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LYAPUNOV-BASED UNIFORM ASYMPTOTIC STABILITY OF CAPUTO FRACTIONAL DYNAMIC SYSTEMS WITH APPLICATIONS TO COMPUTING SYSTEMS

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ABSTRACT. This paper develops a novel framework for uniform asymptotic stability (UAS) analysis of Caputo fractional dynamic equations on time scales using vector Lyapunov functions and the newly introduced Caputo fractional delta Dini derivative. Our approach establishes rigorous UAS criteria that guarantee consistent boundedness and uniform convergence of system trajectories, extending classical stability theory to hybrid systems with memory effects. The theoretical advancements unify continuous and discrete analysis through time-scale calculus while incorporating fractional-order dynamics. We demonstrate the practical efficacy of the framework through two applications: stabilization of a fractional-order robotic arm with PD control and load balancing in distributed computing systems. The results provide both theoretical insights into fractional hybrid systems and practical tools for engineering applications where memory effects and mixed continuous-discrete dynamics coexist.

Key words and phrases: Uniform asymptotic stability; time scale; control systems; Fractional derivative.

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1. INTRODUCTION

In recent decades, fractional calculus has rapidly evolved from a purely theoretical branch of mathematical analysis into a powerful tool for modelling real-world phenomena. Its ability to generalize classical integer-order differentiation and integration to non-integer orders allows it to capture memory effects, hereditary properties, and anomalous dynamics pervasive in disciplines such as control theory, viscoelasticity, biology, signal processing, and electrical engineering. Traditional integer-order models often fall short in accurately representing these complex behaviours, particularly when system responses depend not only on current states but also on historical data. Fractional differential equations (FDEs) thus provide a more nuanced and realistic framework for describing such systems.

However, analysing the stability of fractional dynamic models remains a central challenge. The Lyapunov direct method is a key analytical tool for this purpose, as it avoids the need for explicit solutions by instead relying on the construction of Lyapunov functions to infer long-term behaviour [1]. While effective for many systems, the classical scalar Lyapunov function approach has limitations, especially in high-dimensional or strongly coupled systems, where it may fail to fully capture intricate inter-variable interactions, leading to conservative or incomplete stability conditions.

To address this, recent advancements have turned to vector Lyapunov functions (VLFs) (see [13]), which decompose complex dynamics into component-wise analyses. This framework offers greater flexibility and precision, particularly for multidimensional fractional systems with critical inter-variable dependencies. In this study, we leverage VLFs to analyze the stability of Caputo fractional dynamic equations on time scales (CFrDyE$\mathbb{T}$), a class of hybrid systems exhibiting both continuous and discrete evolution.

A pivotal challenge in this domain is integrating fractional calculus with time scale theory, a unifying framework introduced by [5] that generalizes differential [3] and difference [12] equations into a single coherent theory. Time scale calculus enables the study of systems whose temporal behaviour is neither purely continuous nor discrete but instead defined on a non-empty closed subset of the real line, $\mathbb{T}$. Such hybrid dynamics arise frequently in engineering and natural sciences, where events occur at both fixed instants and over continuous intervals. Combining time scales with the non-local properties of fractional operators introduces significant mathematical complexity, necessitating specialized analytical tools.

To tackle these challenges, we employ the Caputo fractional delta Dini derivative (CFrΔDiD) [10], a generalized derivative designed for fractional systems on time scales. Unlike conventional fractional derivatives, the CFrΔDiD captures both non-local memory effects and discrete-continuous temporal shifts, enabling stability criteria robust across arbitrary time scales. This expands the applicability of fractional calculus to a broader class of real-world systems.

A key focus of this study is UAS, a strong form of stability ensuring that all system trajectories not only remain bounded but also converge uniformly over time, regardless of initial conditions. UAS is critical in control and systems engineering, where reliability, predictability, and convergence speed are paramount. Building on foundational works [1, 6], our core contribution is a novel UAS framework for CFrDyE$\mathbb{T}$ using VLFs and the CFrΔDiD. This approach ensures uniform convergence across all starting times,a practical necessity for real-world implementations requiring consistent long-term performance.

Our proposed UAS framework for CFrDyE$\mathbb{T}$ achieves its practical validation through two implemented case studies that embody the core challenges of hybrid fractional-order systems.

The first application to robotic arm control systems reveals how the approach stabilizes PD-controlled mechanisms with viscoelastic joints, where the fractional-order dynamics capture essential memory effects and the time-scale formulation naturally accommodates mixed continuous-discrete operation. The second implementation in distributed computing systems demonstrates the method's effectiveness for load balancing across server networks, proving particularly adept at handling the inherent memory effects in task queue dynamics while operating across the continuous-discrete spectrum of computing events. These working implementations serve to concretely establish three key practical advantages; the capacity to provide rigorous stability guarantees for systems exhibiting both memory-dependent behaviour and hybrid time domains, the ability to bridge traditional gaps between continuous control theory and discrete-event system analysis, and the formulation of directly applicable stability criteria for engineering design. The robotic application specifically validates the framework's utility for physical systems with compliance-induced memory effects, while the computing case confirms its operational value for discrete resource allocation problems with temporal coupling, together spanning the spectrum of real-world applications that require simultaneous handling of fractional dynamics and time-scale hybridity.

The structure of the paper is as follows: in Section 2, we introduce the basic definitions, notations, and preliminary lemmas that form the foundation of our analysis. In Section 3, we derive new UAS criteria for CFrDyE$\mathbb{T}$. In Section 4, we give detailed mathematical illustrations of our results. To demonstrate the efficacy and real-world relevance of our theoretical contributions, Section 5 provides two practical examples in computing systems; the fractional robotic arm control system with PD feedback and the fractional load balancing in distributed computing systems that illustrate the practical application of the derived results.

2. PRELIMINARY NOTES

Definition 2.1. [4] Let $t \in \mathbb{T}$, then for the course of this work, $\wp(t)$ will denote the usual forward jump operator, $\rho(t)$ will denote the backward jump operator and $\kappa(t) := \wp(t) - t$ will denote the graininess function (see Definition 1.1, table 1.1 and Definition 1.58 in [4] for more details).

Also, C_{rd} will represent right dense continuity, see in [4].

Definition 2.2 **([4]).** A function $\mathfrak{K} \in C([0,\mathfrak{j}],[0,\infty))$ is referred to as a class $\mathcal{K}$ function if it satisfies the following: it is strictly increasing over the interval $[0,\mathfrak{j}]$ and satisfies $\mathfrak{K}(0) = 0$.

STATEMENT OF THE PROBLEM

In this work, we focus on the CFrDyE$\mathbb{T}$ of order τ with $0 < \tau < 1$

$$\begin{aligned} {}^{C\mathbb{T}}D^{\tau}\aleph^{\Delta} &= \Re(t,\aleph),\ t \in \mathbb{T}, \\ \aleph(t_0) &= \aleph_0,\ t_0 \geq 0. \end{aligned} \tag{2.1}$$

Here, $\Re \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$, $\Re(t,0) \equiv 0$, and ${}^{C\mathbb{T}}D^{\tau}\aleph^{\Delta}$ represents the Caputo fractional delta derivative of $\aleph \in \mathbb{R}^N$ of order τ with respect to $t \in \mathbb{T}$. Let $\aleph(t) = \aleph(t,t_0,\aleph_0) \in C^{\tau}_{rd}[\mathbb{T},\mathbb{R}^N]$ (indicating that the fractional derivative of $\aleph(t)$ of order τ exists and is rd-continuous) be a solution of (2.1), assuming the solution is both unique and exists, see [2, 11]. This work focuses on analysing the UAS of the system described in (2.1).

To achieve this, we will utilize the comparison system of lower dimension of the form

$$ {}^{C\mathbb{T}}D^{\tau}\varphi^{\Delta} = \Im(t,\varphi),\ \varphi(t_0) = \varphi_0 \geq 0. \tag{2.2}$$

Here, $\varphi \in \mathbb{R}^n_+$, $\Im : \mathbb{T} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$, and $\Im(t, 0) \equiv 0$. The system (2.2) with $\varphi(t_0) = \varphi_0$ has a unique solution $\varphi(t) = \varphi(t; t_0, \varphi_0) \in C^\tau_{rd}[\mathbb{T}, \mathbb{R}^n_+]$. For more details, see [2, 6].

Remark 2.1. For the purpose of this work, it is important to note that all inequalities between vectors are to be interpreted component-wise.

Definition 2.3. [7] The CFrΔDiD of the LF $\Upsilon(t, \aleph) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ along the solution trajectories of the system (2.1) is expressed as:

$$\begin{aligned} &{}^{C\mathbb{T}}D^\tau_+ \Upsilon^\Delta(t, \aleph) \\ =\ & \limsup_{\kappa \to 0^+} \frac{1}{\kappa^\tau} \Bigg[\sum_{j=0}^{[\frac{t-t_0}{\kappa}]} (-1)^j ({}^\tau C_j) [\Upsilon(\wp(t) - j\kappa, \aleph(\wp(t)) - \kappa^\tau \Re(t, \aleph(t)) \\ & -\Upsilon(t_0, \aleph_0)] \Bigg], \end{aligned}$$

where $t \in \mathbb{T}, \aleph, \aleph_0 \in \mathbb{R}^N$, $\kappa = \wp(t) - t$ and $\aleph(\wp(t)) - \kappa^\tau \Re(t, \aleph) \in \mathbb{R}^N$,
which can further be expanded to obtain

$$\begin{aligned} {}^{C\mathbb{T}}D^\tau_+ \Upsilon^\Delta(t, \aleph) &= \limsup_{\kappa \to 0^+} \frac{1}{\kappa^\tau} \Bigg\{ \Upsilon(\wp(t), \aleph(\wp(t))) \\ &\quad + \sum_{j=1}^{[\frac{t-t_0}{\kappa}]} (-1)^j ({}^\tau C_r) [\Upsilon(\wp(t) - j\kappa, \aleph(\wp(t)) - \kappa^\tau \Re(t, \aleph(t)))] \Bigg\} \\ &\quad - \frac{\Upsilon(t_0, \aleph_0)(t - t_0)^{-\tau}}{\Gamma(1 - \tau)}, \end{aligned} \tag{2.3}$$

where Γ is the gamma function such that

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Definition 2.4. The zero solution of 2.1 is said to be uniformly asymptotically stable if it satisfies the following conditions:

- (S_1) **Uniform Stability:** For every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ (independent of t_0) such that:
$$\|\aleph(t_0)\| < \delta \implies \|\aleph(t)\| < \epsilon \quad \forall t \geq t_0.$$
- (S_2) **Uniform Attractivity:** There exists $\delta_0 > 0$ (independent of t_0) and a function $T(\epsilon) > 0$ (also independent of t_0) such that:
$$\|\aleph(t_0)\| < \delta_0 \implies \|\aleph(t)\| < \epsilon \quad \forall t \geq t_0 + T(\epsilon).$$

Lemma 2.1. [7] *Let $\theta, \phi \in C_{rd}([t_0, T]_\mathbb{T}, \mathbb{R}^n)$, and assume $\exists\ t_1 \in (t_0, T]_\mathbb{T}$: $\theta(t_1) = \phi(t_1)$ and $\theta(t) < \phi(t)$ for $t \in [t_0, t_1)_\mathbb{T}$. If the CFr$\Delta$DiD of θ and ϕ are defined at $t = t_1$, then the inequality*

$${}^{C\mathbb{T}}D^\tau_+ \theta^\Delta(t_1) >^{C\mathbb{T}} D^\tau_+ \phi^\Delta(t_1),$$

holds.

Lemma 2.2. [9] *Assume the following conditions hold:*

- (i) $\Im \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$ *and $\Im(t, \varphi)\kappa$ is non-decreasing with respect to φ.*
- (ii) $\Upsilon \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *is locally Lipschitz with respect to its second argument, and satisfies*

$${}^{C\mathbb{T}}D^\tau_+ \Upsilon^\Delta(t, \aleph) \leq \Im(t, \Upsilon(t, \aleph)), \quad (t, \aleph) \in \mathbb{T} \times \mathbb{R}^N. \tag{2.4}$$

(iii) $\varpi(t) = \varpi(t; t_0, \varphi_0)$ *represents the maximal solution of* (2.2)*, which exists on* $\mathbb{T}$.

Then, the inequality

$$\Upsilon(t, \aleph(t)) \leq \varpi(t), \quad t \geq t_0, \tag{2.5}$$

is satisfied, provided that

$$\Upsilon(t_0, \aleph_0) \leq \varphi_0, \tag{2.6}$$

where $\aleph(t) = \aleph(t; t_0, \aleph_0)$ *denotes any solution of* (2.1) *for* $t \in \mathbb{T}$ *and* $t \geq t_0$.

Proof. See [9] for proof. ∎

Lemma 2.3 (Uniform Stability)**.** [9] *Consider the following assumptions:*

(1) $\Im \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$, *where* $\Im(t, \varphi)$ *is non-decreasing in* φ *and satisfies* $\Im(t, \varphi) \equiv 0$.

(2) *The function* $\Upsilon(t, \aleph(t)) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *satisfies the following properties:*

  (i) Υ *is locally Lipschitz with respect to* $\aleph$*, and* $\Upsilon(t, 0) \equiv 0$.

  (ii) *The bounds* $b(\|\aleph\|) \leq \Upsilon_0(t, \aleph) \leq a(\|\aleph\|)$ *hold, where* $\Upsilon_0(t, \aleph) = \sum_{i=1}^N \Upsilon_i(t, \aleph(t))$*, and* $a, b \in \mathcal{K}$.

  (iii) *For any* $t, t_0 \geq 0$ *and* $\aleph, \aleph_0 \in \mathbb{R}^N$*, the inequality*

$$^{C\mathbb{T}}D^{\tau}_{+}\Upsilon^{\Delta}(t, \aleph(t)) \leq \Im(t, \Upsilon(t, \aleph(t))),$$

  holds.

(3) *The zero solution of the system* (2.2) *is uniformly stable.*

Under these assumptions, the zero solution of the system (2.1) *is also uniformly stable.*

3. MAIN RESULTS

Theorem 3.1 (Uniform asymptotic Stability)**.** *Suppose the following conditions are satisfied:*

1. $\Upsilon(t, \aleph) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$*, and* Υ *is Lipschitz continuous with respect to* $\aleph$.
2. $\Im \in C_{rd}[\mathbb{T} \times \mathbb{R}^N_+, \mathbb{R}^N_+]$*, where* $\Im(t, \varphi)$ *is non-decreasing with respect to* φ*, and* $\Im(t, \varphi) \equiv 0$.
3. *For* $(t, \aleph) \in \mathbb{R}_+ \times \mathbb{R}^n$*, the inequality*

$$^{C\mathbb{T}}D^{\tau}_{+}\Upsilon^{\Delta}_0(t, \aleph) \leq -c(\|\aleph\|),$$

   holds, where $\Upsilon_0(t, \aleph) = \sum_{i=1}^N \Upsilon_i(t, \aleph(t))$*, and* $c \in \mathcal{K}$.
4. *The bounds* $\omega(\|\aleph\|) \leq \Upsilon_0(t, \aleph) \leq \psi(\|\aleph\|)$ *are satisfied, where* $\psi, \omega \in \mathcal{K}$.
5. *The zero solution* $\aleph = 0$ *of system* (2.1) *is uniformly stable.*

Under these assumptions, the zero solution $\aleph = 0$ *of system* (2.1) *is uniformly asymptotically stable.*

Proof. The zero solution $\aleph^* = 0$ of (2.1) is considered uniformly asymptotically stable if it meets the criterion for uniform stability (as established in Lemma 2.3) and, for any given $\eta > 0$, there exists a time $T = T(\eta) > 0$ such that for all $t_0 \in \mathbb{T}$ and $\aleph_0 \in \mathbb{R}^N$ satisfying $\|\aleph_0\| \leq \delta$, the condition $\|\aleph^*(t; t_0, \aleph_0)\| < \eta$ holds whenever $t \geq t_0 + T$.
Let $\lambda \in (0, \delta)$ be a constant such that

$$\psi(\lambda) < \omega(\delta), \quad \text{with} \quad \|\aleph_0\| < \lambda. \tag{3.1}$$

Assumption (4) and (3.1) at $(t_0, \aleph_0)$, gives

$$\omega(\|\aleph_0\|) \leq \Upsilon_0(t_0, \aleph_0) \leq \psi(\|\aleph_0\|) < \psi(\lambda) < \omega(\delta). \tag{3.2}$$

Clearly from (3.2), $\|\aleph_0\| < \delta$.
Let $\eta = \eta(\epsilon), \eta \in (0, \epsilon)$.
We claim that

$$\|\aleph_0\| < \delta \quad \Longrightarrow \quad \|\aleph^*(t)\| < \eta, \quad \text{for } t \geq t_0 + T. \tag{3.3}$$

Assume (3.3) is not true, then there exist at least one $t \geq t_0 + T$, such that

$$\|\aleph_0\| < \delta \quad \Longrightarrow \quad \|\aleph^*(t)\| \geq \eta. \tag{3.4}$$

Since $c \in \mathcal{K}$, condition (2) of the theorem can be written as

$${}^{C\mathbb{T}}D_+^{\tau}\Upsilon_0^{\Delta}(t, \aleph^*) \leq -c(\eta), \tag{3.5}$$

which is equivalent to the Volterra integral equation

$$\Upsilon_0(t, \aleph^*(t)) \leq \Upsilon_0(t_0, \aleph_0) - \frac{c(\eta)}{\Gamma(\tau)} \int_{t_0}^{t} (t-s)^{\tau-1} \Delta s \quad \text{for} \quad t \geq t_0 + T, \tag{3.6}$$

Then, from **??**, we can deduce the following:
If $\mathbb{T} = \mathbb{R}$, then (3.6) becomes

$$\Upsilon_0(t, \aleph^*(t)) \leq \Upsilon_0(t_0, \aleph_0) - \frac{c(\eta)(t - t_0)}{\tau\Gamma(\tau)}, \tag{3.7}$$

and for sufficiently large t, (3.7) contradicts the condition (4) of the Theorem and in-fact definition of $\Upsilon(t, \aleph(t))$ and so (3.3) holds when $\mathbb{T} = \mathbb{R}$.
If $\mathbb{T}$ consists of only l-s and r-s points, then (3.6) becomes

$$\Upsilon_0(t, \aleph^*(t)) \leq \Upsilon_0(t_0, \aleph_0) - \frac{c(\eta)}{\Gamma(\tau)} \sum_{j=1}^{i} \kappa(t_j) r(t_j), \tag{3.8}$$

for all $t_i, t_j \in \mathbb{T}$ and $r(t) = (t-s)^{\tau-1}$.
as $i \to \infty$, the right hand side of (3.8) tends to $-\infty$ which is also a contradiction.
If $\mathbb{T}$ consists of t_i such that $\wp(t) > t$, then (3.6) becomes

$$\Upsilon_0(t, \aleph^*(t)) \leq \Upsilon_0(t_0, \aleph_0) - \frac{c(\eta)}{\Gamma(\tau)} \kappa(t_i) r(t_i) \tag{3.9}$$

where $r(t) = (t-s)^{\alpha-1}$. As $i \to \infty$, the right hand side of (3.9) approaches $-\infty$. Which is another contradiction, implying that (3.3) holds across different time scales.
Therefore the zero solution of (2.1) is uniformly asymptotically stable. ∎

4. ILLUSTRATIONS

Illustration 1

Given the following system

$$\begin{aligned} {}^{C\mathbb{T}}D^{\tau}\theta_1^{\Delta}(t) &= \; -2\theta_1 - \theta_1^2 \exp(\theta_1) - \theta_2^2 \exp(\theta_2) \\ {}^{C\mathbb{T}}D^{\tau}\theta_2^{\Delta}(t) &= \; -\theta_1^2 \exp(\theta_2) - 2\theta_2 - \theta_2^2 \exp(\theta_1) \end{aligned} \tag{4.1}$$

with initial conditions

$$\theta_1(t_0) = \theta_{10} \quad \text{and} \quad \theta_2(t_0) = \theta_{20},$$

Choose a vector LF $\Upsilon = (\Upsilon_1, \Upsilon_2)^T$, such that $\Upsilon_1(t, \theta_1, \theta_2) = |\theta_1|$ and $\Upsilon_2(t, \theta_1, \theta_2) = |\theta_2|$, for $t \in \mathbb{T}$ and $(\theta_1, \theta_2) \in \mathbb{R}^2$.
From (2.3), we compute the CFrΔDiD for $\mathcal{V}_1(t, \theta_1, \theta_2) = |\theta_1|$ as follows

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^{\tau}\Upsilon_1^{\Delta}(t,\aleph) &= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Bigg\{\Upsilon_1(\wp(t),\aleph(\wp(t)) \\
&\quad + \sum_{j=1}^{[\frac{t-t_0}{\kappa}]} (-1)^j ({}^\tau C_j)\left[\Upsilon_1(\wp(t)-j\kappa, \aleph(\wp(t)) - \kappa^\tau \Re_1(t,\aleph(t)))\right]\Bigg\} \\
&\quad - \frac{\Upsilon_1(t_0,\aleph_0)(t-t_0)^{-\tau}}{\Gamma(1-\tau)}
\end{aligned}$$

$$\begin{aligned}
&= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{|\theta_1(\wp(t))| + \sum_{j=1}^{[\frac{t-t_0}{\kappa}]} (-1)^j ({}^\tau C_j)[|\theta_1(\wp(t)) - \kappa^\tau \Re_1(t,\theta_1)|]\right\} \\
&\quad - \frac{|\theta_{10}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \\
&\le \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{|\theta_1(\wp(t))| + \sum_{j=1}^{[\frac{t-t_0}{\kappa}]} (-1)^j ({}^\tau C_j)[|\theta_1(\wp(t))| + |\kappa^\tau \Re_1(t,\theta_1)|]\right\} \\
&\quad - \frac{|\theta_{10}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \\
&\le \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Bigg\{|\theta_1(\wp(t))| + \sum_{j=1}^{\lfloor\frac{t-t_0}{\kappa}\rfloor} (-1)^j {}^\tau C_j |\theta_1(\wp(t))| \\
&\quad + \sum_{j=1}^{\lfloor\frac{t-t_0}{\kappa}\rfloor} (-1)^j {}^\tau C_j |\kappa^\tau \Re_1(t,\theta_1)|\Bigg\} - \frac{|\theta_{10}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)}
\end{aligned}$$

$$\begin{aligned}
&\le |\theta_1(\wp(t))| \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau} \sum_{j=0}^{\lfloor\frac{t-t_0}{\kappa}\rfloor} (-1)^j {}^\tau C_j + |\Re_1(t,\theta_1)| \limsup_{\kappa\to 0^+} \sum_{j=1}^{\lfloor\frac{t-t_0}{\kappa}\rfloor} (-1)^j {}^\tau C_j \\
&\quad - \frac{|\theta_{10}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)}.
\end{aligned}$$

Using (2.12) and (2.14) in [8], we obtain

$$ {}^{C\mathbb{T}}D_+^{\tau}\Upsilon_1^{\Delta}(t,\theta_1) = \frac{|\theta_1(\wp(t))|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} - |\Re_1(t;\theta_1)| - \frac{|\theta_{10}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} $$

$$ {}^{C\mathbb{T}}D_+^{\tau}\Upsilon_1^{\Delta}(t,\theta_1) \le \frac{|\theta_1(\wp(t))|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} - |\Re_1(t;\theta_1)|. $$

when $t \to \infty$, then $\frac{|\theta_1(\wp(t))|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \to 0$, so that

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^\tau \Upsilon_1^\Delta(t,\theta_1) &\leq -|\Re_1(t;\theta_1)| \\
&= -\left[|-2\theta_1 - \theta_1^2\exp(\theta_1) - \theta_2^2\exp(\theta_2)|\right] \\
&\leq -[|-2\theta_1|] \\
&\leq -[2|\theta_1|] \\
{}^{C\mathbb{T}}D_+^\tau \Upsilon_1^\Delta(t,\theta_1) &\leq -2\Upsilon_1 + 0\Upsilon_2. \qquad (4.2)
\end{aligned}$$

Also, we compute the CFrΔDiD for $\Upsilon_2(t,\theta_1,\theta_2) = |\theta_2|$ as follows:

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^\tau \Upsilon_2^\Delta(t,\theta_2) &= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Bigg\{\Upsilon_2(\wp(t),\theta_2(\wp(t))) \\
&\quad + \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j)\left[\Upsilon_2(\wp(t)-j\kappa, \theta_2(\wp(t)) - \kappa^\tau \Re_2(t,\theta_2(t)))\right]\Bigg\} \\
&\quad - \frac{\Upsilon_2(t_0,\theta_{20})(t-t_0)^{-\tau}}{\Gamma(1-\tau)}.
\end{aligned}$$

$$\begin{aligned}
&= \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{|\theta_2(\wp(t))| + \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j)[|\theta_2(\wp(t)) - \kappa^\tau\Re_2(t,\theta_2)|]\right\} \\
&\quad - \frac{|\theta_{20}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \\
&\leq \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{|\theta_2(\wp(t))| + \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j)[|\theta_2(\wp(t))| + |\kappa^\tau\Re_2(t,\theta_2)|]\right\} \\
&\quad - \frac{|\theta_{20}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \\
&\leq \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Bigg\{|\theta_2(\wp(t))| + \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j)|\theta_2(\wp(t))| \\
&\quad + \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j)|\kappa^\tau\Re_2(t,\theta_2)|\Bigg\} - \frac{|\theta_{20}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \\
&\leq |\theta_2(\wp(t))| \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau} \sum_{j=0}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j) \\
&\quad + |\Re_2(t,\theta_2)| \limsup_{\kappa\to 0^+} \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa}\rfloor} (-1)^j({}^\tau C_j) - \frac{|\theta_{20}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)}.
\end{aligned}$$

From (2.12) and (2.14) in [8] we obtain

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_2^\Delta(t,\theta_2) = \frac{|\theta_2(\wp(t))|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} - |\Re_2(t;\theta_2)| - \frac{|\theta_{20}|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} $$

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_2^{\Delta}(t,\theta_2) \leq \frac{|\theta_2(\wp(t))|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} - |\Re_2(t;\theta_2)|.$$

when $t \to \infty$, $\frac{|\theta_2(\wp(t))|(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \to 0$, so that,

$$\begin{aligned}
^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_2^{\Delta}(t,\theta_2) &\leq -|\Re_2(t;\theta_2)| \\
&= -\left[|-\theta_1^2\exp(\theta_2) - 2\theta_2 - \theta_2^2\exp(\theta_1)|\right] \\
&\leq -\left[|-2\theta_2|\right] \\
&\leq -2|\theta_2|.
\end{aligned}$$

Therefore,

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_2^{\Delta}(t,\theta_2) \leq 0\Upsilon_1 - 2\Upsilon_2. \tag{4.3}$$

From (4.2) and (4.3), we have that

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon^{\Delta} \leq \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \Im(t,\Upsilon). \tag{4.4}$$

So that the comparison system for (4.4) becomes

$$^{C\mathbb{T}}D_{+}^{\tau}\varphi^{\Delta} = \Im(t,\varphi) = Y\varphi, \tag{4.5}$$

where $Y = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.
The vector inequality (4.4), along with all the requirements outlined in Theorem 2.3, is fulfilled if the eigenvalues of Y possess negative real parts. Given that the eigenvalues of Y are both -2, it follows that (4.1) is uniformly asymptotically stable.

Illustration 2

Given the CFrDyE$\mathbb{T}$

$$\begin{aligned}
^{C\mathbb{T}}D^{\tau}\zeta_1^{\Delta}(t) &= \quad 3\zeta_1 - \frac{\zeta_2^2 + 2\zeta_3^2}{\zeta_1} \\
^{C\mathbb{T}}D^{\tau}\zeta_2^{\Delta}(t) &= \quad -\frac{\zeta_1^2 +}{\zeta_2} + 4\zeta_2 - \frac{3\zeta_3^2}{\zeta_2} \\
^{C\mathbb{T}}D^{\tau}\zeta_3^{\Delta}(t) &= \quad -\frac{\zeta_1^2 + \zeta_2}{\zeta_3} + 3\zeta_3,
\end{aligned} \tag{4.6}$$

with initial conditions

$$\zeta_1(t_0) = \zeta_{10}, \quad \zeta_2(t_0) = \zeta_{20}, \quad \text{and} \quad \zeta_3(t_0) = \zeta_{30},$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$.

Choose a vector LF $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)^T$, where $\Upsilon_1 = \zeta_1^2$, $\Upsilon_2 = \zeta_2^2$ and $\Upsilon_3 = \zeta_3^2$. From (**??**), we compute the CFrΔDiD for $\Upsilon_1 = \zeta_1^2$ as follows:

$$\begin{aligned}
&{}^{C\mathbb{T}}D_+^{\tau}\Upsilon_1^{\Delta} \\
= &\limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Big\{ \big[(\zeta_1(\wp(t)))^2\big] - \big[(\zeta_{10})^2\big] \\
&+ \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa} \rfloor} (-1)^j \left({}^\tau C_j\right) \left[(\zeta_1(\wp(t)) - \kappa^\tau \Re_1(t,\zeta))^2\right] - \left[((\zeta_{10})^2\right]\Big\} \\
= &\limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Big\{ \big[(\zeta_1(\wp(t)))^2\big] - \big[(\zeta_{10})^2\big] \\
&+ \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa} \rfloor} (-1)^j \left({}^\tau C_j\right) \left[(\zeta_1(\wp(t)))^2 - 2\zeta_1(\wp(t))\kappa^\tau \Re_1(t,\zeta_1,\zeta_2,\zeta_3)\right. \\
&\left. +\kappa^{2\tau}(\Re_1(t,\zeta_1,\zeta_2,\zeta_3))^2\right] - \big[(\zeta_{10})^2\big]\Big\}.
\end{aligned}$$

$$\begin{aligned}
= \; & -\limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{ \sum_{j=0}^{\lfloor \frac{t-t_0}{\kappa} \rfloor} (-1)^j \left({}^\tau C_j\right) \big[(\zeta_{10})^2\big] \right\} \\
& + \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{ \sum_{j=0}^{\lfloor \frac{t-t_0}{\kappa} \rfloor} (-1)^j \left({}^\tau C_j\right) \big[(\zeta_1(\wp(t)))^2\big] \right\} \\
& - \limsup_{\kappa\to 0^+} \left\{ \sum_{j=1}^{\lfloor \frac{t-t_0}{\kappa} \rfloor} (-1)^j \left({}^\tau C_j\right) \left[2\zeta_1(\wp(t))\kappa^\tau \Re_1(t,\zeta_1,\zeta_2,\zeta_3)\right] \right\}.
\end{aligned}$$

From (2.12) and (2.14) in [8], we get

$$\leq \frac{(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \big[(\zeta_1(\wp(t)))^2\big] - \left[2\zeta_1(\wp(t))\Re_1(t,\zeta_1,\zeta_2,\zeta_3)\right].$$

When $t \to \infty$, $\frac{(t-t_0)^{-\tau}}{\Gamma(1-\tau)} \left[(\zeta_1(\wp(t)))^2\right] \to 0$, so that

$$\leq \quad -2[\zeta_1(\wp(t))\Re_1(t,\zeta_1,\zeta_2,\zeta_3)].$$

but $\zeta(\wp(t)) \leq \kappa^{C\mathbb{T}}D^\tau\zeta^\Delta(t) + \zeta(t)$, so that

$$\begin{aligned}
&= -2\Big[\kappa(t)\Re_1^2(t,\zeta_1,\zeta_2,\zeta_3) + \zeta_1(t)\Re_1(t,\zeta_1,\zeta_2,\zeta_3)\Big] \\
&= -2\left[\kappa(t)\left(3\zeta_1 - \frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right)^2 + \zeta_1\left(3\zeta_1 - \frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right)\right] \\
&= -2\kappa(t)\left[\left(3\zeta_1 - \frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right)^2\right] - 2\zeta_1\left[3\zeta_1 - \frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right]. \qquad (4.7)
\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$ we have that $\kappa=0$, so that (4.7) becomes:

$$\begin{aligned} {}^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_1^{\Delta}(\zeta_1,\zeta_2,\zeta_3) &= -2\zeta_1\left[3\zeta_1-\frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right] \\ &= -6\zeta_1^2+2\zeta_2^2+4\zeta_3^2 \\ &= (-6 \quad 2 \quad 4)\cdot(\Upsilon_1 \quad \Upsilon_2 \quad \Upsilon_3)^T. \end{aligned} \tag{4.8}$$

If $\mathbb{T}=\mathbb{N}_0$, we have that $\kappa=1$, so that (4.7) becomes:

$$\begin{aligned} &= -2\left[\left(3\zeta_1-\frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right)^2\right]-2\zeta_1\left[3\zeta_1-\frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right] \\ &\leq -2\zeta_1\left[3\zeta_1-\frac{\zeta_2^2+2\zeta_3^2}{\zeta_1}\right]. \end{aligned}$$

Resulting in the same outcome as (4.8). Evidently, this approach is equally applicable to any other discrete-time scenario.
Next, we compute the CFrΔDiD for $\Upsilon_2(\zeta)=\zeta_2^2$ as follows:

$$\begin{aligned} &{}^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_2^{\Delta}(\zeta) \\ =\;& \limsup_{\kappa\to0^+}\frac{1}{\kappa^\tau}\Big\{\left[(\zeta_2(\wp(t)))^2\right]-\left[(\zeta_{20})^2\right] \\ &+\sum_{j=1}^{\left[\frac{t-t_0}{\kappa}\right]}(-1)^j\,({}^\tau C_j)\left[(\zeta_2(\wp(t))-\kappa^\tau\Re_2(t,\zeta))^2\right]-\left[((\zeta_{20})^2\right]\Big\} \\ =\;& \limsup_{\kappa\to0^+}\frac{1}{\kappa^\tau}\Big\{\left[(\zeta_2(\wp(t)))^2\right]-\left[(\zeta_{20})^2\right] \\ &+\sum_{j=1}^{\left[\frac{t-t_0}{\kappa}\right]}(-1)^j\,({}^\tau C_j)\left[(\zeta_2(\wp(t)))^2-2\zeta_2(\wp(t))\kappa^\tau\Re_2(t,\zeta_1,\zeta_2,\zeta_3)\right. \\ &\left.+\kappa^{2\tau}(\Re_2(t,\zeta_1,\zeta_2,\zeta_3))^2\right]-\left[(\zeta_{20})^2\right]\Big\} \\ =\;& -\limsup_{\kappa\to0^+}\frac{1}{\kappa^\tau}\left\{\sum_{j=0}^{\left[\frac{t-t_0}{\kappa}\right]}(-1)^j\,({}^\tau C_j)\left[(\zeta_{20})^2\right]\right\} \\ &+\limsup_{\kappa\to0^+}\frac{1}{\kappa^\tau}\left\{\sum_{j=0}^{\left[\frac{t-t_0}{\kappa}\right]}(-1)^j\,({}^\tau C_j)\left[(\zeta_2(\wp(t)))^2\right]\right\} \\ &-\limsup_{\kappa\to0^+}\left\{\sum_{j=1}^{\left[\frac{t-t_0}{\kappa}\right]}(-1)^j\,({}^\tau C_j)\left[2\zeta_2(\wp(t))\kappa^\tau\Re_2(t,\zeta_1,\zeta_2,\zeta_3)\right]\right\}. \end{aligned}$$

From (2.12) and (2.14) in [8], we get

$$\leq\frac{(t-t_0)^{-\tau}}{\Gamma(1-\tau)}\left[(\zeta_2(\wp(t)))^2\right]-\left[2\zeta_2(\wp(t))\Re_2(t,\zeta_1,\zeta_2,\zeta_3)\right].$$

As $t \to \infty$, $\frac{(t-t_0)^{-\tau}}{\Gamma(1-\tau)}\left[(\zeta_2(\wp(t)))^2\right] \to 0$, to obtain

$$\leq \quad -2[\zeta_2(\wp(t))\Re_2(t,\zeta_1,\zeta_2,\zeta_3)],$$

introducing $\zeta(\wp(t)) \leq \kappa^{C\mathbb{T}}D^{\tau}\zeta^{\Delta}(t)+\zeta(t)$.

$$\begin{aligned}
&= -2\Big[\kappa(t)\Re_2^2(t,\zeta_1,\zeta_2,\zeta_3)+\zeta_2(t)\Re_2(t,\zeta_1,\zeta_2,\zeta_3)\Big]\\
&= -2\Big[\kappa(t)\left(-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right)^2+\zeta_2\left(-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right)\Big]\\
&= -2\kappa(t)\left[\left(-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right)^2\right]-2\zeta_2\left[-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right].
\end{aligned} \tag{4.9}$$

If $\mathbb{T}=\mathbb{R}$ we have that $\kappa=0$, so that (4.9) becomes:

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^{\tau}\Upsilon_2^{\Delta}(\zeta_1,\zeta_2) &= -2\zeta_2\left[-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right]\\
&= 2\zeta_1^2-8\zeta_2^2+6\zeta_3^2\\
&= (2 \quad -8 \quad 6)\cdot(\Upsilon_1 \quad \Upsilon_2 \quad \Upsilon_3)^T.
\end{aligned} \tag{4.10}$$

If $\mathbb{T}=\mathbb{N}_0$, we have that $\kappa=1$, so that (4.9) becomes:

$$\begin{aligned}
&= -2\left[\left(-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right)^2\right]-2\zeta_2\left[-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right]\\
&\leq -2\zeta_2\left[-\frac{\zeta_1^2}{\zeta_2}+4\zeta_2-\frac{3\zeta_3^2}{\zeta_2}\right].
\end{aligned}$$

This leads to the same conclusion as (4.10), and it is evident that this approach is equally applicable to any other discrete time instance.

Finally, we compute the CFrΔDiD for $\Upsilon_3(\zeta_1,\zeta_2,\zeta_3)=\zeta_3^2$ as follows:

$$\begin{aligned}
&{}^{C\mathbb{T}}D_+^\tau \Upsilon_3^\Delta \\
= \;& \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Bigg\{ \left[(\zeta_3(\wp(t)))^2\right] - \left[(\zeta_{30})^2\right] \\
&+ \sum_{j=1}^{\left[\frac{t-t_0}{\kappa}\right]} (-1)^j \left({}^\tau C_j\right)\left[(\zeta_3(\wp(t)) - \kappa^\tau \Re_3(t,\zeta))^2\right] - \left[((\zeta_{30})^2\right]\Bigg\} \\
= \;& \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\Bigg\{ \left[(\zeta_3(\wp(t)))^2\right] - \left[(\zeta_{30})^2\right] \\
&+ \sum_{j=1}^{\left[\frac{t-t_0}{\kappa}\right]} (-1)^j \left({}^\tau C_j\right)\left[(\zeta_3(\wp(t)))^2 - 2\zeta_3(\wp(t))\kappa^\tau \Re_3(t,\zeta_1,\zeta_2,\zeta_3)\right. \\
&\left. +\kappa^{2\tau}(\Re_3(t,\zeta_1,\zeta_2,\zeta_3))^2\right] - \left[(\zeta_{30})^2\right]\Bigg\} \\
= \;& -\limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{ \sum_{j=0}^{\left[\frac{t-t_0}{\kappa}\right]} (-1)^j \left({}^\tau C_j\right)\left[(\zeta_{30})^2\right]\right\} \\
&+ \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\tau}\left\{ \sum_{j=0}^{\left[\frac{t-t_0}{\kappa}\right]} (-1)^j \left({}^\tau C_j\right)\left[(\zeta_3(\wp(t)))^2\right]\right\} \\
&- \limsup_{\kappa\to 0^+} \left\{ \sum_{j=1}^{\left[\frac{t-t_0}{\kappa}\right]} (-1)^j \left({}^\tau C_j\right)\left[2\zeta_3(\wp(t))\kappa^\tau \Re_3(t,\zeta_1,\zeta_2,\zeta_3)\right]\right\}.
\end{aligned}$$

From (2.12) and (2.14) in [8], we get

$$\leq \frac{(t-t_0)^{-\tau}}{\Gamma(1-\tau)}\left[(\zeta_3(\wp(t)))^2\right] - \left[2\zeta_3(\wp(t))\Re_3(t,\zeta_1,\zeta_2,\zeta_3)\right].$$

As $t\to\infty$, $\frac{(t-t_0)^{-\tau}}{\Gamma(1-\tau)}\left[(\zeta_3(\wp(t)))^2\right]\to 0$, then

$$\leq \; -2[\zeta_3(\wp(t))\Re_3(t,\zeta_1,\zeta_2,\zeta_3)].$$

Using $\zeta(\wp(t)) \leq \kappa^{C\mathbb{T}}D^\tau \zeta^\Delta(t) + \zeta(t)$,

$$\begin{aligned}
&= \; -2\Bigg[\kappa(t)\Re_3^2(t,\zeta_1,\zeta_2,\zeta_3) + \zeta_3(t)\Re_3(t,\zeta_1,\zeta_2,\zeta_3)\Bigg] \\
&= \; -2\Bigg[\kappa(t)\left(-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right)^2 + \zeta_3\left(-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right)\Bigg]
\end{aligned}$$

$$= \; -2\kappa(t)\left[\left(-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right)^2\right] - 2\zeta_3\left[-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right]. \tag{4.11}$$

When $\mathbb{T} = \mathbb{R}$, $\kappa = 0$, then (4.11) becomes:

$$\begin{aligned} {}^{C\mathbb{T}}D_+^{\tau}\Upsilon_3^{\Delta}(\zeta_1,\zeta_2,\zeta_3) &= -2\zeta_3\left[-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right] \\ &= 2\zeta_1^2+2\zeta_2-6\zeta_3^2 \\ &= (2 \quad 2 \quad -6)\cdot(\Upsilon_1 \quad \Upsilon_2 \quad \Upsilon_3)^T. \end{aligned} \tag{4.12}$$

For $\mathbb{T} = \mathbb{N}_0$, $\kappa = 1$, then (4.11) becomes:

$$\begin{aligned} &= -2\left[\left(-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right)^2\right]-2\zeta_1\left[-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right] \\ &\le -2\zeta_1\left[-\frac{\zeta_1^2+\zeta_2}{\zeta_3}+3\zeta_3\right]. \end{aligned}$$

We obtain the same conclusion as (4.12).
From (4.8), (4.10) and (4.12), we obtain

$$ {}^{C\mathbb{T}}D_+^{\tau}\Upsilon^{\Delta} \le \begin{pmatrix} -6 & 2 & 4 \\ 2 & -8 & 6 \\ 2 & 2 & -6 \end{pmatrix}\begin{pmatrix}\Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3\end{pmatrix} = \Im(t,\Upsilon). \tag{4.13}$$

Now, consider the comparison system

$$ {}^{C\mathbb{T}}D_+^{\tau}\varphi^{\Delta} = \Im(t,\varphi) = Y\varphi, \tag{4.14}$$

where $Y = \begin{pmatrix} -6 & 2 & 4 \\ 2 & -8 & 6 \\ 2 & 2 & -6 \end{pmatrix}$.
The vector inequality (4.13), along with all the conditions of Theorem 2.3, are fulfilled if the matrix Y has eigenvalues with negative real parts. Since the eigenvalues of Y are $\beta_1 = -10.57$, $\beta_2 = -8.44$, and $\beta_3 = -0.99$, it follows that (4.6) is uniformly stable. Consequently, we deduce that the trivial solution $\aleph_0 = 0$ of the system (4.6) is uniformly asymptotically stable.

5. APPLICATION TO COMPUTING SYSTEMS

Application I: Fractional Robotic Arm Control System with PD Feedback. Consider a fractional-order model for a robotic arm with angular position $\theta(t)$, angular velocity $\omega(t)$, and control input voltage $V(t)$. The dynamics under PD control are:

$$\begin{aligned} &{}^{C\mathbb{T}}D^{\tau}\theta(t) = \omega, \quad \theta(t_0) = \theta_0, \\ &{}^{C\mathbb{T}}D^{\tau}\omega(t) = -\frac{B}{J}\omega - \frac{K}{J}\theta + \frac{K_m}{J}V, \quad \omega(t_0) = \omega_0, \\ &{}^{C\mathbb{T}}D^{\tau}V(t) = -K_p\theta - K_d\omega - K_i\int_{t_0}^{t}\theta(\tau)\,d\tau, \quad V(t_0) = V_0, \end{aligned} \tag{5.1}$$

where B (damping), J (inertia), K (stiffness), and K_m (motor constant) are positive constants. The PD controller gains $K_p, K_d, K_i > 0$ regulate position, velocity, and integral error.

Define the vector Lyapunov function to be $\Upsilon(t,\theta,\omega,V) = (\Upsilon_1,\Upsilon_2,\Upsilon_3)^T$, where:

$$\Upsilon_1 = \theta^2, \quad \Upsilon_2 = \omega^2, \quad \Upsilon_3 = V^2.$$

Applying Definition 2.3 and performing the operations as in 4 we obtain that:

For $\Upsilon_1 = \theta^2$:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_1 \leq 2\theta \cdot {}^{C\mathbb{T}} D^\tau \theta = 2\theta\omega - \frac{\theta_0^2 (t-t_0)^{-\tau}}{\Gamma(1-\tau)}. $$

As $t \to \infty$:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_1 \leq 2\theta\omega. $$

For $\Upsilon_2 = \omega^2$:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_2 \leq 2\omega \left(-\frac{B}{J}\omega - \frac{K}{J}\theta + \frac{K_m}{J} V \right) - \frac{\omega_0^2 (t-t_0)^{-\tau}}{\Gamma(1-\tau)}. $$

As $t \to \infty$:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_2 \leq -\frac{2B}{J}\aleph_2 - \frac{2K}{J}\theta\omega + \frac{2K_m}{J}\omega V. $$

For $\Upsilon_3 = V^2$:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_3 \leq 2V(-K_p\theta - K_d\omega) - \frac{V_0^2 (t-t_0)^{-\tau}}{\Gamma(1-\tau)}. $$

As $t \to \infty$:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon_3 \leq -2K_p V\theta - 2K_d V\omega. $$

Taking $\theta\omega \leq \frac{\theta^2+\omega^2}{2}$, $\omega V \leq \frac{\omega^2+V^2}{2}$, and $V\theta \leq \frac{V^2+\theta^2}{2}$, we obtain:

$$ {}^{C\mathbb{T}}D_+^\tau \Upsilon \leq \begin{pmatrix} 0 & 1 & 0 \\ -\frac{K}{J} & -\frac{B}{J} + \frac{K_m}{2J} & \frac{K_m}{2J} \\ -K_p & -K_d & 0 \end{pmatrix} \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \end{pmatrix}. $$

Then the Jacobian at equilibrium ($\theta = 0, \omega = 0, V = 0$) is:

$$ \mathcal{J} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{K}{J} & -\frac{B}{J} + \frac{K_m}{2J} & \frac{K_m}{2J} \\ -K_p & -K_d & 0 \end{pmatrix}. $$

So that the eigenvalues satisfy:

$$ \lambda^3 + \left(\frac{B}{J} - \frac{K_m}{2J} \right) \lambda^2 + \left(\frac{K}{J} - \frac{K_m K_d}{2J} \right) \lambda + \frac{K_p K_m}{2J} = 0. $$

Using the Routh-Hurwitz criterion, uniform asymptotic stability requires:

$$ \frac{B}{J} > \frac{K_m}{2J}, \quad \frac{K}{J} > \frac{K_m K_d}{2J}, \quad \frac{K_p K_m}{2J} > 0. $$

Without loss of generality, we conclude that the robotic arm equilibrium ($\theta = 0, \omega = 0, V = 0$) is asymptotically stable if damping dominates motor effects ($B > \frac{K_m}{2}$), stiffness outweighs derivative gain coupling ($K > \frac{K_m K_d}{2}$), and proportional gain ensures positional correction ($K_p > 0$). This clearly aligns with PD control design principles, where gains K_p, K_d must balance system inertia, damping, and motor dynamics to achieve stabilization. The fractional framework extends classical control theory to systems with memory-dependent dynamics, such as viscoelastic robotic joints.

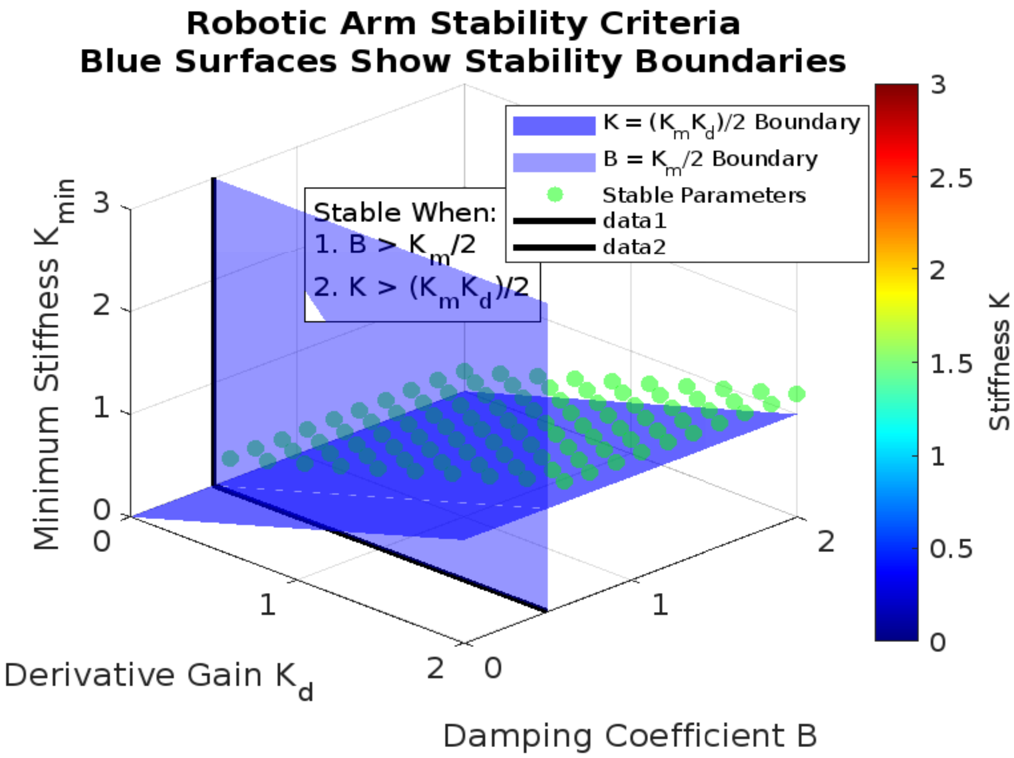

Figure 1: Stability region (green) for robotic arm control showing required damping (B) and stiffness (K) relative to motor constant (K_m) and derivative gain (K_n). Blue surfaces mark critical boundaries from Routh-Hurwitz criteria.

Figure 1 above shows a three-dimensional visualization of the stability criteria for the fractional-order robotic arm system (5.1). The blue surface represents the critical boundary $K = \frac{K_m K_d}{2}$, while the vertical blue plane indicates $B = \frac{K_m}{2}$. The green points show parameter combinations satisfying both stability conditions $(B > \frac{K_m}{2}$ and $K > \frac{K_m K_d}{2})$, guaranteeing uniform asymptotic stability. Black reference lines mark the minimum thresholds for stability. The plot confirms that increasing damping (B) and stiffness (K) beyond these boundaries stabilizes the system, even with high derivative gains (K_n).

Application II: Fractional Load Balancing in Distributed Computing Systems. Consider a distributed computing system with three nodes managing task queues $q_1(t), q_2(t), q_3(t)$. Then, we formulate the Caputo fractional-order dynamic system control for load balancing as:

$$
\begin{aligned}
{}^{C\mathbb{T}}D^{\tau} q_1(t) &= \lambda_1 - \mu_1 q_1 + \gamma(q_2 - q_1) - \gamma(q_1 - q_3), \\
{}^{C\mathbb{T}}D^{\tau} q_2(t) &= \lambda_2 - \mu_2 q_2 + \gamma(q_3 - q_2) - \gamma(q_2 - q_1), \\
{}^{C\mathbb{T}}D^{\tau} q_3(t) &= \lambda_3 - \mu_3 q_3 + \gamma(q_1 - q_3) - \gamma(q_3 - q_2),
\end{aligned}
\tag{5.2}
$$

where λ_i are task arrival rates, μ_i are processing rates, and $\gamma > 0$ is the load-balancing coupling coefficient.

Define the Vector Lyapunov function $\Upsilon(t, q) = (\Upsilon_1, \Upsilon_2, \Upsilon_3)^T$, where:

$$\Upsilon_i = q_i^2 \quad \text{for} \quad i = 1, 2, 3.$$

Applying Definition 2.3 to each Υ_i as computed in the illustrations above, we obtain :

For $\Upsilon_1 = q_1^2$:

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_1 = 2q_1 \cdot ^{C\mathbb{T}} D^{\tau}q_1 = 2q_1\left(\lambda_1 - (\mu_1 + 2\gamma)q_1 + \gamma q_2 + \gamma q_3\right).$$

For $\Upsilon_2 = q_2^2$:

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_2 = 2q_2\left(\lambda_2 - (\mu_2 + 2\gamma)q_2 + \gamma q_3 + \gamma q_1\right).$$

For $\Upsilon_3 = q_3^2$:

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon_3 = 2q_3\left(\lambda_3 - (\mu_3 + 2\gamma)q_3 + \gamma q_1 + \gamma q_2\right).$$

Assuming $\lambda_i = 0$, then for $q_iq_j \le \frac{q_i^2+q_j^2}{2}$ we obtain:

$$^{C\mathbb{T}}D_{+}^{\tau}\Upsilon \le \begin{pmatrix} -2(\mu_1+\gamma) & \gamma & \gamma \\ \gamma & -2(\mu_2+\gamma) & \gamma \\ \gamma & \gamma & -2(\mu_3+\gamma) \end{pmatrix} \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \end{pmatrix}.$$

For symmetric parameters ($\mu_i = \mu$), the eigenvalues are:

$$\lambda_1 = -2\mu, \quad \lambda_{2,3} = -2\mu - 3\gamma.$$

All eigenvalues are negative for $\mu > 0$, ensuring uniform asymptotic stability.

Then we conclude that the equilibrium $q_1 = q_2 = q_3 = 0$ (balanced queues) of system (5.2) is uniform asymptotically stable if processing rates $\mu > 0$ and load-balancing effort $\gamma \ge 0$. This ensures tasks are uniformly distributed, preventing overload in distributed computing systems. The fractional framework captures memory-dependent load adjustments, critical for real-time scheduling in cloud environments.

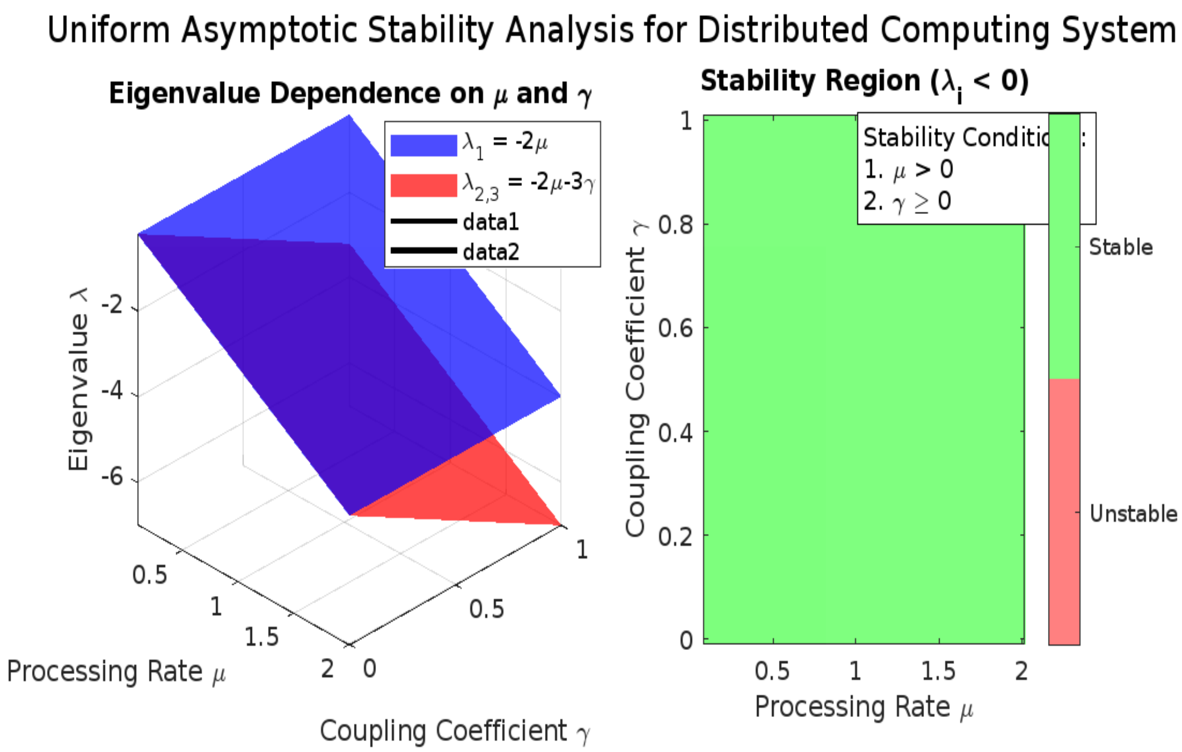

Figure 2: Eigenvalue analysis demonstrating uniform asymptotic stability in distributed computing systems: (Left) Eigenvalue surfaces showing faster convergence with higher processing rates (μ) and load-balancing (γ); (Right) Stability region confirming global stability when $\mu > 0$. The system achieves queue balancing ($q_1 = q_2 = q_3 = 0$) with stronger stability for $\gamma > 0$.

In figure 2 above, the 3D eigenvalue surfaces and stability region plot collectively demonstrate the uniform asymptotic stability criteria for the distributed computing system (5.2) with fractional-order queue dynamics. The blue eigenvalue surface ($\lambda_1 = -2\mu$) reveals that the dominant mode's convergence rate depends linearly on the processing rate μ, where increasing μ exponentially reduces queue backlogs. The red surface ($\lambda_{2,3} = -2\mu - 3\gamma$) shows how load-balancing effort γ further accelerates stabilization by making all eigenvalues more negative, with the -3γ term highlighting how active task redistribution prevents node overloads. The black contours at $\lambda = 0$ confirm the system remains stable for all $\mu > 0$ and $\gamma \geq 0$, as eigenvalues never cross into positive territory. The stability region plot reinforces this by showing the entire $\mu > 0$ half-plane as stable (green), while the theoretically unstable red region ($\mu \leq 0$) has no physical meaning for real systems. These results prove that the queue-balancing equilibrium ($q_1 = q_2 = q_3 = 0$) is globally stable when $\mu > 0$, with stronger stability (faster convergence) achieved through $\gamma > 0$. The fractional-order framework implicitly shapes these dynamics by introducing memory-dependent transient behaviors during load adjustments. For cloud and edge computing systems, this translates to clear design rules: processing capacity μ must exceed input task rates, while load-balancing intensity γ can be tuned to optimize convergence speed and fault tolerance against workload spikes. The visualization thus bridges theoretical stability analysis with practical system design, demonstrating how distributed computing architectures can maintain robust performance even with memory-dependent scheduling delays characteristic of fractional-order dynamics.

6. CONCLUSION

This work has established a novel framework for UAS analysis of CFrDyE$\mathbb{T}$ by introducing the CFrΔDiD as a specialized tool for vector Lyapunov function analysis, enabling rigorous stability proofs for systems with memory effects and hybrid continuousâĂŞdiscrete dynamics. Unlike conventional integer-order methods, which are restricted to either continuous or discrete systems, and traditional time-scale approaches, which cannot capture the long-range dependence induced by fractional operators, our framework unifies fractional calculus with time-scale theory, generalizes comparison principles through vector Lyapunov functions, and derives practical UAS criteria that ensure uniform convergence regardless of initial conditions. Its superiority over existing techniques is further evidenced by applications: in robotic control, it guarantees stability for PD-controlled systems with viscoelastic memory where classical methods fail, and in distributed computing, it ensures stable load balancing with historical workload dependencies where standard models prove inadequate. These results highlight both the theoretical and practical advantages of the proposed framework over existing approaches, establishing it as a robust and general tool for stability analysis of complex hybrid systems.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' contributions. All authors contributed equally to the manuscript.

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