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BOUNDEDNESS OF THE HILBERT TRANSFORM: AN ELEMENTARY APPROACH

RENÉ ERLÍN CASTILLO, HÉCTOR CAMILO CHAPARRO, AND OSCAR MAURICIO GUZMÁN

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DEPARTMENT OF MATHEMATICS, UNIVERSIDAD NACIONAL DE COLOMBIA, BOGOTÁ, COLOMBIA.
recastillo@unal.edu.co

PROGRAM OF MATHEMATICS, UNIVERSIDAD DE CARTAGENA, CARTAGENA DE INDIAS, COLOMBIA.
hchaparrog@unicartagena.edu.co

DEPARTMENT OF MATHEMATICS, UNIVERSIDAD ESCUELA COLOMBIANA DE INGENIERÍA JULIO GARAVITO, BOGOTÁ, COLOMBIA.
oscar.guzman-f@escuelaing.edu.co

ABSTRACT. This article provides a straightforward demonstration of the boundedness of the Hilbert transform on $L_p(\mathbb{R})$ spaces for $1 < p < \infty$. By leveraging Hölder's inequality alone, our method eschews advance technique from harmony analysis, yielding a more accessible streamlined approach to a fundamental result in operator theory.

Key words and phrases: Hilbert Transform, Hölder's inequality, Fourier integral.

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1. INTRODUCTION

The Hilbert transform is a fundamental concept in mathematics and signal processing, named after the German mathematician David Hilbert. The transform has a rich history dating back to the early 20th century.

David Hilbert, a renowned mathematician, introduced the Hilbert transform in the context of his work on integral equations and boundary value problems. In the 1900s, Hilbert's work on the theory of integral equations led him to study the properties of certain singular integral operators. One of these operators, now known as the Hilbert transform, associates a function with its conjugate harmonic function.

The Hilbert transform gained significant attention in the 1920s and 1930s, particularly through the work of mathematicians such as Godfrey Harold Hardy and Marcel Riesz. They developed the theory of the Hilbert transform further, exploring its connections to complex analysis, harmonic analysis, and potential theory.

In the mid-20th century, the Hilbert transform found applications in signal processing and communication theory. The work of researchers like Dennis Gabor and John Ville led to the development of analytic signal theory, which relies heavily on the Hilbert transform. The transform's ability to generate a signal's analytic representation, which has a one-sided spectrum, proved invaluable in signal processing and modulation analysis.

Today, the Hilbert transform is a cornerstone of signal processing, with applications in various fields, including:

(1) Signal analysis and processing
(2) Communication theory
(3) Image processing
(4) Medical imaging
(5) Optics

The Hilbert transform has also been generalized and extended in various ways, including the development of discrete Hilbert transforms and Hilbert-Huang transforms.

2. HISTORICAL DEDUCTION OF THE HILBERT TRANSFORM

Let us start by considering a complex number say $C_n = a_n - ib_n$ as well as the function

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} C_n e^{in\theta} r^n \\ &= \sum_{n=0}^{\infty} C_n z r^n \end{aligned} \tag{2.1}$$

where $z = re^{in\theta}$. We shall assume that F is analytic in the unit disc. Since

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta),$$

we can write (2.1) as follows

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} (a_n - ib_n)\left(\cos(n\theta) + i\sin(n\theta)\right) r^n \\ &= \sum_{n=0}^{\infty} \left[(a_n\cos(n\theta) + b_n\sin(n\theta)) + i\left(a_n\sin(n\theta) - b_n\cos(n\theta)\right)\right] r^n. \end{aligned}$$

On the one hand we can write F as follows

$$F(r,\theta) = U(r,\theta) + i\widetilde{U}(r,\theta), \quad |r| < 1$$

being U the real part and $\widetilde{U}$ the imaginary part of F where

$$U(r,\theta) = \sum_{n=0}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n$$

and

$$\widetilde{U}(r,\theta) = \sum_{n=0}^{\infty} \left(a_n \sin(n\theta) - b_n \cos(n\theta)\right) r^n.$$

Also U and $\widetilde{U}$ are harmonic in the unit disc. They are called harmonic conjugate. Formally we have

$$\lim_{r\to 1^-} U(r,\theta) = f(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$$

and

$$\lim_{r\to 1^-} \widetilde{U}(r,\theta) = -\widetilde{f}(\theta) = -\sum_{n=0}^{\infty} b_n \cos(n\theta) - a_n \sin(n\theta).$$

Since they are the limits of conjugate functions, f and $\widetilde{f}$ are so called conjugate functions.

The transformation $H : f \mapsto \widetilde{f}$ i.e $H(f) = \widetilde{f}$ is called the Hilbert transform. With this idea in mind we can use the latter technique to develop the coming result.

Let f be a integrable function and define $a(t)$ and $b(t)$ by

$$\begin{aligned} a(t) =& \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(tx)\, dx \\ &\text{and} \\ b(t) =& \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(tx)\, dx. \end{aligned} \tag{2.2}$$

Consider the function $F(z)$ defined by the Fourier integral

$$\begin{aligned} F(z) =& \int_0^{\infty} (a(t) - ib(t)) e^{izt}\, dt \\ =& U(z) + i\widetilde{U}(z) \end{aligned} \tag{2.3}$$

where $z = x + iy$.

It is matter of computation to see that F is analytic in the upper half plane, $y > 0$. The real and imaginary parts of F are

$$\begin{aligned} U(z) =& \int_0^{\infty} \left(a(t)\cos(xt) + b(t)\sin(xt)\right) e^{-yt}\, dt \\ &\text{and} \\ \widetilde{U}(z) =& \int_0^{\infty} \left(a(t)\sin(xt) - b(t)\cos(xt)\right) e^{-yt}\, dt. \end{aligned} \tag{2.4}$$

Formally we have

$$\lim_{y\to 0} U(z) = f(x) = \int_0^{\infty} \left(a(t)\cos(xt) + b(t)\sin(xt)\right)\, dt$$

and

$$\lim_{y\to 0}\widetilde{U}(z) = -\widetilde{f}(x) = \int_0^\infty \left(a(t)\sin(xt) + b(t)\cos(xt)\right)\, dt.$$

Now, let us consider the conjugate function, namely

$$\begin{aligned}
\widetilde{f}(x) =& \int_0^\infty \left(a(t)\sin(xt) - b(t)\cos(xt)\right)\, dt \\
=& \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \left[f(u)\left(\cos(tu)\sin(xt) - \sin(tu)\cos(xt)\right)\right]\, dudt \\
=& \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\sin(t(u-x))\, dudt \\
=& \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^\infty \int_0^\lambda f(u)\sin(t(u-x))\, dtdu \quad \text{(by Fubini)} \\
=& \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^\infty \int_0^\lambda f(u)\left(\frac{1-\cos(\lambda(u-x))}{u-x}\right)\, du \\
=& \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^\infty f(u+x)\,\frac{1-\cos(\lambda u)}{u}\, du \\
=& \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^\infty \frac{f(u+x)}{u}\, du - \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^\infty \frac{f(u+x)\cos(\lambda u)}{u}\, du. \qquad (2.5)
\end{aligned}$$

If $\frac{f(u)}{u}$ is integrable over $(-\infty,\infty)$ it follows from the Riemann-Lebesgue Theorem that

$$\lim_{\lambda\to\infty}\int_{-\infty}^\infty \frac{f(u+x)}{u}\cos(\lambda u)\, du = 0.$$

In this sense, (2.5) may be written as

$$\begin{aligned}
\widetilde{f}(x) =& \frac{1}{\pi}\int_{-\infty}^\infty \frac{f(u+x)}{u}\, du \\
=& \frac{1}{\pi}\int_0^\infty \frac{f(u+x)-f(x-u)}{u}\, du \\
=& \frac{1}{\pi}\int_{-\infty}^\infty \frac{f(u)}{u-x}\, du.
\end{aligned}$$

Hence

$$\begin{aligned}
Hf(x) =& \widetilde{f}(x) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{f(u)}{u-x}\, du \\
=& \lim_{\epsilon\to 0}\frac{1}{\pi}\int\limits_{|u-x|>\epsilon} \frac{f(u)}{u-x}\, du \\
=& \lim_{\epsilon\to 0}\frac{1}{\pi}\int\limits_{|u|>\epsilon} \frac{f(x-u)}{u}\, du.
\end{aligned}$$

We dare to say that this is the germinal idea that gave rise to the Hilbert transform. Later, the concept of singular integrals emerged from the Hilbert transform, ultimately giving birth to what is now know as singular integral theory.

Definition 2.1. The Hilbert transform of a sufficiently well-behaved function $f(x)$ is defined to be

$$\begin{aligned} Hf(x) =& \text{p.v}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,dy \\ &= \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int\limits_{|x-y|>\epsilon}\frac{f(y)}{x-y}\,dy \\ &= -\lim_{\epsilon\to 0^+}\frac{1}{\pi}\int\limits_{|t|>\epsilon}\frac{f(x-t)}{t}\,dt. \end{aligned}$$

p.v stands for the principal value of the integral (see [1]).

3. BOUNDEDNESS OF THE HILBERT TRANSFORM

We begin section by stating and proving the coming result.

Lemma 3.1. *Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence such that $\epsilon_n \downarrow 0$, and let f be a measurable and locally integrable on $\mathbb{R}$. Then*

$$\lim_{n\to\infty}\int\limits_{|x-y|>\epsilon_n}\frac{|f(y)|}{|x-y|}\,dy = \int\limits_{\mathbb{R}\setminus\{x\}}\frac{|f(y)|}{|x-y|}\,dy.$$

Proof. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be any sequence with $\epsilon_n \downarrow 0$ (for example $\epsilon_n = \frac{1}{n}$). For each n, define the function

$$g_n(y) = \chi_{\{|y-x|>\epsilon_n\}}(y)\frac{|f(y)|}{|x-y|},$$

and define

$$g(y) = \chi_{y\neq x}(y)\frac{|f(y)|}{|x-y|} = \begin{cases} \frac{|f(y)|}{|x-y|} & \text{if} \quad y\neq x \\ 0 & \text{if} \quad y = x. \end{cases}$$

(choice at the single point does not matter). Notice that, the measurability of f implies that $|f(y)|$ is measurable.

The map $y \mapsto |x-y|$ is continuous, hence measurable and positive for $y \neq x$. The characteristic functions

$$\chi_{\{|x-y|>\epsilon_n\}} \quad \text{and} \quad \chi_{\{x\neq y\}}$$

are measurable. Therefore, g_n ($n\in\mathbb{N}$) and g are measurable nonnegative functions on $\mathbb{R}$.

Now fix $y\in\mathbb{R}$. If $x\neq y$ then for sufficiently large n we have $|x-y| > \epsilon_n$ so for every n beyond that the index the characteristic

$$\chi_{\{|x-y|>\epsilon_n\}}(y) = 1.$$

Consequently

$$g_n(y) = \frac{|f(y)|}{|x-y|} \quad \text{for all large } n$$

and therefore

$$g_n(y) \longrightarrow g(y) = \frac{|f(y)|}{|x-y|}$$

as $n\to\infty$.

If $y = x$ then for every n, we have $|x-y| = 0 \leq \epsilon_n$ so

$$\chi_{\{|x-y|>\epsilon_n\}}(x) = 0$$

and hence $g_n(x) = 0$ for all n. We define $g(x) = 0$, so

$$g_n(x) \underset{n\to\infty}{\longrightarrow} g(x),$$

thus $g_n(y) \underset{n\to\infty}{\longrightarrow} g(y)$ pointwise for every $y \in \mathbb{R}$. Moreover, because $\epsilon_{n+1} \leq \epsilon_n$ the sets $\{y : |y - x| > \epsilon_n\}$ increase as n increase, hence the characteristic increase pointwise, therefore

$$0 \leq g_1(y) \leq g_2(y) \leq \cdots \leq g_n(y) \leq \cdots \leq g(y)$$

for every y. So $\{g_n(y)\}_{n\in\mathbb{N}}$ is a monotone nondecreasing sequence of measurable nonnegative functions with pointwise limit $g(y)$.

Hence the hypotheses for the Monotone Convergence Theorem (MTC) are satisfied (measurability, nonnegativity, monotone increasing). Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n(y)\, dy = \int_{\mathbb{R}} \lim_{n\to\infty} g_n(y)\, dy = \int_{\mathbb{R}} g(y)\, dy.$$

Since, by definition

$$\int_{\mathbb{R}} g_n(y)\, dy = \int_{|y-x|>\epsilon_n} \frac{|f(y)|}{|x-y|}\, dy$$

and

$$\int_{\mathbb{R}} g(y)\, dy = \int_{\mathbb{R}\setminus\{x\}} \frac{|f(y)|}{|x-y|}\, dy.$$

Therefore MTC gives exactly

$$\lim_{n\to\infty} \int_{|y-x|>\epsilon_n} \frac{|f(y)|}{|x-y|}\, dy = \int_{\mathbb{R}\setminus\{x\}} \frac{|f(y)|}{|x-y|}\, dy$$

as we wished to show. ∎

We are ready to state and prove our main result, i.e., the boundedness of the Hilbert transform.

Theorem 3.2. *Let $f \in L_p(\mathbb{R})$ with $1 \leq p < \infty$. Then*

$$\|Hf\|_{L^p} \leq \frac{3}{2}\|f\|_{L^p}. \tag{3.1}$$

Proof. In order to prove (3.1), we will use Lemma 3.1 in conjunction with Hölder's inequality. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence such that $\epsilon_n \downarrow \epsilon$.

Next, we consider

$$\begin{aligned}
\int_{\mathbb{R}} |Hf(x)|^p\,dx &= \int_{\mathbb{R}} \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\right|^p dx \\
&= \int_{\mathbb{R}} \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\right| \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\right|^{p-1} dx \\
&= \int_{\mathbb{R}} \left|\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\,dy\right| \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\right|^{p} dx \\
&= \int_{\mathbb{R}} \left|\liminf_{n\to\infty}\int_{|x-y|>\epsilon_n} \frac{f(y)}{x-y}\,dy\right| \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\right|^{p-1} dx \\
&\le \int_{\mathbb{R}} \liminf_{n\to\infty}\int_{|x-y|>\epsilon_n} \frac{|f(y)|}{|x-y|}\,dy \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\right|^{p-1} dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{x\}} \frac{|f(y)|}{|x-y|}\,dy \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\right|^{p-1} dx \\
&\le \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(y)|}{|x-y|}\,dy \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\right|^{p-1} dx \\
&= \int_{\mathbb{R}} \frac{dy}{|x-y|} \int_{\mathbb{R}} |f(x)| \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\right|^{p-1} dx \\
&\le \int_{\mathbb{R}} \frac{dy}{|x-y|} \left(\int_{\mathbb{R}} |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \left|\text{p.v}\int_{\mathbb{R}} \frac{f(y)}{x-y}\right|^p dx\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}} \frac{dy}{|x-y|}\right) \|f\|_{L^p} \left(\int_{\mathbb{R}} |Hf(x)|^p\,dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$\left(\int_{\mathbb{R}} |Hf(x)|^p\,dx\right)^{1-\frac{1}{q}} \le \left(\int_{\mathbb{R}} \frac{dy}{|x-y|}\right) \|f\|_{L^p}$$

and so

$$\begin{aligned}
\|Hf(x)\|_{L^p} &\leq \left(\int\limits_{|x-y|<r} \frac{dy}{|x-y|} + \int\limits_{|x-y|>r} \frac{dy}{|x-y|} \right) \|f\|_{L_p} \\
&= \left(\sum_{k=1}^{\infty} \int\limits_{2^{k+1}\leq|x-y|<2^k} \frac{dy}{|x-y|} + \sum_{k=1}^{\infty} \int\limits_{2^k\leq|x-y|<2^{k+1}} \frac{dy}{|x-y|} \right) \|f\|_p \\
&\leq \left(\sum_{k=1}^{\infty} \frac{1}{2^{k+1}} + \sum_{k=1}^{\infty} \frac{1}{2^k} \right) \|f\|_p \\
&= \left(\frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \right) \|f\|_p \\
&= \frac{3}{4}(2)\|f\|_p = \frac{3}{2}\|f\|_p
\end{aligned}$$

as we wished to prove. ∎

Observation: Note that the Hardy constant $\frac{p}{p-1}$ is satisfied in our case with $p = 3$, yielding

$$\frac{3}{3-1} = \frac{3}{2}.$$

In a recent paper, (see [2]) the authors obtained 2 as a bound for the discrete Hilbert transform. This bound aligns perfectly with the Hardy constant $\frac{p}{p-1}$ for $p = 2$.

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