multiplication operator. Then M_ψ is power bounded if and only if $\{\psi^n\}_{n \in \mathbb{N}}$ is equicontinuous and pointwise bounded.*

In view of Lemma 2.1 we have the following lemma.

Lemma 2.3. *Let $\psi \in H(B_X)$ and $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ be multiplication operator. Then $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ is power bounded if and only if $\|\psi\|_\infty \leq 1$.*

Proof. Firstly assume that $\|\psi\|_\infty \leq 1$. For $f \in \mathcal{B}(B_X)$ and every compact set $K \subseteq B_X$, we have

$$\sup_{x \in K} |M_\psi^n f(x)| \leq \sup_{x \in B_X} |\psi^n(x)| \sup_{x \in K} |f(x)| \leq \sup_{x \in K} |f(x)|.$$

Thus M_ψ is power bounded. On the other hand assume that M_ψ is power bounded. Since $\{M_\psi^n : n \in \mathbb{N}\}$ is equicontinuous, for every compact set $K \subseteq B_X$, there exist $C > 0$ and a compact set $L \subseteq B_X$ such that

$$\sup_{x \in K} |\psi^n(x) f(x)| \leq C \sup_{x \in L} |f(x)|, \quad \forall f \in \mathcal{B}(B_X), \forall n \in \mathbb{N}.$$

Now, consider the constant function $f \equiv 1$, which belongs to $\mathcal{B}(B_X)$ and the set $K = \{x\}$, which is compact. Substituting this choice of f and K, we obtain

$$\sup_{x \in B_X} |\psi(x)| \leq C^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ yields $\|\psi\|_\infty \leq 1$. ∎

Let $\mathcal{F}$ denote an arbitrary subset of $\mathcal{B}(B_X)$ and let K be a compact subset of B_X. By Lemma 1.3, it follows that for each $f \in \mathcal{F}$, we have

$$|f(x)| \leq C \log \frac{2}{1 - \|x\|^2} \|f\|_\mathcal{B}$$

for all $x \in K$. So there exists a constant $M > 0$ such that

$$\sup\{\|f\|_K : f \in \mathcal{F}\} < M$$

for every compact set $K \subseteq B_X$, where $\|f\|_K = \sup\{|f(x)| : x \in K\}$. Moreover $(\mathcal{B}(B_X), \tau_c)$ is τ_c-relatively compact by Montel's theorem, [14, Theorem 17.21]. Hence $(\mathcal{B}(B_X), \tau_c)$ is semi-Montel. As a consequence of [2, Proposition 3.1], we obtain the following result.

Corollary 2.4. *Let $\psi \in H(B_X)$. If the multiplication operator $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ is power bounded, then it is uniformly mean ergodic (and hence mean ergodic).*

Lemma 2.5. *Let $\psi \in H(B_X)$ and $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ be multiplication operator. If $|\psi(x_0)| > 1$ for some $x_0 \in B_X$, then M_ψ is not mean ergodic (thus, it is not power bounded).*

Proof. Suppose, to the contrary, that M_ψ is mean ergodic on $\mathcal{B}(B_X)$. Then, we have $\frac{1}{n} M_\psi^n \to 0$ pointwise on B_X. Take $1 \in \mathcal{B}(B_X)$ and fix $x_0 \in B_X$. Hence, it follows that

$$\frac{1}{n} |\psi(x_0)|^n \to 0, \quad (n \to \infty).$$

However, if $|\psi(x_0)| > 1$, then

$$\frac{1}{n} |\psi(x_0)|^n \to \infty, \quad (n \to \infty),$$

which is a contradiction. Therefore, M_ψ cannot be mean ergodic. Finally, by Corollary 2.4, it follows that M_ψ is not power bounded. ∎

In view of these results, the main conclusion of this section is the following characterisation.

Theorem 2.6. *Let $\psi \in H(B_X)$ and $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ be multiplication operator. Then the following are equivalent:*

  (i) $\|\psi\|_\infty \leq 1$.
 (ii) M_ψ *is uniformly mean ergodic.*
(iii) M_ψ *is mean ergodic.*
(iv) M_ψ *is power bounded.*

The following corollary is a result of Eberlein's mean ergodic theorem for linear and continuous operators on locally convex spaces (see [5, Corollary 5.6] and [15, Corollary 2.2]). By adapting this result expressed below, in case that the operator M_ψ is Cesàro bounded, we state the mean ergodicity of M_ψ.

Corollary 2.7. *Let $\psi \in H(B_X)$. Assume that $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ is Cesàro bounded. Then M_ψ is mean ergodic if and only if $\frac{\psi^n}{n} \to 0$ as $n \to \infty$ in $\mathcal{B}(B_X)$ and $((M_\psi)_{[n]}f)_{n\in\mathbb{N}}$ is $\sigma(\mathcal{B}(B_X), \mathcal{B}(B_X)')$-relatively compact for each $f \in \mathcal{B}(B_X)$.*

Proof. To prove that M_ψ is mean ergodic, it suffices to show that $\frac{1}{n} M_\psi^n f \to 0$ as $n \to \infty$ for every $f \in \mathcal{B}(B_X)$. For $f \in \mathcal{B}(B_X)$ and a compact subset $K \subseteq B_X$, we obtain

$$\lim_{n\to\infty} \sup_{x\in K} \frac{|M_\psi^n f(x)|}{n} = \lim_{n\to\infty} \sup_{x\in K} \frac{|\psi^n(x)f(x)|}{n} \leq \sup_{x\in K} |f(x)| \lim_{n\to\infty} \sup_{x\in K} \frac{|\psi^n(x)|}{n}$$

and so

$$\frac{1}{n} \|M_\psi^n f\|_K \leq \|\frac{\psi^n}{n}\|_K \|f\|_K.$$

This implies that $\frac{1}{n} M_\psi^n f \to 0$ as $n \to \infty$ for every $f \in \mathcal{B}(B_X)$ and compact subset K of B_X.

Conversely, if M_ψ is mean ergodic, then from (1.2), we obtain that $\frac{1}{n} M_\psi^n f \to 0$ as $n \to \infty$ for every $f \in \mathcal{B}(B_X)$. For the first case, taking $f \equiv 1$ yields the desired result. Also the sequence $((M_\psi)_{[n]}f)_{n\in\mathbb{N}}$ is convergent in $\mathcal{B}(B_X)$ for every $f \in \mathcal{B}(B_X)$. In particular, it is a $\sigma(\mathcal{B}(B_X), \mathcal{B}(B_X)')$-relatively compact for each $f \in \mathcal{B}(B_X)$. ∎

3. DYNAMICS WITH THE NORM TOPOLOGY

Throughout this section, we consider the Bloch space $\mathcal{B}(B_X)$ endowed with the norm topology $\tau_{\|\cdot\|}$ given in (1.1) and study the multiplication operator

$$M_\psi : (\mathcal{B}(B_X), \tau_{\|\cdot\|}) \to (\mathcal{B}(B_X), \tau_{\|\cdot\|}), \quad M_\psi(f) = \psi f.$$

In this context, we investigate the interplay between power boundedness and (uniform) mean ergodicity, relying on the following lemma, which is stated in [12, Lemma 11].

Definition 3.1. *A Banach space $\mathcal{X}$ of complex-valued functions on a set Ω is said to be a functional Banach space if for each $w \in \Omega$, the point evaluation functional $e_w(f) = f(w)$, $f \in \mathcal{X}$, is bounded; that is, there exists a constant $C > 0$ such that $|f(w)| \leq C\|f\|$ for each $f \in \mathcal{X}$.*

Lemma 3.1. *Let $\mathcal{X}$ be a functional Banach space on the set Ω and let ψ be a complex-valued function on Ω such that $\psi \mathcal{X} \subset \mathcal{X}$. Then the operator M_ψ of multiplication by ψ is a bounded operator on $\mathcal{X}$, and $|\psi(w)| \leq \|M_\psi\|$ for all $w \in \Omega$. In particular, ψ is a bounded function.*

By Lemma 1.3, $\mathcal{B}(B_X)$ is a functional Banach space, and $\|\psi\|_\infty \leq \|M_\psi\|$.

3.1. **Bloch functions on the unit ball of a Banach space.** In this subsection, Bloch functions on the unit ball B_X of an infinite dimensional complex Banach space X are discussed.

Proposition 3.2. *Let $\psi \in H(B_X)$ and M_ψ is a bounded operator on $\mathcal{B}(B_X)$. If $\|\psi\|_\infty > 1$, then $M_\psi : \mathcal{B}(B_X) \to \mathcal{B}(B_X)$ is neither mean ergodic (uniformly mean ergodic) nor power bounded.*

Proof. Observe that, since M_ψ is bounded and satisfies $M_\psi(1) = \psi$ when applied to the constant function $f \equiv 1$, it follows that $\psi \in \mathcal{B}(B_X)$. Now assume that the operator M_ψ is mean ergodic. Then $\|\frac{1}{n}\psi^n f\|_\mathcal{B} \to 0$ as $n \to \infty$ for all $f \in \mathcal{B}(B_X)$. In particular, for $f \equiv 1$, we have $\|\frac{1}{n}\psi^n\|_\mathcal{B} \to 0$ as $n \to \infty$. Moreover by Lemma 1.3, we obtain $\frac{1}{n}|\psi^n(x)| \to 0$ as $n \to \infty$ for all $x \in B_X$. Thus $\|\psi\|_\infty \leq 1$.

Suppose $\|\psi\|_\infty > 1$. Then we can choose $\lambda \in \mathbb{R}$ such that $\|\psi\|_\infty > \lambda > 1$. So there exists $x_0 \in B_X$ such that $|\psi(x_0)| > \lambda$. From this it follows that

$$\|M_\psi^n\| = \|M_{\psi^n}\| \geq \|\psi^n\|_\infty \geq |\psi^n(x_0)| = |\psi(x_0)|^n > \lambda^n > 1.$$

Since $\lim_{n \to \infty} \lambda^n = +\infty$, therefore M_ψ is not power bounded. ∎

Proposition 3.3. *Suppose that M_ψ is a bounded operator on $\mathcal{B}(B_X)$. If $\psi(x) = \xi$ for all $x \in B_X$, where $\xi \in \partial\mathbb{D}$, then M_ψ is uniformly mean ergodic (hence mean ergodic) and power bounded on $\mathcal{B}(B_X)$.*

Proof. Firstly suppose $\psi(x) = \xi$ for $x \in B_X$, where $\xi \in \partial\mathbb{D}$. If $\xi = 1$ then $M_{\psi_{[n]}}f = f$ for every $n \in \mathbb{N}$ and $f \in \mathcal{B}(B_X)$. So clearly M_ψ is uniformly mean ergodic. Now if $\xi \neq 1$ then

$$M_{\psi_{[n]}}f = \frac{\xi}{n}\frac{1 - \xi^n}{1 - \xi}f.$$

In this case for $f \in \mathcal{B}(B_X)$ with $\|f\|_\mathcal{B} \leq 1$ we have

$$
\begin{aligned}
\|M_{\psi_{[n]}}\| &= \sup_{\|f\|_\mathcal{B} \leq 1} \|M_{\psi_{[n]}}f\|_\mathcal{B} \\
&= \sup_{\|f\|_\mathcal{B} \leq 1} \left\{ \frac{1}{n}\frac{|1 - \xi^n|}{|1 - \xi|}|f(0)| + \frac{1}{n}\frac{|1 - \xi^n|}{|1 - \xi|}\sup_{x \in B_X}(1 - \|x\|^2)\|f'(x)\| \right\} \\
&\leq \frac{2}{n|1 - \xi|}\|f\|_\mathcal{B} + \frac{2}{n|1 - \xi|}\|f\|_\mathcal{B} \\
&\leq \frac{4}{n|1 - \xi|}.
\end{aligned}
$$

So $\|M_{\psi_{[n]}}\| \to 0$ when $n \to \infty$ and M_ψ is uniformly mean ergodic.

If $\psi \equiv \xi$ for some $\xi \in \partial\mathbb{D}$, we have that, for any $f \in \mathcal{B}(B_X)$, $M_{\psi^n}f = \xi^n f$ and so $\|M_\psi^n f\|_\mathcal{B} = \|f\|_\mathcal{B}$. Hence M_ψ is power bounded. ∎

3.2. **Bloch functions on the unit ball of a Hilbert space.** In this subsection, Bloch functions on the unit ball B_H of an infinite dimensional complex Hilbert space H are discussed.

A function $\psi \in H(B_H)$ is said to be *multiplier* of $\mathcal{B}(B_H)$ if $M_\psi f \in \mathcal{B}(B_H)$ for all $f \in \mathcal{B}(B_H)$. An application of the closed graph theorem shows that M_ψ is a bounded linear operator on $\mathcal{B}(B_H)$ if and only if ψ is multiplier of $\mathcal{B}(B_H)$.

As mentioned above, Arazy [4] and Brown and Shields [8] characterized the bounded multiplication operators M_ψ on the classical Bloch space $\mathcal{B}(\mathbb{D})$. Now a natural generalization for infinite dimensional Hilbert space H of [8, Theorem 1] proved by Brown and Shields in [8] arises as follows such that we here give a slightly different proof.

Lemma 3.4. *The following are equivalent:*

(i) ψ *is a multiplier on* $\mathcal{B}(B_H)$

(ii) $\psi \in H^\infty(B_H)$ *and*

$$\|\psi^{'}(x)\| = O\left(\frac{1}{(1 - \|x\|^2)\log(2/(1 - \|x\|^2))}\right).$$

Proof. (i) $\Rightarrow$ (ii): Suppose that ψ is multiplier of $\mathcal{B}(B_H)$. First by Lemma 3.1, we have $\psi \in H^\infty(B_H)$ and $\|\psi\|_\infty \leq \|M_\psi\|$. Now let $a \in H$ with $\|a\| < 1$, define the mapping

$$L_a : B_H \to \mathbb{C}, \quad L_a(x) := \frac{1 + \langle x, a\rangle^2}{2}.$$

For every $x \in B_H$, we have $|L_a(x)| < 1$ and therefore $L_a(B_H) \subset \mathbb{D}$. In consequence $1 - L_a(x) \neq 0$ for all $x \in B_H$. Moreover $\|L\| < 1$ and L_a is holomorphic on B_H; that is $L_a \in H(B_H)$ (see also [6, Lemma 8.13]). Finally $L_a^{'}(x)(h) = \langle x, a\rangle\langle h, a\rangle$ for $h \in H$. Now, using the function L_a, define

$$f_a : B_H \to \mathbb{C}, \quad f_a(x) = \log\frac{2}{1 - L_a(x)}.$$

Since f_a is composition of the holomorphic maps $x \to \langle x, a\rangle$, $z \to z^2$ and $z \to \log\frac{4}{1-z}$, it follows that $f_a \in H(B_H)$. By the chain rule, for each $h \in H$, we have

$$f_a^{'}(x)(h) = \frac{L_a^{'}(x)(h)}{1 - L_a(x)} = \frac{2\langle x, a\rangle\langle h, a\rangle}{1 - \langle x, a\rangle^2}.$$

Furthermore, by using $\|L_a^{'}(x)\| = \langle x, a\rangle\|a\|$, we obtain that

$$(1 - \|x\|^2)\|f_a^{'}(x)\| = (1 - \|x\|^2)\frac{2|\langle x, a\rangle|\|a\|}{|1 - \langle x, a\rangle^2|} \leq 2\|a\|^2$$

so

$$\|f_a\|_\mathcal{B} = |f_a(0)| + (1 - \|x\|^2)\|f_a^{'}(x)\| \leq 2\log 2 + 2\|a\|^2 < 3.$$

In consequence $f_a \in \mathcal{B}(B_H)$. Since f_a is in $\mathcal{B}(B_H)$ and ψ is multiplier of $\mathcal{B}(B_H)$, we have

$$\|\psi f_a\|_\mathcal{B} = \|M_\psi f_a\|_\mathcal{B} \leq \|M_\psi\|\|f_a\|_\mathcal{B} \leq 3\|M_\psi\|.$$

From this fact, we obtain

$$
\begin{aligned}
(1 - \|x\|^2)\|\psi^{'}(x)\||f_a(x)| &\leq 3\|M_\psi\| + (1 - \|x\|^2)\|f_a^{'}(x)\||\psi(x)| \\
&\leq 3\|M_\psi\| + \|f_a\|_\mathcal{B}\|\psi\|_\infty \\
&\leq 6\|M_\psi\|.
\end{aligned}
$$

Hence

$$(1 - \|x\|^2)\|\psi^{'}(x)\|\left|\log\frac{4}{1 - \langle x, a\rangle^2}\right| \leq 6\|M_\psi\|.$$

Now, for each $x \in B_H \setminus \{0\}$, consider $a = rx/\|x\|$ with $0 < r < 1$. Substituting into the above inequality and taking the limit as $r \to 1^-$, it follows that

$$(1 - \|x\|^2)\|\psi'(x)\|\log\frac{2}{1 - \|x\|^2} \leq 6\|M_\psi\|.$$

This gives us the desired inequality.

(ii) $\Rightarrow$ (i): Observe that $(\psi f)^{'}(x) = \psi^{'}(x)f(x) + \psi(x)f^{'}(x)$ for all $x \in B_H$. Then by Lemma 1.3, we have

$$
\begin{aligned}
(1 - \|x\|^2)\|(\psi f)^{'}(x)\| &\leq (1 - \|x\|^2)\|\psi^{'}(x)\||f(x)| + |\psi(x)|(1 - \|x\|^2)\|f^{'}(x)\| \\
&\leq \left((1 - \|x\|^2)\|\psi^{'}(x)\|C\log\frac{2}{1 - \|x\|^2} + \|\psi\|_\infty\right)\|f\|_\mathcal{B}.
\end{aligned}
$$

Thus $\psi f \in \mathcal{B}(B_H)$. ∎

Note that the method of the proof of [13, Theorem 2.2] may be adapted to infinite dimensional Hilbert space, so we get the following corollary.

Corollary 3.5. *ψ is an analytic function on B_H inducing a bounded multiplication operator M_ψ on $\mathcal{B}(B_H)$. Then*

$$\|M_\psi\| \leq \max\{\|\psi\|_\mathcal{B}, \|\psi\|_\infty + \sigma_\psi\}$$

where

$$\sigma_\psi := \sup_{x \in B_H} \frac{1}{2}(1 - \|x\|^2)\|\psi^{'}(x)\| \log \frac{1 + \|x\|}{1 - \|x\|}.$$

Lemma 3.6. *Suppose ψ is a holomorphic function on B_H inducing a bounded multiplication operator M_ψ on $\mathcal{B}(B_H)$. If $\|\psi\|_\infty < 1$ or $\psi(x) = \xi$ for $x \in B_H$, where $\xi \in \partial\mathbb{D}$, then $M_\psi : \mathcal{B}(B_H) \to \mathcal{B}(B_H)$ is power bounded.*

Proof. First suppose that $\|\psi\|_\infty < 1$ and let $f \in \mathcal{B}(B_H)$. Then

$$
\begin{aligned}
\|M_\psi^n f\|_\mathcal{B} &= |\psi^n(0)||f(0)| + (1 - \|x\|^2)\|f^{'}(x)\||\psi^n(x)| \\
&+ (1 - \|x\|^2)|f(x)|\|\psi^{'}(x)\|n|\psi^{n-1}(x)| \\
&\leq \|f\|_\mathcal{B} + (1 - \|x\|^2)C \log \frac{2}{1 - \|x\|^2}\|\psi^{'}(x)\|n|\psi^{n-1}(x)|\|f\|_\mathcal{B}
\end{aligned}
$$

for all $x \in B_H$. By Lemma 3.4, since

$$M := \sup\{(1 - \|x\|^2)\|\psi^{'}(x)\| \log \frac{2}{1 - \|x\|^2} : x \in B_H\} < \infty,$$

we have

$$\|M_\psi^n f\|_\mathcal{B} \leq \|f\|_\mathcal{B}\big(1 + CMn|\psi^{n-1}(x)|\big)$$

for all $x \in B_H$. But since $\|\psi\|_\infty < 1$, we have $\lim_n n\|\psi\|_\infty^{n-1} = 0$ and therefore $\{n\|\psi\|_\infty^{n-1}\}_{n \in \mathbb{N}}$ is a bounded sequence. Let $K = \sup_{n \in \mathbb{N}} n\|\psi\|_\infty^{n-1}$, then we conclude

$$\|M_\psi^n f\|_\mathcal{B} \leq \|f\|_\mathcal{B}(1 + CMK).$$

So, in this first case M_ψ is power bounded.

Moreover, if $\psi \equiv \gamma$ where $\gamma \in \partial\mathbb{D}$, M_ψ is an isometry on $\mathcal{B}(B_H)$, and for all $n \geq 1$, $\|M_\psi^n\| = 1$, therefore M_ψ is power bounded. ∎

We show that a multiplication operator M_ψ on the Bloch space $\mathcal{B}(B_H)$ of an infinite-dimensional Hilbert space H is not necessarily power bounded, even if the mapping ψ satisfies $\|\psi\|_\infty = 1$ and is non-constant.

Example 3.1. Let $u \in H$ be a unit vector and define the mapping $\psi(x) = \langle x, u \rangle$ for $x \in B_H$. It is evident that $\|\psi\|_\infty = 1$. Consider the test function $f(x) = (-1/2) \log(1 - \langle x, u \rangle)$, which belongs to $\mathcal{B}(B_H)$ with

$$\sup_{x \in B_H} (1 - \|x\|^2)\|f^{'}(x)\| \leq \sup_{x \in B_H} \frac{1 - \|x\|^2}{2|1 - \langle x, u \rangle|} \leq 1.$$

and take $x_n = (1 - \frac{1}{n})u \in B_H$ for each $n \in \mathbb{N}$. Then

$$
\begin{aligned}
\|M_\psi^n f\|_{\mathcal{B}} &\geq \sup_{x \in B_H} (1 - \|x\|^2)\|(\psi^n f)^{'}(x)\| \\
&\geq \frac{1}{2}(1 - \|x_n\|^2) \left| n\langle x_n, u\rangle^{n-1} \log \frac{1}{1 - \langle x_n, u\rangle} + \frac{\langle x_n, u\rangle^n}{1 - \langle x_n, u\rangle} \right| \\
&\geq \frac{1}{2}(1 - \frac{1}{n})^{n-1} \log n.
\end{aligned}
$$

Thus $\|M_\psi^n f\|_{\mathcal{B}} \geq C \log n \to \infty$ as $n \to \infty$, where $C > 0$ is a constant. This proves that $M_\psi : \mathcal{B}(B_H) \to \mathcal{B}(B_H)$ is not power bounded.

We show that the result obtained in [3] for the classical Bloch space remains valid even if H is an infinite dimensional Hilbert space. Moreover by generalizing Proposition 3.3, we establish the following result for Hilbert spaces, which allows for a larger admissible class of multipliers ψ compared to the Banach spaces.

Proposition 3.7. *Suppose that M_ψ is a bounded operator on $\mathcal{B}(B_H)$. If $\|\psi\|_\infty < 1$ or $\psi(x) = \xi$ for $x \in B_H$, where $\xi \in \partial\mathbb{D}$, then M_ψ is uniformly mean ergodic (and hence mean ergodic) on $\mathcal{B}(B_H)$.*

Proof. The case $\psi(x) = \xi$ for $x \in B_H$, where $\xi \in \partial\mathbb{D}$, has already been established in Proposition 3.3. We now turn to the remaining case. Let's use Corollary 3.5 to show that M_ψ is uniformly mean ergodic if $\|\psi\|_\infty < 1$. There are two cases to consider:

If $\|M_{\psi^n}\| \leq \|\psi^n\|_{\mathcal{B}}$ for every $n \in \mathbb{N}$, since $\|\psi\|_{\mathcal{B}} \leq \|\psi\|_\infty$, we have

$$
(3.1) \qquad \|M_\psi^n\| \leq \|\psi^n\|_{\mathcal{B}} \leq |\psi^n(0)| + \|\psi^n\|_\infty \leq |\psi^n(0)| + \|\psi\|_\infty^n.
$$

If $\|M_{\psi^n}\| \leq \|\psi^n\|_\infty + \sigma_{\psi^n}$, by Lemma 3.4, we obtain

$$
\begin{aligned}
\|M_\psi^n\| &\leq \|\psi^n\|_\infty + \sup_{x \in B_H} \frac{n}{2}(1 - \|x\|^2)\|\psi^{'}(x)\|\, |\psi(x)|^{n-1} \log \frac{1 + \|x\|}{1 - \|x\|} \\
&\leq \|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1} \sup_{x \in B_H} \frac{1}{2}(1 - \|x\|^2)\|\psi^{'}(x)\| \log \frac{1 + \|x\|}{1 - \|x\|} \\
&\leq \|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1} \left[\sup_{x \in B_H} (1 - \|x\|^2)\|\psi^{'}(x)\| \log \frac{2}{1 - \|x\|^2} \right] \\
(3.2) \qquad &\leq \|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1} O(1).
\end{aligned}
$$

So by (3.1) and (3.2) for all $n \in \mathbb{N}$, we obtain

$$
\|M_\psi^n\| \leq \max\left\{ |\psi^n(0)| + \|\psi\|_\infty^n, \|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1} O(1) \right\}.
$$

In this here we have

$$
\lim_{n\to\infty} (|\psi^n(0)| + \|\psi\|_\infty^n) = 0
$$

and similarly

$$
\lim_{n\to\infty} (\|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1}) = 0.
$$

This implies that M_ψ^n is convergent to zero in operator norm on $\mathcal{B}(B_H)$. Using Proposition 1.1, we obtain that M_ψ is uniformly mean ergodic. ∎

We give an example showing that, on the Bloch space $\mathcal{B}(B_H)$ of an infinite-dimensional Hilbert space H, the multiplication operator M_ψ need not be mean ergodic, even when the mapping ψ is non-constant and satisfies $\|\psi\|_\infty = 1$.

Example 3.2. Let $u \in H$ be a unit vector and define the mapping $\psi(x) = \langle x, u \rangle^2$ for $x \in B_H$. Clearly that $\|\psi\|_\infty = 1$. Consider $f(x) = (-1/2) \log(1 - \langle x, u \rangle)$, $x \in B_H$, which belongs to $\mathcal{B}(B_H)$ and satisfies $\|f\|_\mathcal{B} \le 1$. Take $x_n = \left(1 - \frac{1}{n}\right) u \in B_H$ for each $n \in \mathbb{N}$. Moreover, define the Cesàro means

$$(M_\psi)_{[n]} f = \frac{1}{n} \sum_{k=1}^n M_\psi^k f = \frac{1}{n} \sum_{k=1}^n \langle x, u \rangle^{2k} f(x), \quad (n \in \mathbb{N}).$$

It follows that

$$\|(M_\psi)_{[n]} f\|_\mathcal{B} \ge (1 - \|x_n\|^2) \frac{1}{n} \left| \sum_{k=1}^n \left(2k (1 - \frac{1}{n})^{2k-1} f(x_n) + (1 - \frac{1}{n})^{2k} f'(x_n) \right) \right|$$

$$\ge \frac{2n-1}{n^2} \frac{\log n}{n} \sum_{k=1}^n k \left(1 - \frac{1}{n} \right)^{2k-1}.$$

Since $(1 - \frac{1}{n})^{2k-1} \ge (1 - \frac{1}{n})^{2n}$ for $1 \le k \le n$, we obtain that

$$\|(M_\psi)_{[n]} f\|_\mathcal{B} \ge (1 - \frac{1}{n})^{2n} \log n.$$

Consequently, $M_\psi : \mathcal{B}(B_H) \to \mathcal{B}(B_H)$ is neither mean ergodic nor uniformly mean ergodic.

Considered together with previous results, the following theorem is the main conclusion of this section.

Theorem 3.8. *Suppose that M_ψ is a bounded operator on $\mathcal{B}(B_H)$. If $\|\psi\|_\infty < 1$ or $\psi(x) = \xi$ for $x \in B_H$, where $\xi \in \partial \mathbb{D}$, then the following holds.*
   (i) *M_ψ is uniformly mean ergodic.*
   (ii) *M_ψ is mean ergodic.*
   (iii) *M_ψ is power bounded.*

Recall that it was proved in [11] that the multiplication operator M_ψ on $\mathcal{B}(B_H)$ given by $M_\psi(f) = \psi f$ is invertible if and only if $\frac{1}{\psi} \in H^\infty(B_H)$.

Theorem 3.9. *Let M_ψ be a bounded linear operator on $\mathcal{B}(B_H)$. Then $\sigma(M_\psi) = \overline{\psi(B_H)}$.*

Proof. Let $\lambda \notin \sigma(M_\psi)$. This implies that $M_{\psi - \lambda}$ is invertible so $M_{\psi - \lambda}^{-1} = M_{(\psi - \lambda)^{-1}}$. This means that the function $(\psi - \lambda)^{-1}$ will not possess a pole at any point of B_H. Therefore, $\psi(x) - \lambda \ne 0$ for every $x \in B_H$, that is, $\lambda \notin \psi(B_H)$. This line of reasoning establishes that the image set $\psi(B_H)$ is a subset of M_ψ's spectrum which is a closed set: $\overline{\psi(B_H)} \subseteq \sigma(M_\psi)$.

Conversely, let $\lambda \notin \overline{\psi(B_H)}$. Then there exists $c > 0$ such that $|\psi(x) - \lambda| \ge c$ for all $x \in B_H$. Define $g(x) = (\psi(x) - \lambda)^{-1}$. Then $g \in H^\infty(B_H)$ and $\|g\|_\infty \le 1/c$. Moreover,

$$g'(x) = -\frac{\psi'(x)}{(\psi(x) - \lambda)^2},$$

and hence

$$\sigma_g \le \frac{1}{c^2} \sup_{x \in B_H} \frac{1}{2} (1 - \|x\|^2) \|\psi'(x)\| \log \frac{1 + \|x\|}{1 - \|x\|} = \frac{1}{c^2} \sigma_\psi < \infty.$$

Thus $g \in \mathcal{B}(B_H)$ and $M_{\psi - \lambda}$ is invertible, which yields $\lambda \notin \sigma(M_\psi)$. So $\sigma(M_\psi) \subseteq \overline{\psi(B_H)}$. ∎

Definition 3.2. For a holomorphic function $f : B_H \to \mathbb{C}$, consider

$$\|f\|_\mathcal{R} := \sup_{x \in B_H} (1 - \|x\|^2) |\mathcal{R}f(x)|, \qquad \mathcal{R}f(x) := f'(x)(x),$$

where $\mathcal{R}f(x)$ denotes the radial derivative of f at x. Moreover, consider

$$\|f\|_{\text{weak}} := \sup_{\|y\|=1} \|f_y\|_{\mathcal{B}}, \qquad f_y(z) := f(zy),\ |z| < 1,$$

for each $y \in H$ with $\|y\| = 1$, and

$$\|f\|_{\text{inv}} := \sup_{x \in B_H} \|\tilde{\nabla}f(x)\|.$$

Here the invariant gradient $\tilde{\nabla}f$ is defined by $\tilde{\nabla}f(x) = \nabla(f \circ \varphi_x)(0)$ for $x \in B_H$ and φ_x denotes the automorphism of B_H associated with x (for these topics, see [9]).

We denote $\mathcal{B}_{\mathcal{R}}(B_H)$, $\mathcal{B}_{weak}(B_H)$ and $\mathcal{B}_{inv}(B_H)$ the space of holomorphic functions on B_H for which $\|f\|_{\mathcal{R}} < \infty$, $\|f\|_{weak} < \infty$ and $\|f\|_{inv} < \infty$, respectively. As usual, $|f(0)| + \|f\|_{\mathcal{R}}$, $|f(0)| + \|f\|_{weak}$ and $|f(0)| + \|f\|_{inv}$ are complete norms in these spaces. In the proofs of Theorem 2.6 and Theorem 3.8 in [9] , it is shown that the four spaces $\mathcal{B}(B_H)$, $\mathcal{B}_{\mathcal{R}}(B_H)$, $\mathcal{B}_{weak}(B_H)$ and $\mathcal{B}_{inv}(B_H)$ coincide and that their norms are actually equivalent.

Let $\mathcal{H}$ stand for the spaces $\mathcal{B}(B_H)$, $\mathcal{B}_{\mathcal{R}}(B_H)$, $\mathcal{B}_{weak}(B_H)$ or $\mathcal{B}_{inv}(B_H)$ for the following corollaries and theorem.

Corollary 3.10. *Suppose that M_ψ is a bounded operator on $\mathcal{H}$. If $\|\psi\|_\infty < 1$ or $\psi(x) = \xi$ for $x \in B_H$, where $\xi \in \partial\mathbb{D}$, then the following holds.*

    (i) *M_ψ is uniformly mean ergodic.*
    (ii) *M_ψ is mean ergodic.*
    (iii) *M_ψ is power bounded.*

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