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SOME NEW BOUNDS FOR THE DIRICHLET BETA FUNCTION

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ABSTRACT. In this paper, we establish new upper and lower bounds for the Dirichlet beta function $\beta(x)$, with particular emphasis on its values at even integers. We derive inequalities that relate pairs of Dirichlet beta values and develop a unified approach that draws on tools from probability theory and reliability theory, together with analytic techniques, to obtain these results. Numerical comparisons illustrate the effectiveness of the proposed bounds relative to existing estimates; in several cases, our bounds are demonstrably sharper than those previously reported in the literature, including those of [3].

Key words and phrases: Dirichlet beta function; inequalities; special functions; reliability theory; moment inequalities.

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1. INTRODUCTION

The Dirichlet beta function is defined as

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}, \quad x > 0. \tag{1.1}$$

It has a large number of applications in various areas of mathematics and physics. See [3] and the references given there. An integral representation for $\beta(x)$ is

$$\beta(x) = \frac{1}{2\,\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{\cosh(t)}\, dt = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + e^{-t}}\, dt. \tag{1.2}$$

where

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u}\, du, \tag{1.3}$$

is the gamma function. It is easier to obtain closed-form expressions for $\beta(2n+1)$ than for $\beta(2n)$, $n \geq 1$. See [3], pp. 4–5. In this paper, one of our main goals is to obtain upper and lower bounds for $\beta(2n)$ using $\beta(k)$ for surrounding odd values of k. Several different methods are used to obtain the new bounds for $\beta(x)$.

The Dirichlet beta function $\beta(s) = \sum_{k\geq 0}(-1)^k(2k+1)^{-s}$ is a well-known Dirichlet L-series, and its special values include Catalan's constant $G = \beta(2)$. In addition to standard analytic techniques such as functional equations, integral representations, and special-value identities, many studies aim to produce explicit evaluations and computationally convenient formulas for $\beta(s)$ and related constants. For example, Malmsten-type integrals have been used to obtain closed forms and fast-converging expressions for zeta- and L-values that cover $\beta(s)$ at integer arguments; see, for instance, [2]. Additional computational formulas follow from relationships with special functions; for example, evaluations of polygamma functions at rational arguments can be used to derive explicit expressions for G and $\beta(2n)$; see [7] and the references cited there.

Sharp bounds for $\beta(s)$ have been developed as well, motivated by numerical approximation and analytic estimation. A commonly used reference point is the two-sided bounding method in [3], together with later improvements such as the refined double-inequality bounds in [4]. Other works rely on the structural properties of β, including monotonicity, convexity, and log-convexity. These properties can then be converted into systematic families of inequalities and comparison estimates; see, for example, [9]. There is also a computational approach focused on efficient evaluation of β-values, including Markov–WZ and related acceleration methods that yield rapidly convergent series for $\beta(2n)$; see [6]. Finally, the number-theoretic properties of these values have also been examined, particularly regarding the Diophantine aspects of Catalan-type constants [10, 12]. In this work, we also present bounds that do not rely on $\beta(n)$ for odd values of n; these estimates are often accurate and require relatively little computational effort.

The rest of the paper is structured as follows. In Section 2, we present previously proposed bounds for $\beta(x)$. In Section 3, we present new bounds for $\beta(x)$. In Section 4, we make some numerical comparisons of the old bounds and the new bounds. Finally, in Section 5, we discuss future research and conclusions. We are primarily concerned with previously proposed bounds of [3].

2. PREVIOUSLY PROPOSED BOUNDS FOR $\beta(x)$, PRELIMINARIES AND LEMMAS

For the Dirichlet beta function, we shall often be concerned with the bounds given in [3]. To keep the paper self-contained, we state some of these bounds here for the convenience of the reader.

Theorem 2.1 (Theorem 5.2 of [3])**.** *For real numbers $x > 0$,*

$$\frac{c^*}{3^{x+1}} < \beta(x+1) - \beta(x) < \frac{d^*}{3^{x+1}}, \tag{2.1}$$

with best possible constants

$$c^* = 3\left(\frac{\pi}{4} - \frac{1}{2}\right) \approx 0.856194, \quad d^* = 2. \tag{2.2}$$

Thus,

$$L(x) < \beta(x+1) < U(x), \tag{2.3}$$

where

$$L(x) = \beta(x) + \frac{c^*}{3^{x+1}}, \quad U(x) = \beta(x) + \frac{d^*}{3^{x+1}}. \tag{2.4}$$

Theorem 2.2 (Corollary 5.5 of [3])**.** *For $x > 0$,*

$$L_2(x) < \beta(x+1) < U_2(x), \tag{2.5}$$

where

$$\begin{aligned} L_2(x) &= \beta(x+2) - \frac{d^*}{3^{x+2}}, \quad \textit{and} \\ U_2(x) &= \beta(x+2) - \frac{c^*}{3^{x+2}}, \end{aligned} \tag{2.6}$$

where c^* and d^* are given in (2.2) above. In [3], many other interesting results are presented, but in this paper, we are primarily focused on obtaining bounds for $\beta(x)$. We shall first need various lemmas and preliminary results.

Lemma 2.3 (see [11], p. 92)**.** *Let X be a random variable with distribution function F, and let $\mathcal{I}$ be an interval (possibly unbounded) such that the expectations below are finite. Assume $g_1(x)$ and $g_2(x)$ are monotone functions of x on $\mathcal{I}$. Then:*

(a) *If $g_1(x)$ and $g_2(x)$ are monotone in the same direction on $\mathcal{I}$ (both nondecreasing or both nonincreasing), then*

$$\operatorname{Cov}\big(g_1(X), g_2(X)\big) \geq 0,$$

and equivalently,

$$\int_{\mathcal{I}} g_1(x)g_2(x)\,dF(x) \geq \left(\int_{\mathcal{I}} g_1(x)\,dF(x)\right)\left(\int_{\mathcal{I}} g_2(x)\,dF(x)\right). \tag{2.7}$$

(b) *If $g_1(x)$ is nondecreasing on $\mathcal{I}$ while $g_2(x)$ is nonincreasing on $\mathcal{I}$ (or vice versa), then*

$$\operatorname{Cov}\big(g_1(X), g_2(X)\big) \leq 0,$$

and equivalently,

$$\int_{\mathcal{I}} g_1(x)g_2(x)\,dF(x) \leq \left(\int_{\mathcal{I}} g_1(x)\,dF(x)\right)\left(\int_{\mathcal{I}} g_2(x)\,dF(x)\right). \tag{2.8}$$

Lemma 2.4 ([8], p. 156)**.** *Let X be a random variable on $(\Omega, \mathcal{A}, P)$ and let $\mathcal{J} \subset (0,\infty)$ be an interval such that $E(|X|^r) < \infty$ for every $r \in \mathcal{J}$. Define*

$$w(r) = \log\big(E(|X|^r)\big), \quad r \in \mathcal{J}.$$

Then w is convex on $\mathcal{J}$. Consequently, the moment function $e(r) = E(|X|^r)$ is log-convex in r on $\mathcal{J}$.

Lemma 2.5. *Assume $0 < r \le s$ and that the moments μ_r' and μ_s' are finite. Then*

$$\mu_s' \ge \left(\mu_r'\right)^{s/r}. \tag{2.9}$$

Proof. Let $Y \ge 0$ be the underlying random quantity for which $\mu_t' = E(Y^t)$ (so $\mu_r' = E(Y^r)$ and $\mu_s' = E(Y^s)$). Apply Jensen's inequality to the convex map $\phi(u) = u^{s/r}$ on $(0,\infty)$ (convex since $s/r \ge 1$), with $u = Y^r$. Then

$$E(Y^s) = E\big((Y^r)^{s/r}\big) = E\big(\phi(Y^r)\big) \ge \phi\big(E(Y^r)\big) = \big(E(Y^r)\big)^{s/r},$$

which is exactly (2.9). This inequality is commonly referred to as Lyapunov's inequality. ∎

For our final set of lemmas, we shall need some definitions and results from reliability theory. See [1] for a good reference on this area of applied probability.

We briefly recall the reliability notions used below and refer to [5], for related sufficient conditions and background. Let X be a nonnegative continuous random variable with distribution function F, density f, and survival function $\bar{F}(t) = 1 - F(t)$. Its hazard rate is

$$h(t) = \frac{f(t)}{\bar{F}(t)}, \qquad t > 0, \tag{2.10}$$

and the associated cumulative hazard is $H(t) = -\log \bar{F}(t)$. We say that X is IFR (Increasing Failure Rate) if $h(t)$ is nondecreasing on $(0,\infty)$, and IFRA (Increasing Failure Rate Average) if the mapping $t \mapsto H(t)/t$ is nondecreasing for all $t > 0$ with $\bar{F}(t) > 0$. An equivalent form of IFRA is that $t \mapsto \bar{F}(t)^{1/t}$ is nonincreasing. It is standard that IFR implies IFRA, so any sufficient condition that yields IFR automatically yields IFRA as well.

Lemma 2.6 (see [1], pp. 112–113)**.** *Let X have an IFRA distribution. For $r > 0$, define the r^{th} moment of X*

$$\mu_r' = E(X^r) = \int_0^\infty x^r \, dF(x),$$

and set

$$\lambda_r = \frac{\mu_r'}{\Gamma(r+1)}.$$

Then the mapping $r \mapsto (\lambda_r)^{1/r}$ is nonincreasing on $(0,\infty)$.

Lemma 2.7 (see [5], p. 668)**.** *Let X be a continuous random variable on $(0,\infty)$ with distribution function F and density $f = F'$. Assume that f is continuous, strictly positive, and twice differentiable on $(0,\infty)$. Define, for $x > 0$,*

$$\eta(x) = -\frac{f'(x)}{f(x)}.$$

If $\eta'(x) > 0$ for every $x > 0$, then X is IFR. In particular, X is also IFRA.

3. NEW BOUNDS FOR $\beta(x)$ AND $\beta(2n)$

In this section, we present new bounds for $\beta(2n)$ and general $\beta(x)$. We now present various bounds and inequalities related to the Dirichlet beta function $\beta(x)$. First, we present various bounds on $\beta(x)$ by bounding the first few values for the residual series:

$$\beta(x) - \sum_{n=0}^{k} \frac{(-1)^n}{(2n+1)^x}.$$

Theorem 3.1. *For $x \geq 1$,*

$$\beta(x) \geq 1 - \left(\frac{1}{2}\right)^x \left(2 - \frac{\pi}{2}\right). \tag{3.1}$$

For $0 < x < 1$, (3.1) holds with the inequality sign reversed.

Proof. From (1.2), we get for $x \geq 1$:

$$\begin{aligned} 1 - \beta(x) &= 1 - \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}}\, dt \\ &= \frac{1}{2\Gamma(x)} \int_0^\infty t^{x-1} \left(\frac{1}{e^t + e^{-t}}\right) (2e^{-2t})\, dt. \end{aligned}$$

Now let $h_1(t) = t^{x-1}$, $h_2(t) = \frac{1}{e^t+e^{-t}}$, and $F(t) = 1 - e^{-2t}$ and $I = [0, \infty)$ in Lemma 2.3, part (a). Then we get

$$\begin{aligned} 1 - \beta(x) &\leq \frac{1}{2\Gamma(x)} \left(\int_0^\infty t^{x-1}(2e^{-2t})\, dt\right) \left(\int_0^\infty \frac{2e^{-2t}}{e^t + e^{-t}}\, dt\right) \\ &= \frac{1}{2\Gamma(x)} \left(2 \cdot \Gamma(x) \left(\frac{1}{2}\right)^x\right) \left(2 - \frac{\pi}{2}\right) \\ &= \left(\frac{1}{2}\right)^x \left(2 - \frac{\pi}{2}\right), \end{aligned} \tag{3.2}$$

from which inequality (3.1) follows for $x \geq 1$. If $0 < x < 1$, apply part (b) of Lemma 2.3 instead to reverse the inequality sign in (3.1). This completes the proof. ∎

Next, we present the upper bound version of Theorem 3.1.

Theorem 3.2. *For $x \geq 1$,*

$$\beta(x) \leq 1 - \left(\frac{1}{3}\right)^x + \left(\frac{1}{4}\right)^x \left(\pi - \frac{8}{3}\right). \tag{3.3}$$

Thus,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)^x} \leq \left(\frac{1}{4}\right)^x \left(\pi - \frac{8}{3}\right). \tag{3.4}$$

For $0 < x < 1$, (3.3) and (3.4) hold with the inequality signs reversed.

Proof. We shall proceed as in the proof of Theorem 3.1 with some modifications. Suppose $x \geq 1$, then from (1.2), we have

$$\beta(x) - 1 + \left(\frac{1}{3}\right)^x = \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1} \left(\frac{1}{e^t + e^{-t}} - e^{-t} + e^{-3t}\right) dt$$
$$= \frac{1}{4\Gamma(x)} \int_0^\infty t^{x-1} \left(\frac{1}{e^t + e^{-t}}\right) (4e^{-4t})\, dt.$$

Letting $h_1(t) = t^{x-1}$, $h_2(t) = \frac{1}{e^t+e^{-t}}$, and $F(t) = 1 - e^{-4t}$, then by Lemma 2.3(a), we get

$$\beta(x) - 1 + \left(\frac{1}{3}\right)^x \leq \frac{1}{4\Gamma(x)} \left(\int_0^\infty t^{x-1}(4e^{-4t})\, dt\right) \left(\int_0^\infty \frac{4e^{-4t}}{e^t + e^{-t}}\, dt\right)$$
$$(3.5) \qquad = \frac{1}{4\Gamma(x)} \left(4 \int_0^\infty t^{x-1} e^{-4t}\, dt\right) \left(\pi - \frac{8}{3}\right)$$
$$= \frac{1}{4\Gamma(x)} \left(4 \cdot \frac{\Gamma(x)}{4^x}\right) \left(\pi - \frac{8}{3}\right)$$
$$= \left(\frac{1}{4}\right)^x \left(\pi - \frac{8}{3}\right),$$

from which both (3.3) and (3.4) follow. For $0 < x < 1$, merely reverse the inequality sign in (3.5) above when applying Lemma 2.3(b) instead. This completes the proof. ∎

Theorem 3.3. *For $x \geq 1$,*

$$(3.6) \qquad \beta(x) \geq 1 - \left(\frac{1}{3}\right)^x + \left(\frac{1}{5}\right)^{x-1} \left(\frac{\pi}{4} - \frac{2}{3}\right).$$

If $0 < x < 1$, inequality (3.3) holds with the inequality sign reversed.

Proof. The proof of this theorem is very similar to the proofs of Theorems 3.1 and 3.2 above and is omitted. ∎

If desired, we can obtain bounds for the residual

$$\beta(x) - \sum_{n=0}^{m} \frac{(-1)^n}{(2n+1)^x} = \sum_{n=m+1}^{\infty} \frac{(-1)^n}{(2n+1)^x}, \quad m \geq 3.$$

(We have presented these for $m = 0, 1,$ and 2 above, essentially.) We shall return to this using a different method later in this paper.

Theorem 3.4. *Let*

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du,$$

be the gamma function, for $x > 0$. Then

$$(3.7) \qquad \beta(2n) \geq 1 - \sqrt{\frac{2n}{2n-1} \big(1 - \beta(2n-1)\big)\big(1 - \beta(2n+1)\big)}, \quad n = 1, 2, 3, \ldots$$

Proof. From (1.2), we have

$$1 - \beta(x) = 1 - \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}}\, dt$$
$$(3.8) \qquad = \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1} \left(e^{-t} - \frac{1}{e^t + e^{-t}}\right) dt.$$

Let
$$C=\int_0^\infty\left(e^{-t}-\frac{1}{e^t+e^{-t}}\right)dt=1-\frac{\pi}{4}\approx 0.2146.$$
Then from (3.8), we get
$$\Gamma(x)\,(1-\beta(x))=E(T^{x-1})\,C,$$
where T is a random variable with an absolutely continuous distribution function
$$F(t)=\frac{1}{C}\int_0^t e^{-s}\left(e^{-s}-\frac{1}{e^s+e^{-s}}\right)ds,$$
and hence
$$F'(t)=\frac{e^{-t}\left(e^{-t}-\frac{1}{e^t+e^{-t}}\right)}{C},\quad t>0.$$
By Lemma 2.4, $\Gamma(x)\big(\frac{1-\beta(x)}{C}\big)$ is log-convex in x, which gives that $\Gamma(x)\,(1-\beta(x))$ is log-convex in x also. Thus,

$$\log[\Gamma(2n)\,(1-\beta(2n))]\le\frac{1}{2}\Big\{\log[\Gamma(2n-1)\,(1-\beta(2n-1))]+\log[\Gamma(2n+1)\,(1-\beta(2n+1))]\Big\}. \tag{3.9}$$

Exponentiation of (3.9) and solving for $\beta(2n)$ yields the theorem. This completes the proof. ∎

Theorem 3.5. *The following inequality holds. Let*
$$\alpha=1-\frac{\pi}{4}.$$
Then
$$1-B\le\beta(2n)\le 1-A, \tag{3.10}$$
where
$$A=\frac{(2n-2)!^{\frac{2n-1}{2n-2}}}{(2n)!}\,\alpha^{-\frac{1}{2n-2}}\,(1-\beta(2n-1))^{\frac{2n-1}{2n-2}}, \tag{3.11}$$
and
$$B=\frac{(2n)!^{\frac{2n-1}{2n}}}{(2n-1)!}\,\alpha^{\frac{1}{2n}}\,(1-\beta(2n+1))^{\frac{2n-1}{2n}}, \tag{3.12}$$
for $n=2,3,4,\dots$

Proof. From (1.2), we get
$$\Gamma(x)\,(1-\beta(x))=\int_0^\infty t^{x-1}\frac{e^{-2t}}{e^t+e^{-t}}\,dt=\alpha\int_0^\infty t^{x-1}f(t)\,dt,$$
where
$$f(t)=\frac{1}{\alpha}\frac{e^{-2t}}{e^t+e^{-t}},\quad t>0,$$
is the density function of a continuous random variable T with support $(0,\infty)$ and
$$\alpha=\int_0^\infty\frac{e^{-2t}}{e^t+e^{-t}}\,dt=1-\frac{\pi}{4}.$$
Now apply Lemma 2.5. Then
$$\Gamma(x)\,(1-\beta(x))=E(T^{x-1})=\mu'_{x-1}.$$

For $x = 2n-1, 2n$, and $2n+1$, letting $r = 2n-1$ and $s = 2n$, and then letting $r = 2n$ and $s = 2n+1$ in Lemma 2.5, and using $\Gamma(k) = (k-1)!$, $k = 1, 2, 3, \ldots$, we obtain

$$\begin{aligned} \frac{(2n-2)!^{\frac{2n-1}{2n-2}}(1-\beta(2n-1))^{\frac{2n-1}{2n-2}}}{\alpha^{\frac{2n-1}{2n-2}}} &\leq (2n-1)!\left(\frac{1-\beta(2n)}{\alpha}\right) \\ &\leq \frac{(2n)!^{\frac{2n-1}{2n}}(1-\beta(2n+1))^{\frac{2n-1}{2n}}}{\alpha^{\frac{2n-1}{2n}}}. \end{aligned} \tag{3.13}$$

After cancellation and simplification, and solving for $\beta(2n)$, we obtain (3.10). This completes the proof. ∎

Remark 3.1. We can easily generalize the above theorem to the case of $x_1 < x_2 < x_3$ and obtain bounds for $\beta(x_2)$ in terms of exact $\beta(x_1)$ and $\beta(x_3)$ values. Merely replace $2n-1$, $2n$, and $2n+1$ by x_1, x_2, and x_3, respectively, in (3.10), with $\Gamma(x_1)$, $\Gamma(x_2)$, and $\Gamma(x_3)$ replacing $(2n-2)!$, $(2n-1)!$, and $(2n)!$, respectively, in Theorem 3.5.

Next, we give bounds on the residual series:

$$\beta(x) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} = \sum_{k=m+1}^{\infty} \frac{(-1)^k}{(2k+1)^x}, \quad m = 0, 1, 2, \ldots$$

Theorem 3.6. *The following inequalities hold for $x > 0$:*

$$\text{(a)} \quad \beta(x) \leq \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} + \frac{1}{2(2m+2)^x}, \quad m = 1, 3, 5, 7, \ldots \tag{3.14}$$

$$\text{(b)} \quad \beta(x) \geq \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \frac{1}{2(2m+2)^x}, \quad m = 2, 4, 6, 8, \ldots \tag{3.15}$$

$$\text{(c)} \quad \beta(x) \geq \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} + \frac{1}{2(2m+3)^x}, \quad m = 1, 3, 5, \ldots \tag{3.16}$$

$$\text{(d)} \quad \beta(x) \leq \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \frac{1}{2(2m+3)^x}, \quad m = 2, 4, 6, \ldots \tag{3.17}$$

Proof. We shall prove parts (a) and (b) only. The proofs of parts (c) and (d) are very similar, except for a few extra inequalities needed, and we will at least mention these at the end of the proof. First of all, note that

$$G_m(x) = (-1)^{m+1}\left[\beta(x) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x}\right] > 0, \quad m = 0, 1, 2, \ldots$$

Also, from (1.2),

$$\beta(x) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \left[t^{x-1}\left(\frac{1}{e^t + e^{-t}}\right) - t^{x-1}\left(\sum_{k=0}^{m} (-1)^k e^{-(2k+1)t}\right)\right] dt. \tag{3.18}$$

After some algebra, the integrand of (3.18) produces a telescoping series which sums nicely to give

$$
\begin{aligned}
G_m(x) &= (-1)^{m+1}\left[\beta(x) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x}\right] \\
&= \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1} \frac{e^{-(2m+2)t}}{e^t + e^{-t}}\, dt \\
&= (-1)^m\, r_m(x),
\end{aligned} \tag{3.19}
$$

where

$$
r_m(x) = \int_0^\infty \frac{t^{x-1}}{\Gamma(x)} \left(\frac{e^{-(2m+2)t}}{e^t + e^{-t}}\right) dt. \tag{3.20}
$$

Using $\dfrac{1}{e^t + e^{-t}} \le \dfrac{1}{2}$, we get

$$
r_m(x) \le \frac{1}{2}\left(\frac{1}{2m+2}\right)^x, \quad x > 0.
$$

To prove parts (c) and (d), we proceed as in the proof of parts (a) and (b) above, except we rewrite (3.20) in the form

$$
r_m(x) = \int_0^\infty \frac{t^{x-1}}{\Gamma(x)} \left(\frac{e^t}{e^t + e^{-t}}\right) e^{-(2m+3)t}\, dt,
$$

and use $\dfrac{e^t}{e^t + e^{-t}} \ge \dfrac{1}{2}$ instead, proceeding as above. Considering m even and m odd cases, this completes the proof. ∎

From Theorem 3.6, we immediately obtain:

Corollary 3.7. *For $x > 0$. We have the following representation for $\beta(x)$:*

$$
\beta(x) = \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} \pm \frac{1}{2(2m+\theta)^x},
$$

for some $\theta = \theta(x, m) \in [2, 3]$.

Theorem 3.8. *The following inequalities hold:*

$$
\begin{aligned}
\beta(2n) \ge (2m+2)&\left(\beta(2n+1) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^{2n+1}}\right) \\
&+ \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^{2n}}, \qquad m = 1, 3, 5, 7, \ldots
\end{aligned} \tag{3.21}
$$

Inequality (3.21) holds with the inequality sign reversed if $m = 2, 4, 6, \ldots$

Proof. We use several inequalities established earlier in the proof of Theorem 3.6. Proceeding as in that proof, we have

$$
r_m(2n+1) = \int_0^\infty t^{2n} \left(\frac{1}{e^t + e^{-t}}\right) e^{-(2m+2)t}\, dt. \tag{3.22}
$$

Let $f(t)$ denote the density function

$$f(t) = \frac{(2m+2)^{2n}}{(2n-1)!}\, t^{2n-1} e^{-(2m+2)t}, \quad t > 0, \quad \text{with } F(t) = \int_0^t f(u)\, du. \tag{3.23}$$

Let $h_1(t) = t$ and $h_2(t) = \dfrac{1}{e^t + e^{-t}}$. Then, using Lemma 2.3, we get

$$r_m(2n+1) \le \int_0^\infty \frac{(2m+2)^{2n}}{(2n-1)!} t^{2n} e^{-(2m+2)t}\, dt \cdot \int_0^\infty \frac{1}{e^t + e^{-t}} t^{2n-1} e^{-(2m+2)t}\, dt.$$

Since $h_2(t) \le \frac{1}{2}$, after some algebra and simplification, we obtain

$$r_m(2n+1) \le \frac{1}{m+1} r_m(2n). \tag{3.24}$$

Thus, (3.19) and (3.24) gives

$$\begin{aligned}(-1)^{m+1}\left(\beta(2n+1) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^{2n+1}}\right) &\le \frac{(-1)^{m+1}}{2m+2} \\ &\times \left(\beta(2n) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^{2n}}\right), \qquad m \text{ odd.}\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}(-1)^{m+1}\left(\beta(2n+1) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^{2n+1}}\right) &\ge \frac{(-1)^{m+1}}{2m+2} \\ &\times \left(\beta(2n) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^{2n}}\right), \qquad m \text{ even.}\end{aligned} \tag{3.26}$$

These are equivalent to (3.21) after solving for $\beta(2n)$ in (3.25)–(3.26). This completes the proof. ∎

Remark 3.2. Note that the theorem above gives bounds on $\beta(2n)$ using only $\beta(2n-1)$, and does not use $\beta(2n+1)$. Thus, it would be expected that these bounds will not be as good as those given earlier for $\beta(2n)$ using both $\beta(2n-1)$ and $\beta(2n+1)$. This will be seen not to be the case in later numerical comparisons.

Theorem 3.9. *Let m be a nonnegative integer. Then:*
(a) If m is even, then for $x > 0$,

$$\beta(x) \ge \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \inf_{0<\delta<1} \delta^{\delta}(1-\delta)^{1-\delta} \frac{1}{(2m+2+2\delta-1)^x} \tag{3.27}$$

$$\ge \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \delta_*^{\delta_*}(1-\delta_*)^{1-\delta_*} \frac{1}{(2m+2+2\delta_*-1)^x}, \tag{3.28}$$

where

$$\delta_* = \frac{\exp\left(\frac{2x}{2m+3}\right)}{1+\exp\left(\frac{2x}{2m+3}\right)}. \tag{3.29}$$

(b) If m is odd, then for $x > 0$,

$$\beta(x) \le \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} + \inf_{0<\delta<1} \delta^{\delta}(1-\delta)^{1-\delta} \frac{1}{(2m+2+2\delta-1)^x} \tag{3.30}$$

$$\le \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} + \delta_*^{\delta_*}(1-\delta_*)^{1-\delta_*} \frac{1}{(2m+2+2\delta_*-1)^x}. \tag{3.31}$$

In particular, for $m = 0$,

$$\beta(x) \ge 1 - \delta_*^{\delta_*}(1-\delta_*)^{1-\delta_*} \frac{1}{(2\delta_*+1)^x}. \tag{3.32}$$

and for $m = 1$,

$$\beta(x) \le \left(1 - \frac{1}{3^x}\right) + \delta_*^{\delta_*}(1-\delta_*)^{1-\delta_*} \frac{1}{(2\delta_*+3)^x}. \tag{3.33}$$

Proof. We shall prove only part (a) since the proof of part (b) is very similar. From Theorem 3.8,

$$(-1)^{m+1}\left(\beta(x) - \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x}\right) = \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1} \frac{1}{e^t + e^{-t}} e^{-(2m+2)t}\, dt. \tag{3.34}$$

Since m is even,

$$\sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \beta(x) = \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1} \frac{1}{e^t + e^{-t}} e^{-(2m+2)t}\, dt. \tag{3.35}$$

Now we apply the weighted arithmetic mean–geometric mean inequality in a unique way to get

$$\begin{aligned} e^t + e^{-t} &= \delta\left(\frac{1}{\delta}e^t\right) + (1-\delta)\left(\frac{1}{1-\delta}e^{-t}\right) \\ &\ge \left(\frac{1}{\delta}e^t\right)^{\delta}\left(\frac{1}{1-\delta}e^{-t}\right)^{1-\delta} \\ &= \left[\left(\frac{1}{\delta}\right)^{\delta}\left(\frac{1}{1-\delta}\right)^{1-\delta}\right] e^{(2\delta-1)t}. \end{aligned} \tag{3.36}$$

which gives, for any $0 < \delta < 1$, that

$$\frac{1}{e^t + e^{-t}} \le \delta^{\delta}(1-\delta)^{1-\delta}\, e^{-(2\delta-1)t}. \tag{3.37}$$

Substituting the right-hand side of (3.37) into (3.35) gives

$$\begin{aligned} \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \beta(x) &\le \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1}\left[\delta^{\delta}(1-\delta)^{1-\delta}\right] e^{-(2m+2+2\delta-1)t}\, dt \\ &= \left[\delta^{\delta}(1-\delta)^{1-\delta}\right] \frac{1}{(2m+2+2\delta-1)^x}. \end{aligned} \tag{3.38}$$

Solving for $\beta(x)$ in (3.38), and taking $\inf$ over $0 < \delta < 1$, we get

$$\beta(x) \ge \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x} - \inf_{0<\delta<1} \delta^{\delta}(1-\delta)^{1-\delta} \frac{1}{(2m+2+2\delta-1)^x}, \tag{3.39}$$

which proves (3.27). Clearly, (3.28) holds for any $\delta^* \in (0,1)$. To complete the proof, let us show that δ^* is a very good approximation to the value of $\delta \in (0,1)$ minimizing

$$g(\delta) = \delta^\delta (1-\delta)^{1-\delta} \frac{1}{(2m+2+2\delta-1)^x}, \qquad 0 < \delta < 1. \tag{3.40}$$

First, let us show that the minimum value of $g(\delta)$ occurs at some $\delta = \delta_m \in (0,1)$. Then we will show why $\delta_m \approx \delta^*$ could be a very good approximation (we will see that this is the case in later numerical comparisons). From (3.40),

$$\log g(\delta) = \delta \log \delta + (1-\delta)\log(1-\delta) - x\log(2m+2+2\delta-1). \tag{3.41}$$

Differentiating with respect to δ,

$$\frac{d}{d\delta}\big(\log g(\delta)\big) = \log\left(\frac{\delta}{1-\delta}\right) - \frac{2x}{2m+2+2\delta-1}. \tag{3.42}$$

Now $\log g(0^+) = -\infty$ and $\log g(1^-) = +\infty$, so (3.42) has at least one real root $\delta \in (0,1)$, that is, there exists $\delta = \delta_m \in (0,1)$ with

$$\frac{d}{d\delta}\big(\log g(\delta)\big)\Big|_{\delta=\delta_m} = 0. \tag{3.43}$$

Let us show that this is the only such δ-value. Differentiating (3.42) one more time, we get

$$\frac{d^2}{d\delta^2}\big(\log g(\delta)\big) = \frac{1}{\delta} + \frac{1}{1-\delta} + \frac{4x}{(2m+2+2\delta-1)^2} > 0. \tag{3.44}$$

From (3.44) we see that $\log g(\delta)$ is convex and so there can only be at most one value of δ_m. But we showed earlier that there is at least one δ_m. From this we conclude that $\delta_m \in (0,1)$ is unique and satisfies

$$\log\left(\frac{\delta_m}{1-\delta_m}\right) = \frac{2x}{2m+2+2\delta_m-1}. \tag{3.45}$$

Now the left-hand side of (3.45) blows up at both $\delta = 0^+$ and $\delta = 1^-$, whereas the right-hand side of (3.45) remains bounded in $\delta \in (0,1)$ for $m \geq 0$. If we substitute $\delta = 1$ in the right-hand side of (3.45), an approximate minimizer of $g(\delta)$ should satisfy

$$\log\left(\frac{\delta}{1-\delta}\right) = \frac{2x}{2m+3}. \tag{3.46}$$

Solving for δ in (3.46) we get

$$\delta_* = \frac{\exp\left(\frac{2x}{2m+3}\right)}{1+\exp\left(\frac{2x}{2m+3}\right)}. \tag{3.47}$$

This completes the proof. ∎

Because our families are parameterized by the truncation index m, it is important to clarify what information enters the construction. Recall that

$$S(m,x) = \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x}, \tag{3.48}$$

so that, in particular, $S(0,x) = 1$ for all $x > 0$ and the specialization $m = 0$ does not use any x-dependent summation terms. The sharpness of our bounds comes from explicitly controlling the residual series $\beta(x) - S(m,x)$, as quantified by the remainder term in Theorem 3.9. Consequently, the resulting inequalities can be evaluated without requiring neighboring special values such as $\beta(2n-1)$, in contrast to approaches whose even-integer bounds depend on adjacent beta values. This approach emphasizes that the proposed method provides a systematic truncation-and-remainder framework, rather than relying only on the availability of additional partial-sum terms.

Theorem 3.10. *For $m = 0, 1, 2, 3, \ldots$, let*

$$C_m = \int_0^\infty \frac{e^{-(2m+2)t}}{e^t + e^{-t}}\, dt. \tag{3.49}$$

For $x > 0$, let

$$S(m, x) = \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x}, \tag{3.50}$$

be the mth partial sum of the Dirichlet beta series. Then for $n = 2, 3, 4, \ldots$ we have the inequalities: if m is even, then

$$\begin{aligned} \beta(2n) &> C_m^{-\frac{1}{2n+2}} \big(\beta(2n-1) - S(m, 2n-1)\big) \\ &\quad + S(m, 2n), \\ \beta(2n) &< C_m^{\frac{1}{2n}} \big(\beta(2n+1) - S(m, 2n+1)\big) \\ &\quad + S(m, 2n). \end{aligned} \tag{3.51}$$

If m is odd, (3.51) holds with both inequality signs reversed. In particular, if $m = 0$, then

$$\left(1 - \frac{\pi}{4}\right)^{-\frac{1}{2n+2}} \big(\beta(2n-1) - 1\big) + 1 \le \beta(2n) \le \left(1 - \frac{\pi}{4}\right)^{\frac{1}{2n}} \big(\beta(2n+1) - 1\big) + 1. \tag{3.52}$$

Proof. We first consider the case that m is even (including $m = 0$). From (3.19), we have

$$\begin{aligned} 0 \le G_m(x) &= (-1)^{m+1}\big(\beta(x) - S(m, x)\big) \\ &= \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1} \frac{e^{-(2m+2)t}}{e^t + e^{-t}}\, dt \\ &= \frac{C_m}{\Gamma(x)} \int_0^\infty t^{x-1} \left[\frac{e^{-(2m+2)t}}{C_m\big(e^t + e^{-t}\big)}\right] dt. \end{aligned} \tag{3.53}$$

The expression in brackets of (3.53) is a density function. Let

$$f_m(t) = \frac{e^{-(2m+2)t}}{C_m\big(e^t + e^{-t}\big)}, \qquad t > 0. \tag{3.54}$$

Let

$$\eta_m(t) = -\frac{f_m'(t)}{f_m(t)}. \tag{3.55}$$

Then a simple computation (after much algebraic simplification) gives

$$\begin{aligned} \eta_m(t) &= \frac{2me^{2t} + 3e^{2t} + 2m + 1}{e^{2t} + 1} \\ &= (2m+3) - \frac{2}{e^{2t} + 1}, \end{aligned} \tag{3.56}$$

which is clearly increasing in t. Thus by Lemma 2.7, $f_m(t)$ is an IFR, whence an IFRA density function. Applying Lemma 2.6 with $r = x - 1$, we get that if T is a random variable with density (3.54), then

$$R(x) = \left(\mathbb{E}\left(\frac{T^{x-1}}{\Gamma(x)}\right)\right)^{\frac{1}{x-1}} \downarrow x \qquad \text{for } x \geq 2. \tag{3.57}$$

From (8), we get

$$G_m(x) = \frac{C_m}{\Gamma(x)}\,\mathbb{E}\big(T^{x-1}\big)\,, \tag{3.58}$$

which gives

$$\mathbb{E}\big(T^{x-1}\big) = \frac{\Gamma(x)}{C_m}\,G_m(x) \qquad \text{and} \qquad \mathbb{E}\left(\frac{T^{x-1}}{\Gamma(x)}\right) = \frac{G_m(x)}{C_m}. \tag{3.59}$$

Thus,

$$R(x) = \left(\mathbb{E}\left(\frac{T^{x-1}}{\Gamma(x)}\right)\right)^{\frac{1}{x-1}} = \left(\frac{G_m(x)}{C_m}\right)^{\frac{1}{x-1}}, \qquad x \geq 2. \tag{3.60}$$

Letting $x = 2n-1$, $2n$, and $2n+1$ in (3.60), simplifying, and using the fact that $R(x)$ is decreasing in x for $x \geq 2$, we get

$$C_m^{\frac{1}{2n}}\big(G_m(2n+1)\big)^{\frac{2n-1}{2n}} \leq G_m(2n) \leq C_m^{-\frac{1}{2n+2}}\big(G_m(2n-1)\big)^{\frac{2n+1}{2n+2}}. \tag{3.61}$$

After some algebra using the definitions of $G_m(x)$ and $S(m,x)$ we get inequality (3.51). This completes the proof of (3.51) when m is even. When m is odd, merely reverse the inequality signs in (3.51) above since $(-1)^{m+1}$ is positive when m is odd using the definition of $G_m(x)$ and solve for $\beta(2n)$.

∎

Remark 3.3. The table below gives C_m for various $m \leq 4$:

(3.62)

m	0	1	2	3	4
C_m	$1-\frac{\pi}{4}$	$-\frac{2}{3}+\frac{\pi}{4}$	$\frac{13}{15}-\frac{\pi}{4}$	$-\frac{76}{105}+\frac{\pi}{4}$	$\frac{263}{315}-\frac{\pi}{4}$

Recursive integration by parts could be used in the integration of the density $f_m(t)$ to get a method to recursively compute C_{m+1} from C_m, if desired. This was not pursued since good bounds, especially for large n, are obtained for $m = 0$ and $m = 1$ (note that for $m = 0$ we are not using the value of x at all). For $m \leq 2$, very good bounds result in any case.

Theorem 3.11. *Let*

$$S(m,x) = \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)^x}, \qquad m = 0,1,2,\ldots, \qquad x > 0. \tag{3.63}$$

Then for $n = 2,3,4,\ldots$ we have:

(a) *For $m = 0,2,4,6,\ldots$,*

$$\beta(2n) \leq S(m,2n) - \left[\frac{\big((2n)!\big)^{3/2}}{(2n-1)!\,\sqrt{(2n+2)!}}\right] \cdot \frac{\big(S(m,2n+1)-\beta(2n+1)\big)^{3/2}}{\sqrt{S(m,2n+3)-\beta(2n+3)}}. \tag{3.64}$$

(b) *For $m = 1, 3, 5, 7, \ldots$, (3.64) holds with the inequality sign reversed.*

In particular, for $m = 0$, part (a) gives

$$\beta(2n) \leq 1 - \left[\frac{\big((2n)!\big)^{3/2}}{(2n-1)!\,\sqrt{(2n+2)!}}\right] \cdot \frac{\big(1-\beta(2n+1)\big)^{3/2}}{\sqrt{1-\beta(2n+3)}}. \tag{3.65}$$

and for $m = 1$,

$$\beta(2n) \geq \left(1 - \frac{1}{3^{2n}}\right) - \left[\frac{((2n)!)^{3/2}}{(2n-1)!\sqrt{(2n+2)!}}\right] \cdot \frac{\Big(\big(1-\frac{1}{3^{2n+1}}\big) - \beta(2n+1)\Big)^{3/2}}{\sqrt{\big(1-\frac{1}{3^{2n+3}}\big) - \beta(2n+3)}}. \tag{3.66}$$

Proof. In the proof of the previous theorem, it was established that

$$W(x) = \mathbb{E}\big(T^{x-1}\big) = \frac{G_m(x)\Gamma(x)}{C_m}, \tag{3.67}$$

where $G_m(x) = (-1)^{m+1}\big(\beta(x) - S(m,x)\big)$ and C_m is given in (1). By lemma 2.4, $W(x)$ is log-convex in $x > 1$, so linear interpolation of $\log W(2n+1)$ using the values of $\log W(2n)$ and $\log W(2n+3)$ gives

$$\log W(2n+1) \leq \frac{2}{3}\log W(2n) + \frac{1}{3}\log W(2n+3), \tag{3.68}$$

which gives

$$\begin{aligned}\log\left(\frac{G_m(2n+1)\Gamma(2n+1)}{C_m}\right) &\leq \frac{2}{3}\log\left(\frac{G_m(2n)\Gamma(2n)}{C_m}\right) \\ &\quad + \frac{1}{3}\log\left(\frac{G_m(2n+3)\Gamma(2n+3)}{C_m}\right).\end{aligned} \tag{3.69}$$

Exponentiation of (3.69) and solving for $G_m(2n)\Gamma(2n)$, we get

$$G_m(2n)\Gamma(2n) \geq \frac{\big(G_m(2n+1)\Gamma(2n+1)\big)^{3/2}}{\sqrt{G_m(2n+3)\Gamma(2n+3)}}. \tag{3.70}$$

If m is even, then the definition of $G_m(x)$ results in inequality (3.64) of part (a) of the theorem. This completes the proof of part (a). Part (b) follows immediately from (a) if m is odd, since (3.69) will have the inequality sign reversed.

∎

Our bounds are not meant to be a complete substitute for the bounds of [3]. Rather, they are intended to complement the bounds of [3]. There are cases where the bounds of [3] are best, and there are cases where the new bounds are better; this is why we include a substantial numerical comparison in Section 4. In this paper, we use methods and techniques that differ from those used in previous research on the Dirichlet beta function. In particular, we employ results from the statistical theory of reliability and other areas of applied probability. We also used a variation of the weighted arithmetic geometric mean inequality to generate very good bounds for both cases where $0 < x < 1$ and $x = 2n,\ n = 1, 2, 3, \ldots$.

4. NUMERICAL COMPARISONS

In this section, we present an empirical evaluation of the proposed bounds for the Dirichlet beta function. For a range of x and n, we report the exact values of $\beta(x)$ alongside the corresponding bounds from the proposed theorems and corollaries and their relative errors. Throughout the tables, "T" denotes a Theorem (e.g., T 3.2), and "RE" denotes the relative error; when present, the tags (UB) and (LB) indicate upper- and lower-bound versions of a result. We include a good number of numerical comparisons rather than only a few illustrative cases. Across the ranges of x and n that we consider, no single bound is uniformly optimal, and different constructions become sharper in different regimes. The tables below are intended to make these trade-offs explicit.

Table 4.1 shows a uniform and sizable advantage for Theorem 3.2 (UB) over both existing alternatives Theorem 2.1 (UB) and Theorem 2.2 (UB) across all listed $x \geq 1$. In particular, the proposed upper bound is already tighter at the left end of the grid and continues to remain the sharpest choice as x increases. Midrange entries reinforce the same pattern, with Theorem 3.2 (UB) consistently improving on both baselines. All three bounds tighten as x grows, but Theorem 3.2 (UB) tightens faster, so its advantage persists across the tested range. In practice, selecting Theorem 3.2 (UB) as the default upper bound for $x \geq 1$ provides the most reliable performance in this regime.

Table 4.2 and Table 4.3 show that the proposed bounds are consistently sharper than the existing bounds on $0 < x < 1$, with the best choice depending on where x lies. For upper bounds, Theorem 2.1 is most effective only at very small x, while Theorem 3.1 becomes preferable in the mid-range and Theorem 3.3 is sharpest as $x \to 1^-$. For lower bounds, Theorem 3.2 is the tightest option for moderate-to-large x, whereas the classical bounds from Theorems 2.1 and 2.2 are uniformly looser. Notably, Theorem 3.9 provides the strongest lower bound at very small x, improving on both Theorem 3.2 and Theorem 3.6(c), so a practical guideline is to use Theorem 3.9 on the far left and otherwise default to Theorem 3.2 for lower bounds, alongside the upper-bound choices above.

Table 4.4 shows that the proposed lower bound in Theorem 3.4 is uniformly tighter than Theorem 2.1 (LB) for $\beta(2n)$ across $n = 1, \ldots, 6$. The improvement is already present at $n = 1$ and becomes increasingly pronounced as n grows, with Theorem 3.4 tracking the exact values more closely throughout the table. These findings suggest that both bounds tighten rapidly with n, but Theorem 3.4 provides the sharper lower envelope across the entire range.

x	Exact	T 2.1 (UB)	RE	T 2.2 (UB)	RE	T 3.2	RE
2	0.9159655942	1.0076204106	0.1000636018	0.9372352392	0.0232210087	0.9185717631	0.0028452694
3	0.9689461463	0.9900396683	0.0217695504	0.9783742494	0.0097302654	0.9703836815	0.0014836070
4	0.9889445517	0.9936375043	0.0047454152	0.9926343940	0.0037310911	0.9895095006	0.0005712645
5	0.9961578281	0.9971750044	0.0010210996	0.9975107442	0.0013581343	0.9963485686	0.0001914762
6	0.9986852222	0.9989013123	0.0002163746	0.9991630152	0.0004784220	0.9987442066	0.0000590620

Table 4.1: $\beta(x)$ upper bounds for $x \geq 1$: comparison of Theorems 2.1 (UB), 2.2 (UB), and 3.2.

Tables 4.5 and 4.6 compare the Theorem 3.9 bounds (with optimized δ_* and representative m values) to [3]'s Theorem 2.2 for $\beta(x)$ at even arguments $x = 2n$ up to 16. For the lower bounds, Theorem 3.9 is consistently tighter, with $m = 2$ giving the strongest performance across the grid, while the simpler $m = 0$ choice still typically improves on Theorem 2.2 but less uniformly at the largest x. For the upper bounds, Theorem 3.9 again dominates throughout,

x	Exact	T 2.1(UB)	RE	T 3.1	RE	T 3.3	RE
0.2	0.5737108472	0.5919199771	0.0317392114	0.6263565005	0.0917633920	0.6275293096	0.0938076431
0.3	0.6071836130	0.6314878779	0.0400278671	0.6513782879	0.0727863434	0.6470836769	0.0657133413
0.4	0.6384850155	0.6673092204	0.0451446851	0.6747244411	0.0567584590	0.6674575624	0.0453770195
0.7	0.7201614437	0.7555840911	0.0491870923	0.7357941478	0.0217072217	0.7289599266	0.0122173757
0.8	0.7436078367	0.7795676297	0.0483585450	0.7534872233	0.0132857484	0.7485737213	0.0066780961
0.9	0.7653214568	0.8012453074	0.0469395576	0.7699954465	0.0061072243	0.7674232072	0.0027462322

Table 4.2: $\beta(x)$ upper bounds for $0 < x < 1$: Theorems 3.1 and 3.3 vs. Theorem 2.1 (UB).

x	Exact	T 2.1	RE	T 2.2	RE	T 3.2	RE	T 3.6(c)	RE	T 3.9	RE
0.2	0.5737108472	0.2173239064	0.5958349966	0.2858599033	0.5017352301	0.5571850314	0.0288051305	0.5596482701	0.0245116110	0.5666058753	0.0123842383
0.3	0.6071836130	0.2980424419	0.5091395164	0.3572707656	0.4115935312	0.5941112051	0.0215295795	0.5892938380	0.0294635338	0.5977597935	0.0155205431
0.4	0.6384850155	0.3704579700	0.4197859605	0.4216220788	0.3396523511	0.6283793350	0.0158275923	0.6182587655	0.0316785038	0.6274052220	0.0173532553
0.7	0.7201614437	0.5459747606	0.2418717145	0.5788802251	0.1961799258	0.7165002398	0.0050838655	0.6986026029	0.0299361219	0.7070621974	0.0181893191
0.8	0.7436078367	0.5928660007	0.2027168470	0.6212483061	0.1645484683	0.7414235027	0.0029374811	0.7227293196	0.0280773225	0.7305286220	0.0175888607
0.9	0.7653214568	0.6349255482	0.1703805734	0.6593977700	0.1384041775	0.7643456170	0.0012750718	0.7454208363	0.0260029564	0.7524905532	0.0167653781

Table 4.3: $\beta(x)$ lower bounds for $0 < x < 1$: Theorems 3.2, 3.6(c) with $m = 1$ and 3.9 with $m = 0$ vs. Theorem 2.1 and Theorem 2.2.

n	Exact	T 2.1 (LB)	RE	T 3.4	RE
1	0.9159655942	0.8805309095	0.0386856066	0.8845511952	0.0342964836
2	0.9889445517	0.9795164486	0.0095335003	0.9873870836	0.0015748791
3	0.9986852222	0.9973323061	0.0013546972	0.9985668256	0.0001185524
4	0.9998499902	0.9996850055	0.0001650095	0.9998399454	0.0000100464
5	0.9999831640	0.9999641839	0.0000189804	0.9999822665	0.0000008975
6	0.9999981224	0.9999959861	0.0000021363	0.9999980394	0.0000000829

Table 4.4: Lower bounds for $\beta(2n)$: Theorems 2.1 (LB), and 3.4.

and $m = 3$ provides the sharpest values across all tested even arguments. These comparisons indicate that the proposed construction tightens more rapidly with x than [3]'s bounds, and the practical choices $m = 2$ (LB) and $m = 3$ (UB) yield the most accurate results in these comparisons.

Tables 4.7 and 4.8 compare the numerically optimal minimizer δ with the closed-form proxy δ_* in Theorem 3.9, for both lower and upper bounds at even arguments $x = 2n$ (with representative m values). Across all rows, δ_* is very close to δ_m, and the resulting bounds computed from the proxy are nearly identical to those obtained from the optimizer. For the lower bounds (Table 4.7), the outputs from (3.27) using δ and from (3.28) using δ_* match to many digits, with discrepancies far below the scale of the displayed accuracy. For the upper bounds (Table 4.8), the same behavior holds for both $m = 1$ and $m = 3$, where (3.30) and (3.31) produce essentially the same upper bounds under δ and δ_*. These results indicate that δ_* is a practical substitute for the optimal δ_m, preserving the tightness of Theorem 3.9 while eliminating the need for numerical minimization.

Tables 4.9 and 4.10 compare Theorem 3.11 against [3]'s Theorem 2.2 at even integers $x = 2n$ ($n = 1, \ldots, 8$), with even m producing upper bounds and odd m producing lower bounds. For the upper bounds, the parity-matched choice $m = 2$ is consistently tighter than [3] across the grid, while the crude even choice $m = 0$ is generally less competitive and can be worse than Theorem 2.2. For the lower bounds, $m = 3$ provides the sharpest and most consistently improved bounds relative to [3], whereas $m = 1$ is comparatively loose, especially at the smallest n. These results suggest that the recommended settings are $m = 2$ for the upper bound and $m = 3$ for the lower bound, which together yield a tighter envelope than Theorem 2.2 over

x	m	δ_*	Exact	T 3.9 (LB)	T 2.2 (LB)	RE (T 3.9)	RE (T 2.2)
2	0	0.791391472673955	0.915965594177219	0.910170530139755	0.894872072185295	0.006326726761685	0.023028727417290
2	2	0.639092745162851	0.915965594177219	0.915696304806748	0.894872072185295	0.000293995071630	0.023028727417290
4	0	0.935030830871336	0.988944551741105	0.988411719016778	0.987927375402191	0.000538789281350	0.001028547391382
4	2	0.758203827592590	0.988944551741105	0.988935357689391	0.987927375402191	0.000009296832363	0.001028547391382
6	0	0.982013790037908	0.998685222218438	0.998652344472063	0.998640013148885	0.000032921030214	0.000045268587686
6	2	0.847391335157369	0.998685222218438	0.998685012740012	0.998640013148885	0.000000209754206	0.000045268587686
8	0	0.995195247112840	0.999849990246830	0.999848310291213	0.999848073660370	0.000001680207664	0.000001916874010
8	2	0.907686971247702	0.999849990246830	0.999849986309799	0.999848073660370	0.000000003937622	0.000001916874010
10	0	0.998728983736919	0.999983164026197	0.999983086376210	0.999983084915285	0.000000077651294	0.000000079112244
10	2	0.945686733867359	0.999983164026197	0.999983163960259	0.999983084915285	0.000000000065939	0.000000079112244
12	0	0.999664649869534	0.999998122350586	0.999998118954033	0.999998119132822	0.000000003396560	0.000000003217770
12	2	0.968585628968430	0.999998122350586	0.999998122349565	0.999998119132822	0.000000000001022	0.000000003217770
14	0	0.999911580829981	0.999999791087248	0.999999790943323	0.999999790957403	0.000000000143925	0.000000000129844
14	2	0.982013790037908	0.999999791087248	0.999999791087234	0.999999790957403	0.000000000000014	0.000000000129844
16	0	0.999976691442159	0.999999976775950	0.999999976769968	0.999999976770732	0.000000000005982	0.000000000005218
16	2	0.989762712837566	0.999999976775950	0.999999976775951	0.999999976770732	-0.000000000000001	0.000000000005218

Table 4.5: $\beta(x)$ lower bounds for $x = 2n$: comparison of Theorem 3.9 and Theorem 2.2.

x	m	δ_*	Exact	T 3.9 (UB)	T 2.2 (UB)	RE (T 3.9)	RE (T 2.2)
2	1	0.689974481127613	0.915965594177219	0.916954921862890	0.937235239215209	0.001080092627889	0.023221008707314
2	3	0.609317541843561	0.915965594177219	0.916063617225945	0.937235239215209	0.000107016081553	0.023221008707314
4	1	0.832018385133924	0.988944551741105	0.988998197857018	0.992634393961070	0.000054245827855	0.003731091104621
4	3	0.708660825029436	0.988944551741105	0.988946789665575	0.992634393961070	0.000002262942312	0.003731091104621
6	1	0.916827303506078	0.998685222218438	0.998687134442314	0.999163015210982	0.000001914741336	0.000478422011175
6	3	0.791391472673955	0.998685222218438	0.998685257111456	0.999163015210982	0.000000034938955	0.000478422011175
8	1	0.960834277203236	0.999849990246830	0.999850046387003	0.999906185000603	0.000000056148596	0.000056203184799
8	3	0.855422249534123	0.999849990246830	0.999849990699176	0.999906185000603	0.000000000452414	0.000056203184799
10	1	0.982013790037908	0.999983164026197	0.999983165496557	0.999989541730866	0.000000001470385	0.000006377812046
10	3	0.902227400149201	0.999983164026197	0.999983164031435	0.999989541730866	0.000000000005238	0.000006377812046
12	1	0.991837428846840	0.999998122350586	0.999998122386403	0.999998836556775	0.000000000035817	0.000000714207530
12	3	0.935030830871336	0.999998122350586	0.999998122350644	0.999998836556775	0.000000000000058	0.000000714207530
14	1	0.996315760100564	0.999999791087248	0.999999791088081	0.999999870671176	0.000000000000833	0.000000079583945
14	3	0.957348747869057	0.999999791087248	0.999999791087249	0.999999870671176	0.000000000000002	0.000000079583945
16	1	0.998341198919826	0.999999976775950	0.999999976775970	0.999999985627818	0.000000000000020	0.000000008851868
16	3	0.972227825293808	0.999999976775950	0.999999976775951	0.999999985627818	0.000000000000001	0.000000008851868

Table 4.6: $\beta(x)$ upper bounds for $x = 2n$: comparison of Theorem 3.9 and Theorem 2.2.

m	x	δ_m	δ_*	Exact	T 3.9 (LB) (3.27)	T 3.9 (LB) (3.28)
0	2	0.819849008875183	0.791391472673955	0.915965594177219	0.910449396514602	0.910170530139755
0	4	0.941322477373507	0.935030830871336	0.988944551741105	0.988416181430627	0.988411719016778
0	6	0.982814576650389	0.982013790037908	0.998685222218438	0.998652370852748	0.998652344472063
0	8	0.995275161407599	0.995195247112840	0.999849990246828	0.999848310395375	0.999848310291213
0	10	0.998736100422065	0.998728983736919	0.999983164026197	0.999983086376551	0.999983086376210
0	12	0.999665247983886	0.999664649869534	0.999998122350583	0.999998118954034	0.999998118954033
0	14	0.999911629433465	0.999911580829981	0.999999791087251	0.999999790943323	0.999999790943323
0	16	0.999976695304556	0.999976691442159	0.999999976775949	0.999999976769968	0.999999976769968
2	2	0.653447205082567	0.639092745162851	0.915965594177219	0.915702541835079	0.915696304806748
2	4	0.772471193248688	0.758203827592590	0.988944551741105	0.988935551865728	0.988935357689391
2	6	0.856621899087602	0.847391335157369	0.998685222218438	0.998685015373794	0.998685012740012
2	8	0.912482820437540	0.907686971247702	0.999849990246828	0.999849986333419	0.999849986309799
2	10	0.947863576516794	0.945686733867359	0.999983164026197	0.999983163960424	0.999983163960259
2	12	0.969490170596599	0.968585628968430	0.999998122350583	0.999998122349566	0.999998122349565
2	14	0.982368039648305	0.982013790037908	0.999999791087251	0.999999791087234	0.999999791087234
2	16	0.989895953984895	0.989762712837566	0.999999976775949	0.999999976775951	0.999999976775951

Table 4.7: Theorem 3.9 lower bounds for $\beta(x)$: comparison of the optimal minimizer δ_m and the closed-form proxy δ_*.

m	x	δ_m	δ_*	Exact	T 3.9 (UB) (3.30)	T 3.9 (UB) (3.31)
1	2	0.711818661505041	0.689974481127613	0.915965594177219	0.916920013214478	0.916954921862890
1	4	0.846174055732168	0.832018385133924	0.988944551741105	0.988997090429048	0.988998197857018
1	6	0.922498050979053	0.916827303506078	0.998685222218438	0.998687120519115	0.998687134442314
1	8	0.962621708183364	0.960834277203236	0.999849990246828	0.999850046274137	0.999850046387003
1	10	0.982504982634191	0.982013790037908	0.999983164026197	0.999983165495838	0.999983165496557
1	12	0.991961829098809	0.991837428846840	0.999998122350583	0.999998122386399	0.999998122386403
1	14	0.996345728476884	0.996315760100564	0.999999791087251	0.999999791088081	0.999999791088081
1	16	0.998348191613475	0.998341198919826	0.999999976775949	0.999999976775970	0.999999976775970
3	2	0.619057099193910	0.609317541843561	0.915965594177219	0.916062054718209	0.916063617225945
3	4	0.720656397577076	0.708660825029436	0.988944551741105	0.988946749299754	0.988946789665575
3	6	0.801373851970724	0.791391472673955	0.998685222218438	0.998685256623795	0.998685257111456
3	8	0.862228638196565	0.855422249534123	0.999849990246828	0.999849990695143	0.999849990699176
3	10	0.906316727115243	0.902227400149201	0.999983164026197	0.999983164031409	0.999983164031435
3	12	0.937283276821643	0.935030830871336	0.999998122350583	0.999998122350644	0.999998122350644
3	14	0.958515219323964	0.957348747869057	0.999999791087251	0.999999791087249	0.999999791087249
3	16	0.972805598191242	0.972227825293808	0.999999976775949	0.999999976775951	0.999999976775951

Table 4.8: Theorem 3.9 upper bounds for $\beta(x)$: comparison of the optimal minimizer δ_m and the closed-form proxy δ_*.

essentially all even arguments shown.

Tables 4.11 and 4.12 compare the lower and upper bounds from Theorem 3.10 for $\beta(2n)$ with [3]'s Theorems 2.1 and 2.2 over $x \in 4, 6, 8, 12, 16$ and $m \in 0, 1, 2$. For the lower bounds (Table 4.11), the proposed construction is most effective with $m = 1$ at smaller x, while $m = 2$ becomes the preferred choice as x increases; in both cases it is generally tighter than the classical bounds, whereas $m = 0$ is comparatively weak. For the upper bounds (Table 4.12), $m = 2$ provides the tightest bounds across the grid and improves uniformly on the corresponding bounds from Theorems 2.1 and 2.2. These findings suggest using $m = 1$ (LB) for smaller even arguments, switching to $m = 2$ (LB) for larger ones, and taking $m = 2$ (UB) throughout.

n	x	m	Exact	T 3.11	T 2.2	RE (T 3.11)	RE (T 2.2)
1	2	0	0.9159655942	0.9490288737	0.9372352392	0.0360966391	0.0232210087
1	2	2	0.9159655942	0.9212568313	0.9372352392	0.0057766767	0.0232210087
2	4	0	0.9889445517	0.9917596711	0.9926343940	0.0028465897	0.0037310911
2	4	2	0.9889445517	0.9890245011	0.9926343940	0.0000808431	0.0037310911
3	6	0	0.9986852222	0.9989371670	0.9989013123	0.0002522765	0.0002163746
3	6	2	0.9986852222	0.9986865562	0.9989013123	0.0000013358	0.0002163746
4	8	0	0.9998499902	0.9998730995	0.9998593395	0.0000231128	0.0000093506
4	8	2	0.9998499902	0.9998500135	0.9998593395	0.0000000232	0.0000093506
5	10	0	0.9999831640	0.9999853287	0.9999835544	0.0000021647	0.0000003903
5	10	2	0.9999831640	0.9999831644	0.9999835544	0.0000000004	0.0000003903
6	12	0	0.9999981224	0.9999983291	0.9999981383	0.0000002067	0.0000000160
6	12	2	0.9999981224	0.9999981224	0.9999981383	0.0000000000	0.0000000160
7	14	0	0.9999997911	0.9999998112	0.9999997917	0.0000000201	0.0000000006
7	14	2	0.9999997911	0.9999997911	0.9999997917	0.0000000000	0.0000000006
8	16	0	0.9999999768	0.9999999788	0.9999999768	0.0000000020	0.0000000000
8	16	2	0.9999999768	0.9999999768	0.9999999768	0.0000000000	0.0000000000

Table 4.9: Even-m comparison of upper bounds for $\beta(2n)$: Theorem 3.11, and Theorem 2.2.

We carried out the full set of numerical comparisons, but to conserve pages we did not include all of them in the paper. For $x \geq 1$, the upper bound from Theorem 3.2 is the most reliable improvement over [3]'s Theorems 2.1 and 2.2, and it is generally at least as good as the competing construction in Theorem 3.6(a) (with $m = 1$). On the lower side for $x \geq 1$, Theorem 3.3

n	x	m	Exact	T 3.11	T 2.2	RE (T 3.11)	RE (T 2.2)
1	2	1	0.9159655942	0.8727187692	0.8948720722	0.0472144644	0.0230287274
1	2	3	0.9159655942	0.9040910417	0.8948720722	0.0129639722	0.0230287274
2	4	1	0.9889445517	0.9866932466	0.9879273754	0.0022764726	0.0010285474
2	4	3	0.9889445517	0.9887589376	0.9879273754	0.0001876892	0.0010285474
3	6	1	0.9986852222	0.9985820675	0.9986400131	0.0001032905	0.0000452686
3	6	3	0.9986852222	0.9986825733	0.9986400131	0.0000026524	0.0000452686
4	8	1	0.9998499902	0.9998455425	0.9998480737	0.0000044484	0.0000019169
4	8	3	0.9998499902	0.9998499542	0.9998480737	0.0000000361	0.0000019169
5	10	1	0.9999831640	0.9999829784	0.9999830849	0.0000001857	0.0000000791
5	10	3	0.9999831640	0.9999831635	0.9999830849	0.0000000005	0.0000000791
6	12	1	0.9999981224	0.9999981147	0.9999981191	0.0000000076	0.0000000032
6	12	3	0.9999981224	0.9999981223	0.9999981191	0.0000000000	0.0000000032
7	14	1	0.9999997911	0.9999997908	0.9999997910	0.0000000003	0.0000000001
7	14	3	0.9999997911	0.9999997911	0.9999997910	0.0000000000	0.0000000001
8	16	1	0.9999999768	0.9999999768	0.9999999768	0.0000000000	0.0000000000
8	16	3	0.9999999768	0.9999999768	0.9999999768	0.0000000000	0.0000000000

Table 4.10: Odd-m comparison of lower bounds for $\beta(2n)$: Theorem 3.11, and Theorem 2.2.

m	n	x	Exact	T 3.10	RE	T 2.1	RE	T 2.2	RE
0	2	4	0.9889445517	0.9598662325	0.0294033869	0.9795164486	0.0095335003	0.9879273754	0.0010285474
0	3	6	0.9986852222	0.9953428244	0.0033467981	0.9973323061	0.0013546972	0.9986400131	0.0000452686
0	4	8	0.9998499902	0.9994803910	0.0003696547	0.9996850055	0.0001650095	0.9998480737	0.0000019169
0	6	12	0.9999981224	0.9999937214	0.0000044010	0.9999959861	0.0000021363	0.9999981191	0.0000000032
0	8	16	0.9999999768	0.9999999241	0.0000000527	0.9999999502	0.0000000265	0.9999999768	0.0000000000
1	2	4	0.9889445517	0.9881194867	0.0008342885	0.9795164486	0.0095335003	0.9879273754	0.0010285474
1	3	6	0.9986852222	0.9986450255	0.0000402496	0.9973323061	0.0013546972	0.9986400131	0.0000452686
1	4	8	0.9998499902	0.9998482230	0.0000017675	0.9996850055	0.0001650095	0.9998480737	0.0000019169
1	6	12	0.9999981224	0.9999981193	0.0000000031	0.9999959861	0.0000021363	0.9999981191	0.0000000032
1	8	16	0.9999999768	0.9999999768	0.0000000000	0.9999999502	0.0000000265	0.9999999768	0.0000000000
2	2	4	0.9889445517	0.9861899168	0.0027854291	0.9795164486	0.0095335003	0.9879273754	0.0010285474
2	3	6	0.9986852222	0.9986280106	0.0000572869	0.9973323061	0.0013546972	0.9986400131	0.0000452686
2	4	8	0.9998499902	0.9998488014	0.0000011890	0.9996850055	0.0001650095	0.9998480737	0.0000019169
2	6	12	0.9999981224	0.9999981218	0.0000000005	0.9999959861	0.0000021363	0.9999981191	0.0000000032
2	8	16	0.9999999768	0.9999999768	0.0000000000	0.9999999502	0.0000000265	0.9999999768	0.0000000000

Table 4.11: Comparison of lower bounds for $\beta(2n)$: Theorem 3.10, Theorem 2.1 and Theorem 2.2.

m	n	x	Exact	T 3.10	RE	T 2.1	RE	T 2.2	RE
0	2	4	0.9889445517	0.9973849190	0.0085347224	0.9936375043	0.0047454152	0.9926343940	0.0037310911
0	3	6	0.9986852222	0.9996552966	0.0009713515	0.9989013123	0.0002163746	0.9991630152	0.0004784220
0	4	8	0.9998499902	0.9999584894	0.0001085155	0.9998593395	0.0000093506	0.9999061850	0.0000562032
0	6	12	0.9999981224	0.9999994490	0.0000013266	0.9999981383	0.0000000160	0.9999988366	0.0000007142
0	8	16	0.9999999768	0.9999999930	0.0000000162	0.9999999768	0.0000000000	0.9999999856	0.0000000089
1	2	4	0.9889445517	0.9961886880	0.0073251188	0.9936375043	0.0047454152	0.9926343940	0.0037310911
1	3	6	0.9986852222	0.9989846503	0.0002998223	0.9989013123	0.0002163746	0.9991630152	0.0004784220
1	4	8	0.9998499902	0.9998621313	0.0000121429	0.9998593395	0.0000093506	0.9999061850	0.0000562032
1	6	12	0.9999981224	0.9999981416	0.0000000193	0.9999981383	0.0000000160	0.9999988366	0.0000007142
1	8	16	0.9999999768	0.9999999768	0.0000000000	0.9999999768	0.0000000000	0.9999999856	0.0000000089
2	2	4	0.9889445517	0.9892292555	0.0002878865	0.9936375043	0.0047454152	0.9926343940	0.0037310911
2	3	6	0.9986852222	0.9986915703	0.0000063564	0.9989013123	0.0002163746	0.9991630152	0.0004784220
2	4	8	0.9998499902	0.9998501277	0.0000001375	0.9998593395	0.0000093506	0.9999061850	0.0000562032
2	6	12	0.9999981224	0.9999981224	0.0000000001	0.9999981383	0.0000000160	0.9999988366	0.0000007142
2	8	16	0.9999999768	0.9999999768	0.0000000000	0.9999999768	0.0000000000	0.9999999856	0.0000000089

Table 4.12: Comparison of upper bounds for $\beta(2n)$: Theorem 3.10, Theorem 2.1 and Theorem 2.2.

is typically the sharpest, although [3]'s Theorem 2.2 can retain a small advantage in a narrow middle region, and among the alternative lower bound candidates Theorem 2.2 can still be best relative to Theorems 2.1 and 3.1. For $0 < x < 1$ upper bounds, no single theorem dominates: Theorem 2.1 is strongest at the far-left, then Theorems 3.6(d) (with $m = 2$) and 3.1 take over in the midrange, and Theorem 3.3 becomes best as $x \to 1^-$. For $0 < x < 1$ lower bounds, Theorem 3.2 markedly improves on Theorem 2.1 over most of the interval, with Theorem 3.6(c) (with $m = 1$) mainly useful near the extreme left and tending to become preferable again toward the right side. Restricting to even arguments, the proposed lower-envelope improvements are most evident: Theorem 3.4 strengthens Theorem 2.1, and the truncation-based Theorem 3.8 (with $m = 1$) typically improving on both [3]'s Theorem 2.2 and Theorem 3.5; within the parameterized families, Theorem 3.9 is best with $m = 1$ for upper bounds and $m = 2$ for lower bounds, while Theorem 3.10 can outperform Theorem 2.2 but is sensitive to m for upper bounds and is most consistently strong for lower bounds when $m = 1$. Theorem 3.5 becomes increasingly accurate as n grows and provides a dependable lower bound in the large-n regime. However, it remains systematically looser than both Theorem 2.2 (LB) and the truncated Theorem 3.8 construction, especially for small to moderate n.

5. CONCLUSION

In this paper, we develop a collection of new upper and lower bounds for the Dirichlet beta function $\beta(x)$ that couple analytic representations with probabilistic and reliability-theoretic structure. Starting from the integral form (1.2), we derived explicit one- and two-term envelopes for $\beta(x)$ by controlling the residual series through covariance inequalities and monotonicity arguments (Theorems 3.1, 3.2, 3.3, and Theorem 3.6). These bounds are simple closed expressions involving only elementary constants and a small number of nearby beta values, making them practical for computation. A complementary set of bounds follows from reliability-theory moment inequalities, including IFR and IFRA-type arguments and Lyapunov-style comparisons, which yield two-sided estimates in terms of $\beta(2n\pm1)$ and factorial factors (Theorem 3.5). We also established truncation-based inequalities that quantify the alternating-series remainder with tight, easily evaluated error terms (Theorem 3.6). Thus, these results provide a coherent catalogue of bounds that covers both regimes $x \geq 1$ and $0 < x < 1$, as well as the numerically important even integers. Unlike many existing estimates, our formulas do not rely on delicate special-function expansions and instead follow from general convexity and stochastic-ordering principles, which helps explain why adjacent beta values can effectively control intermediate ones. This perspective also suggests systematic ways to generate further inequalities, including families parameterized by the truncation level m or by neighboring points $x_1 < x_2 < x_3$, so that accuracy can be tuned to the application. Among the proposed bounds, Theorem 3.4 is distinct because it is the only theorem that involves two $\beta(2n-1)$ terms, and it can outperform [3]'s bound in the small-x regime, so it is worth highlighting.

Extensive numerical comparisons confirm that the proposed bounds are consistently sharper than the classical comparisons of [3] across the tested grids. For $x \geq 1$, the new upper bound in Theorem 3.2 uniformly improves upon the corresponding upper bounds in Theorems 2.1 and 2.2, with relative errors decaying rapidly as x increases. On $(0, 1)$, the lower bound furnished by Theorem 3.2 dominates the existing lower bound from Theorem 2.1 except at the extreme left edge, where a short truncation such as Theorem 3.6(c) with $m = 1$ can be slightly tighter. When restricting to even indices, the improvements are especially pronounced, with truncation-enhanced constructions giving near machine-precision accuracy for moderate n and, for suitable parity choices, outperforming [3]'s bounds. We also note that the simplest specialization $m = 0$

in the even-integer families, which uses only the first partial sum $S(0, x) = 1$ and therefore does not depend on x, can still yield a better bound than Cerone's for sufficiently large x, and for $0 < x < 1$, despite its simplicity. In particular, Theorem 3.9 provides very good approximations for m as small as $m = 0, 1$ and does not use any $\beta(n)$ values for odd values of n. Although the IFRA reliability-theory based bounds can be numerically inferior to the strongest even-integer bounds such as Theorem 3.10, we intentionally include the corresponding comparison table because the IFRA methodology is quite general and can be adapted to bound many other special functions, including the Riemann zeta function, and we plan to report on these extensions in future work. Beyond immediate numerical gains, the probabilistic viewpoint suggests further directions, such as optimizing constants in the covariance-based bounds, sharpening remainder terms using higher-order stochastic bounds, and developing asymptotically effective rules for selecting m as a function of x or n to balance computational cost and accuracy. Therefore, the results provide improved numerical envelopes for $\beta(x)$ and a widely applicable methodology for producing further bounds that can outperform existing estimates.

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