Aust. J. Math. Anal. Appl.
Vol. **23** (2026), No. 1, Art. 11, 9 pp.
AJMAA

FUNCTIONAL IDENTITIES OF ADDITIVE MAPS WITHIN SPECIALIZED RING STRUCTURES

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Received 12 June, 2026; accepted 23 July, 2026; published 14 August, 2026.

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ABSTRACT. Let $\mathcal{S}$ be a $(p+m)$!-torsion-free semiprime ring, and let $\mathcal{B}, \Bbbk : \mathcal{S} \to \mathcal{S}$ be additive mappings satisfying, for all $y \in \mathcal{S}$, the identity

$$2\mathcal{B}(y^{p+m}) \ = \ \mathcal{B}(y^p)\,\gamma(y^m) \ + \ \theta(y^p)\,\Bbbk(y^m) \ + \ \mathcal{B}(y^m)\,\gamma(y^p) \ + \ \theta(y^m)\,\Bbbk(y^p),$$

where γ and θ are automorphisms of $\mathcal{S}$. Then $\mathcal{B}$ is a generalized (γ,θ)-derivation on $\mathcal{S}$, with $\Bbbk$ serving as its associated (γ,θ)-derivation. Moreover, analogously, we establish an additional result demonstrating that, if two additive mappings satisfy a prescribed identity, then each such mapping constitutes a generalized left (γ,θ)-derivative having left (γ,θ)-derivative. Furthermore, the appropriateness of this hypothesis is substantiated by explicit illustrative examples.

Key words and phrases: Algebraic identities; Semiprime rings; generalized (γ,θ)-derivation; generalized left (γ,θ)-derivation.

2010 *Mathematics Subject Classification.* Primary 16B99, 16N60. Secondary 16W25.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Throughout, the ring $\mathcal{S}$ will be regarded as an associative ring with identity, and $Z(\mathcal{S})$ will denote its center. A ring $\mathcal{S}$ is called m-torsion-free if, for a fixed integer $m > 1$, the relation $my = 0$ implies $y = 0$ for all $y \in \mathcal{S}$. The commutator of elements $y, z \in \mathcal{S}$ is defined by $[y, z] = yz - zy$. The ring $\mathcal{S}$ is said to be prime if $y\mathcal{S}z = \{0\}$ implies that either $y = 0$ or $z = 0$. It is called semiprime if $y\mathcal{S}y = \{0\}$ implies $y = 0$.

A mapping $\Bbbk : \mathcal{S} \to \mathcal{S}$ is called a derivation if $\Bbbk$ is additive and satisfies

$$\Bbbk(yz) = \Bbbk(y)z + y\Bbbk(z), \quad \text{for all } y, z \in \mathcal{S}.$$

It is called a Jordan derivation if, for all $z \in \mathcal{S}$, it satisfies

$$\Bbbk(z^2) = \Bbbk(z)z + z\Bbbk(z).$$

Every derivation is, in particular, a Jordan derivation; however, the converse does not hold in general.

A classical theorem of Herstein [10] states that, in a prime ring of characteristic not equal to two, every Jordan derivation is in fact a derivation. Cusack [9] extended this final assertion of Herstein to the setting of 2-torsion free semiprime rings. Following Zalar [12], an additive map $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ is called a left centralizer if $\mathcal{T}(yz) = \mathcal{T}(y)z$ holds for all $y, z \in \mathcal{S}$, and it is called a right centralizer if $\mathcal{T}(yz) = y\mathcal{T}(z)$ for every $y, z \in \mathcal{S}$. In particular, $\mathcal{T}$ is termed a Jordan left (respectively, Jordan right) centralizer of $\mathcal{S}$ when these identities are required only in the special case $y = z$. An additive mapping $\mathcal{B} : \mathcal{S} \to \mathcal{S}$ satisfying

$$\mathcal{B}(yz) = \mathcal{B}(y)z + y\Bbbk(z) \quad \text{for all } y, z \in \mathcal{S}$$

is called a generalized derivation associated with a derivation $\Bbbk$ on $\mathcal{S}$. In particular, $\mathcal{B}$ is referred to as a ‘generalized Jordan derivation’ if there is a ‘Jordan derivation’ $\Bbbk$ on $\mathcal{S}$ such that the above relation is imposed only for $y = z$.

The text introduces several types of derivations on the ring $\mathcal{S}$. For two given automorphisms θ and γ, an additive map $\Bbbk : \mathcal{S} \to \mathcal{S}$ is called an (γ, θ)-derivation if

$$\Bbbk(yz) = \Bbbk(y)\gamma(z) + \theta(y)\Bbbk(z)$$

for all $y, z \in \mathcal{S}$. It is called a Jordan (γ, θ)-derivation if

$$\Bbbk(y^2) = \Bbbk(y)\gamma(y) + \theta(y)\Bbbk(y)$$

holds for every $y \in \mathcal{S}$. Any (γ, θ)-derivation is automatically a Jordan (γ, θ)-derivation, though the converse need not hold.

A generalized (γ, θ)-derivation is an additive map $\mathcal{B} : \mathcal{S} \to \mathcal{S}$ associated with an (γ, θ)-derivation $\Bbbk$ such that

$$\mathcal{B}(yz) = \mathcal{B}(y)\gamma(z) + \theta(y)\Bbbk(z).$$

Likewise, a generalized Jordan (γ, θ)-derivation is defined relative to a Jordan (γ, θ)-derivation $\Bbbk$ via the identity

$$\mathcal{B}(y^2) = \mathcal{B}(y)\gamma(y) + \theta(y)\Bbbk(y).$$

Every generalized (γ, θ)-derivation is, in particular, a generalized Jordan (γ, θ)-derivation, but, in general, the reverse implication does not hold. Remarkably, a key theorem of BreÅąar and Vukman [8] shows that on a prime ring that is 2-torsion-free, each Jordan (γ, θ)-derivation is actually an (γ, θ)-derivation. This was later extended in [11], where it was proved that on a 2-torsion-free semiprime ring, every Jordan (γ, θ)-derivation is likewise an (γ, θ)-derivation.

According to [1], every generalized Jordan (γ,θ)-derivation on a 2-torsion-free semiprime ring is in fact a generalized (γ,θ)-derivation. If $\mathcal{B}$ is a generalized derivation having a derivation $\Bbbk$ on $\mathcal{S}$, then for each $y\in\mathcal{S}$ the algebraic relation

$$2\mathcal{B}(y^{p+m})=\mathcal{B}(y^p)\gamma(y^m)+y^p\Bbbk(y^m)+\mathcal{B}(y^m)\gamma(y^p)+y^m\Bbbk(y^p)$$

holds. However, the converse implication does not generally hold. In this paper, we investigate the conditions under which the converse is valid. More precisely, we show that if $\mathcal{B},\Bbbk:\mathcal{S}\to\mathcal{S}$ are two additive mappings satisfying

$$2\mathcal{B}(y^{p+m})=\mathcal{B}(y^p)\gamma(y^m)+y^p\Bbbk(y^m)+\mathcal{B}(y^m)\gamma(y^p)+y^m\Bbbk(y^p)$$

for all $y\in\mathcal{S}$, then $\mathcal{B}$ is a generalized derivation with $\Bbbk$ as its associated derivation on $\mathcal{S}$, provided that $\mathcal{S}$ is a $(p+m)!$-torsion-free semiprime ring.

2. GENERALIZED (γ,θ)-DERIVATION

Let us start with the preceding theorem:

Theorem 2.1. *Suppose that $p,m\geq 1$ are any two fixed integers and $\mathcal{S}$ is a $(p+m)!$-torsion-free semiprime ring. If $\mathcal{B}$ and $\Bbbk$ are two additive mappings from $\mathcal{S}$ to itself which satisfy the algebraic equation $2\mathcal{B}(y^{p+m})=\mathcal{B}(y^p)\gamma(y^m)+\theta(y^p)\Bbbk(y^m)+\mathcal{B}(y^m)\gamma(y^p)+\theta(y^m)\Bbbk(y^p)$, then $\mathcal{B}$ will be a generalized (γ,θ)-derivation linked with a (γ,θ)-derivation $\Bbbk$ on $\mathcal{S}$, where γ and θ are automorphism on $\mathcal{S}$.*

Proof. We have given that

$$2\mathcal{B}(y^{p+m})=\mathcal{B}(y^p)\gamma(y^m)+\theta(y^p)\Bbbk(y^m)+\mathcal{B}(y^m)\gamma(y^p)+\theta(y^m)\Bbbk(y^p). \tag{2.1}$$

Notice that $\Bbbk(e)=0$ and if we putting $y+qz$ for y in (2.1), we find
$2\mathcal{B}\Big(y^{m+p}+\binom{m+p}{1}y^{m+p-1}qb+\binom{m+p}{2}y^{m+p-2}q^2z^2+\cdots+q^{m+p}z^{m+p}\Big)=F\Big(y^p+\binom{p}{1}y^{p-1}qb+\binom{p}{2}y^{p-2}q^2z^2+\cdots+q^pz^p\Big)\gamma\Big(y^m+\binom{m}{1}y^{m-1}qb+\binom{n}{2}y^{m-2}q^2z^2+\cdots+q^mz^m\Big)+\theta\Big(y^p+\binom{p}{1}y^{p-1}qb+\binom{p}{2}y^{p-2}q^2z^2+\cdots+q^pz^p\Big)d\Big(y^m+\binom{m}{1}y^{m-1}qb+\binom{m}{2}y^{m-2}q^2z^2+\cdots+q^mz^m\Big)+F\Big(y^m+\binom{m}{1}y^{m-1}qb+\binom{m}{2}y^{m-2}q^2z^2+\cdots+q^mz^m\Big)\gamma\Big(y^p+\binom{p}{1}y^{p-1}qb+\binom{p}{2}y^{p-2}q^2z^2+\cdots+q^pz^p\Big)+\theta\Big(y^m+\binom{m}{1}y^{m-1}qb+\binom{m}{2}y^{m-2}q^2z^2+\cdots+q^mz^m\Big)d\Big(y^p+\binom{p}{1}y^{p-1}qb+\binom{p}{2}y^{p-2}q^2z^2+\cdots+q^pz^p\Big),$ for all $y,z\in\mathcal{S}$ and $q\in\mathbb{Z}^+$.

Rewrite the above expression by using (2.1) as

$$qf_1(y,z)+q^2f_2(y,z)+\cdots+q^{m+p-1}f_{m+p-1}(y,z)=0,$$

where $f_i(y,z)$ stand for the coefficients of m^i's for each $i=1,2,\cdots,(m+p-1)$. If we replace k by $1,2,\cdots,m+p-1$, then we find a system of $m+p-1$ homogeneous equations. It gives us a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+p-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ m+p-1 & (m+p-1)^2 & \cdots & (m+p-1)^{m+p-1} \end{bmatrix}.$$

This implies that $f_i(y,z)=0$ for all $y,z\in\mathcal{S}$ and for every $i=1,2,\ldots,(m+p-1)$. In particular, for all $y,z\in\mathcal{S}$, we obtain for $i=1$ that

$$\begin{aligned}2\binom{m+p}{1}\mathcal{B}(y^{m+p-1}z) &= \binom{m}{1}\mathcal{B}(y^p)\gamma(y^{m-1}z)+\binom{p}{1}\mathcal{B}(y^{p-1}z)\gamma(y^m)\\ &+\binom{m}{1}\theta(y^p)\Bbbk(y^{m-1}z)+\binom{p}{1}\theta(y^{p-1}z)\Bbbk(y^m)\\ &+\binom{p}{1}\mathcal{B}(y^m)\gamma(y^{p-1}z)+\binom{m}{1}\mathcal{B}(y^{m-1}z)\gamma(y^p)\\ &+\binom{p}{1}\theta(y^m)\Bbbk(y^{p-1}z)+\binom{m}{1}\theta(y^{m-1}z)\Bbbk(y^p),\ \text{for all } y,z\in\mathcal{S}.\end{aligned}$$

By substituting $y=e$ and using the relation $\Bbbk(e)=0$, we obtain

$$2(m+p)\mathcal{B}(z) = (m+p)\mathcal{B}(e)\,\gamma(z) + (m+p)\mathcal{B}(z) + (m+p)\Bbbk(z).$$

Subsequently, applying the torsion restriction of $\mathcal{S}$ yields that

$$\mathcal{B}(z)=\mathcal{B}(e)\gamma(z)+\Bbbk(z) \text{ for all } z \text{ in } \mathcal{S}. \tag{2.2}$$

Next, from $f_2(y,z)=0$, we have

$$\begin{aligned}2\binom{m+p}{2}\mathcal{B}(y^{m+p-2}z^2) &= \binom{m}{2}\mathcal{B}(y^p)\gamma(y^{m-2}z^2)+\binom{p}{1}\binom{m}{1}\mathcal{B}(y^{p-1}z)\gamma(y^{m-1}z)\\ &+\binom{p}{2}\mathcal{B}(y^{p-2}z^2)\gamma(y^m)+\binom{m}{2}\theta(y^p)\Bbbk(y^{m-2}z^2)\\ &+\binom{p}{1}\binom{m}{1}\theta(y^{p-1}z)\Bbbk(y^{m-1}z)+\binom{p}{2}\theta(y^{p-2}z^2)\Bbbk(y^m)\\ &+\binom{p}{2}\mathcal{B}(y^m)\gamma(y^{p-2}z^2)+\binom{m}{1}\binom{p}{1}\mathcal{B}(y^{m-1}z)\gamma(y^{p-1}z)\\ &+\binom{m}{2}\mathcal{B}(y^{m-2}z^2)\gamma(y^p)+\binom{p}{2}\theta(y^m)\Bbbk(y^{p-2}z^2)\\ &+\binom{m}{1}\binom{p}{1}\theta(y^{m-1}z)\Bbbk(y^{p-1}z)+\binom{m}{2}\theta(y^{m-2}z^2)\Bbbk(y^p),\end{aligned}$$

for all y,z in $\mathcal{S}$. Rewrite the above expression by substituting e for y to obtain

$$\begin{aligned}2\binom{m+p}{2}\mathcal{B}(z^2) &= \binom{m}{2}\mathcal{B}(e)\gamma(z^2)+\binom{p}{1}\binom{m}{1}\mathcal{B}(z)\gamma(z)+\binom{p}{2}\mathcal{B}(z^2)\\ &+\binom{m}{2}\Bbbk(z^2)+\binom{p}{1}\binom{m}{1}\theta(z)\Bbbk(z)+\binom{p}{2}\mathcal{B}(e)\gamma(z^2)\\ &+\binom{m}{1}\binom{p}{1}\mathcal{B}(z)\gamma(z)+\binom{m}{2}\mathcal{B}(z^2)+\binom{p}{2}\Bbbk(z^2)\\ &+\binom{m}{1}\binom{p}{1}\theta(z)\Bbbk(z),\ \text{for all } z\in\mathcal{S}.\end{aligned}$$

This implies that

$$\begin{aligned}(m+p-1)(m+p)\mathcal{B}(z^2) &= \left[\tfrac{m(m-1)}{2}+\tfrac{p(p-1)}{2}\right][\mathcal{B}(e)\gamma(z^2)+\Bbbk(z^2)]\\ &+pm\mathcal{B}(z)\gamma(z)+\tfrac{p(p-1)}{2}\mathcal{B}(z^2)+pm\theta(z)\Bbbk(z)\\ &+pm\mathcal{B}(z)\gamma(z)+\tfrac{m(m-1)}{2}\mathcal{B}(z^2)+mp\theta(z)\Bbbk(z).\end{aligned}$$

Replacing z by z^2 in (2.2), we arrive at $\mathcal{B}(z^2)=\mathcal{B}(e)\gamma(z^2)+\Bbbk(z^2)$ for all z in $\mathcal{S}.$ Using this relation in the above relation, we obtain

$$\begin{aligned}(m+p-1)(m+p)\mathcal{B}(z^2) &= [m(m-1)+p(p-1)]\mathcal{B}(z^2)\\ &+2pm\mathcal{B}(z)\gamma(z)+2pm\theta(z)\Bbbk(z).\end{aligned}$$

After some calculation, we arrive at

$$2mp\mathcal{B}(z^2) = 2mp[\mathcal{B}(z)\gamma(z)+\theta(z)\Bbbk(z)].$$

Using the torsion condition, we have

$$\mathcal{B}(z^2)=\mathcal{B}(z)\gamma(z)+\theta(z)\Bbbk(z),\ \text{for all } z\in\mathcal{S}. \tag{2.3}$$

Next, from (2.2)

$$\begin{aligned}\Bbbk(z^2) &= \mathcal{B}(z^2)-\mathcal{B}(e)\gamma(z^2)\\ &= \mathcal{B}(z)\gamma(z)+\theta(z)\Bbbk(z)-\mathcal{B}(e)\gamma(z^2)\\ &= [\mathcal{B}(z)-\mathcal{B}(e)\gamma(z)]\gamma(z)+\theta(z)\Bbbk(z)\\ &= \Bbbk(z)\gamma(z)+\theta(z)\Bbbk(z)\end{aligned}$$

Hence, $\mathbb{k}$ is a Jordan (γ, θ)-derivation on $\mathcal{S}$. Therefore, by (2.3), we conclude that $\mathcal{B}$ is a generalized Jordan (γ, θ)-derivation associated with the Jordan derivation $\mathbb{k}$ on $\mathcal{S}$. Finally, the desired assertion follows from the main theorem in [1]. ∎

The above theorem has immediate consequences.:

Corollary 2.2. [5] *Suppose that $p, m \geq 1$ are any two fixed integers and $\mathcal{S}$ is a semiprime ring with $(m+p-1)!$-torsion-free condition. If $\mathcal{B}$, $\mathbb{k}$ and $\mathcal{T}$ are three additive mappings from $\mathcal{S}$ to itself which satisfy the respective algebraic equation*

(1) $\mathcal{B}(y^{m+p}) = \mathcal{B}(y^p)y^m + y^p\mathbb{k}(y^m)$ *for all $y \in \mathcal{S}$, then $\mathcal{B}$ will be a generalized derivation linked with y derivation $\mathbb{k}$ on $\mathcal{S}$,*
(2) $\mathbb{k}(y^{m+p}) = \mathbb{k}(y^p)y^m + y^p\mathbb{k}(y^m)$ *for every y in $\mathcal{S}$, then $\mathbb{k}$ will be a derivation on $\mathcal{S}$,*
(3) $\mathcal{T}(y^{m+p}) = \mathcal{T}(y^p)y^m$ *for all $y \in \mathcal{S}$, then $\mathcal{T}$ is a centralizer on $\mathcal{S}$.*

Proof. One can prove it by using the same technique as was proved earlier, considering $\gamma = \theta = I$. ∎

Corollary 2.3. [2] *Suppose that $m \geq 1$ is any fixed integer and $\mathcal{S}$ is a semiprime ring having $(2m-1)!$-torsion freeness. If $\mathcal{B}$, $\mathbb{k}$ and $\mathcal{T}$ are three additive mappings from $\mathcal{S}$ to itself which satisfy the respective algebraic equation*

(1) $\mathcal{B}(y^{2m}) = \mathcal{B}(y^m)y^m + y^m\mathbb{k}(y^m)$ *for all $y \in \mathcal{S}$, then $\mathcal{B}$ will be a generalized derivation linked with a derivation $\mathbb{k}$ on $\mathcal{S}$,*
(2) $\mathbb{k}(y^{2m}) = \mathbb{k}(y^m)y^m + y^m\mathbb{k}(y^m)$ *for every y in $\mathcal{S}$, then $\mathbb{k}$ will be derivation on $\mathcal{S}$,*
(3) $\mathcal{T}(y^{2m}) = \mathcal{T}(y^m)y^m$ *for all $y \in \mathcal{S}$, then $\mathcal{T}$ is a centralizer on $\mathcal{S}$.*

Proof. Taking into account $p = m$ in the above result, we will conclude. ∎

The example that ensues shows that the significant results of the article are not redundant.

Example 2.1. *Consider a ring $\mathcal{S} = \left\{ \begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix} \mid \eta, \zeta \in 2\mathbb{Z}_8 \right\}$, $\mathbb{Z}_8$ has its usual meaning. Define mappings $\mathcal{B}, \mathbb{k}, : \mathcal{S} \to \mathcal{S}$ by $\mathcal{B}\begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \zeta \end{pmatrix}$, $\mathbb{k}\begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix} = \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}$, $\gamma\begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix} = \begin{pmatrix} \zeta & 0 \\ 0 & \eta \end{pmatrix}$ and $\theta\begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix} = \begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix}$. It is clear that $\mathcal{B}$ does not form a generalized (γ, θ)-derivation on $\mathcal{S}$; nevertheless, $\mathcal{B}$ and $\mathbb{k}$ satisfy the algebraic assumptions of Theorem 2.1. Hence, this example demonstrates that the semiprimeness hypothesis is indispensable in the main theorem of this paper.*

3. GENERALIZED LEFT (γ, θ)-DERIVATION

This section focuses on another generalization of a derivation, called a left (γ, θ)-derivation, which is defined as follows: a mapping $\delta : \mathcal{S} \to \mathcal{S}$ is called a left (γ, θ)-derivation (respectively, a Jordan left (γ, θ)-derivation) if it is additive and satisfies

$$\delta(yz) = \gamma(y)\delta(z) + \theta(z)\delta(y)$$

$$\big(\text{respectively } \delta(z^2) = \gamma(z)\delta(z) + \theta(z)\delta(z)\big)$$

for all $y, z \in \mathcal{S}$.

An additive mapping $\mathcal{B}_1 : \mathcal{S} \to \mathcal{S}$ is referred to as a generalized left (γ, θ)-derivation (respectively, a generalized Jordan left (γ, θ)-derivation) provided there exists a corresponding Jordan left (γ, θ)-derivation δ satisfying

$$\mathcal{B}_1(yz) = \gamma(y)\mathcal{B}_1(z) + \theta(z)\delta(y)$$
$$\big(\text{respectively } \mathcal{B}_1(z^2) = \gamma(z)\mathcal{B}_1(z) + \theta(z)\delta(z)\big)$$

for all $y, z \in \mathcal{S}$.

It is observed that $\mathcal{B}_1$ is a generalized left (γ, θ)-derivation if and only if $\mathcal{B}_1 = \delta + \mathcal{T}$ on $\mathcal{S}$, where $\mathcal{T}$ is a right γ-centralizer and δ is a left (γ, θ)-derivation on $\mathcal{S}$.

The notion of left (γ, θ)-derivations is subsumed within the broader framework of generalized left (γ, θ)-derivations. Conversely, if we set $\delta = 0$, a generalized left (γ, θ)-derivation coincides with the theory of right γ-centralizers on $\mathcal{S}$. For any fixed element $z \in \mathcal{S}$, every map of the form $\mathcal{B}_1(y) = yz + \delta(y)$ is a generalized left derivation, where δ denotes an arbitrary left (γ, θ)-derivation. Furthermore, if $\mathcal{B}_1$ is a generalized left (γ, θ)-derivation and δ is the associated left (γ, θ)-derivation of $\mathcal{B}_1$ on $\mathcal{S}$, then for each $y \in \mathcal{S}$ one has

$$2\mathcal{B}_1(y^{p+m}) = \gamma(y^p)\mathcal{B}_1(y^m) + \theta(y^m)\delta(y^p) + \gamma(y^m)\mathcal{B}_1(y^p) + \theta(y^p)\delta(y^m).$$

The converse of this implication holds under additional assumptions on $\mathcal{S}$. More precisely, it has been shown that if $m, p \geq 1$ are fixed integers and $\mathcal{S}$ is an $(m+p)!$-torsion-free semiprime ring, then whenever two additive mappings $\mathcal{B}_1, \delta : \mathcal{S} \to \mathcal{S}$ satisfy the above identity, $\mathcal{B}_1$ is necessarily a generalized left (γ, θ)-derivation associated with a left (γ, θ)-derivation δ on $\mathcal{S}$.

Theorem 3.1. *Let $m, p \geq 1$ be any two fixed integers and $\mathcal{S}$ be $(m+p)!$-torsion free semiprime ring. If two mapping $\mathcal{B}_1, \delta : \mathcal{S} \longrightarrow \mathcal{S}$ are additive and fulfilling the algebraic identity $2\mathcal{B}_1(y^{p+m}) = \gamma(y^p)\mathcal{B}_1(y^m) + \theta(y^m)\delta(y^p) + \gamma(y^m)\mathcal{B}_1(y^p) + \theta(y^p)\delta(y^m)$ for all $y \in \mathcal{S}$, then $\mathcal{B}_1$ is a generalized left (γ, θ)-derivation having a left (γ, θ)-derivation δ on $\mathcal{S}$.*

Proof. Since

$$2\mathcal{B}_1(y^{p+m}) = \gamma(y^p)\mathcal{B}_1(y^m) + \theta(y^m)\delta(y^p) + \gamma(y^m)\mathcal{B}_1(y^p) + \theta(y^p)\delta(y^m) \text{ for all } y \in \mathcal{S}, \tag{3.1}$$

then, we put $y + nb$ in place of y to get

$$2\mathcal{B}_1\Big(y^{m+p} + \tbinom{m+p}{1}y^{m+p-1}nb + \tbinom{m+p}{2}y^{m+p-2}n^2z^2 + \cdots + n^{m+p}z^{m+p}\Big) = \gamma\Big(y^m + \tbinom{m}{1}y^{m-1}nb + \tbinom{m}{2}y^{m-2}n^2z^2 + \cdots + n^mz^m\Big)\mathcal{B}_1\Big(y^p + \tbinom{p}{1}y^{p-1}nb + \tbinom{p}{2}y^{p-2}n^2z^2 + \cdots + n^pz^p\Big) + \theta\Big(y^p + \tbinom{p}{1}y^{p-1}nb + \tbinom{p}{2}y^{p-2}n^2z^2 + \cdots + n^pz^p\Big)\delta\Big(y^m + \tbinom{m}{1}y^{m-1}nb + \tbinom{m}{2}y^{m-2}n^2z^2 + \cdots + n^mz^m\Big) + \gamma\Big(y^p + \tbinom{p}{1}y^{p-1}nb + \tbinom{p}{2}y^{p-2}n^2z^2 + \cdots + n^pz^p\Big)\mathcal{B}_1\Big(y^m + \tbinom{p}{1}y^{m-1}nb + \tbinom{m}{2}y^{m-2}n^2z^2 + \cdots + n^mz^m\Big) + \theta\Big(y^m + \tbinom{m}{1}y^{m-1}nb + \tbinom{m}{2}y^{m-2}n^2z^2 + \cdots + n^mz^m\Big)\delta\Big(y^p + \tbinom{p}{1}y^{p-1}nb + \tbinom{p}{2}y^{p-2}n^2z^2 + \cdots + n^pz^p\Big),$$

where n is any positive integer. Using (3.1), rewrite the expression as

$$nP_1(y,z) + n^2P_2(y,z) + \cdots + n^{m+p-1}P_{m+p-1}(y,z) = 0,$$

where $P_i(y,z)$ is the coefficient of n^i for $i = 1, 2, \ldots, m+p-1$.

Substituting $n = 1, 2, \ldots, m+p-1$ yields a system of $m+p-1$ homogeneous linear equations, whose coefficient matrix is a Vandermonde matrix of order $m+p-1$, as in the previous section. Hence $P_i(y,z) = 0$ for all $y, z \in \mathcal{S}$ and all $i = 1, 2, \ldots, (m+p-1)$. In particular, for

$i = 1$ we obtain:

$$\begin{array}{rcl} 2\binom{m+p}{1}\mathcal{B}_1(y^{m+p-1}z) & = & \binom{m}{1}\gamma(y^{m-1}z)\mathcal{B}_1(y^m) + \binom{m}{1}\gamma(y^m)\mathcal{B}_1(y^{m-1}z) \\ & & +\binom{m}{1}\gamma(y^m)\delta(y^{m-1}z) + \binom{m}{1}\gamma(y^{m-1}z)\delta(y^m) \\ & & +\binom{p}{1}\theta(y^{p-1}z)\mathcal{B}_1(y^p) + \binom{p}{1}\theta(y^p)\mathcal{B}_1(y^{p-1}z) \\ & & +\binom{p}{1}\theta(y^p)\delta(y^{p-1}z) + \binom{p}{1}\theta(y^{p-1}z)\delta(y^p) \text{ for all } y, z \in \mathcal{S}. \end{array}$$

Putting $y = e$ and making use of $\delta(e) = 0$ (by replacing e for y in (3.1)) and torsion restriction of $\mathcal{S}$, we arrive at

$$\mathcal{B}_1(z) = \gamma(z)\mathcal{B}_1(e) + \delta(z) \text{ for every } z \in \mathcal{S}. \tag{3.2}$$

Next, $P_2(y, z) = 0$ implies the following:

$$\begin{array}{rcl} 2\binom{m+p}{2}\mathcal{B}_1(y^{m+p-2}z^2) & = & \binom{p}{2}\gamma(y^m)\mathcal{B}_1(y^{p-2}z^2) + \binom{p}{1}\binom{m}{1}\gamma(y^{m-1}z)\mathcal{B}_1(y^{p-1}z) \\ & & +\binom{m}{2}\gamma(y^{m-2}z^2)\mathcal{B}_1(y^p) + \binom{m}{2}\gamma(y^p)\delta(y^{m-2}z^2) \\ & & +\binom{p}{1}\binom{m}{1}\gamma(y^{p-1}z)\delta(y^{m-1}z) + \binom{p}{2}\gamma(y^{p-2}z^2)\delta(y^m) \\ & & +\binom{m}{2}\theta(y^m)\mathcal{B}_1(y^{m-2}z^2) + \binom{m}{1}\binom{p}{1}\theta(y^{p-1}z)\mathcal{B}_1(y^{m-1}z) \\ & & +\binom{p}{2}\theta(y^{p-2}z^2)\mathcal{B}_1(y^m) + \binom{p}{2}\theta(y^m)\delta(y^{p-2}z^2) \\ & & +\binom{m}{1}\binom{p}{1}\theta(y^{m-1}z)\delta(y^{p-1}z) + \binom{m}{2}\theta(y^{m-2}z^2)\delta(y^p), \end{array}$$

for every $y, z \in \mathcal{S}$. Rewrite the above expression by substituting e for y to obtain

$$\begin{array}{rcl} (m+p-1)(m+p)\mathcal{B}_1(z^2) & = & \frac{m(m-1)}{2}\gamma(z^2)\mathcal{B}_1(e) + pm\gamma(z)\mathcal{B}_1(z) + \frac{p(p-1)}{2}\mathcal{B}_1(z^2) \\ & & +\frac{m(m-1)}{2}\delta(z^2) + pm\gamma(z)\delta(z) + \frac{p(p-1)}{2}z^2\mathcal{B}_1(e) \\ & & +mp\theta(z)\mathcal{B}_1(z) + \frac{m(m-1)}{2}\mathcal{B}_1(z^2) + \frac{p(p-1)}{2}\delta(z^2) \\ & & +mp\theta(z)\delta(z), \text{ for every } y, z \in \mathcal{S}. \end{array}$$

That is,

$$\begin{array}{rcl} 2(m+p)(m+p-1)\mathcal{B}_1(z^2) & = & [m(m-1) + p(p-1)][\gamma(z^2)\mathcal{B}_1(e) + \delta(z^2)] \\ & & +4pq\gamma(z)\mathcal{B}_1(z) + [m(m-1) + p(p-1)]\mathcal{B}_1(z^2) \\ & & +4pq\theta(z)\delta(z), \text{ for every } y, z \in \mathcal{S}. \end{array}$$

Using (3.2) after replacing z by z^2 in the above relation, we get

$$\begin{array}{rcl} 2(m+p)(m+p-1)\mathcal{B}_1(z^2) & = & 4pq\gamma(z)\mathcal{B}_1(z) + 4pq\theta(z)\delta(z) \\ & & +2[m(m-1) + p(p-1)]\mathcal{B}_1(z^2), \text{ for every } y, z \in \mathcal{S}. \end{array}$$

Simplify the above expression and make use of the torsion freeness of $\mathcal{S}$, we have

$$\mathcal{B}_1(z^2) = \gamma(z)\mathcal{B}_1(z) + \theta(z)\delta(z), \text{ for every } z \in \mathcal{S}. \tag{3.3}$$

Consider (3.2), we have

$$\begin{array}{rcl} \delta(z^2) & = & \mathcal{B}_1(z^2) - \theta(z^2)\mathcal{B}_1(e) \\ & = & \gamma(z)\mathcal{B}_1(z) + \theta(z)\delta(z) - \theta(z^2)\mathcal{B}_1(e) \\ & = & \gamma(z)\delta(z) + \theta(z)[\mathcal{B}_1(z) - \theta(z)\mathcal{B}_1(e)] \\ & = & \gamma(z)\delta(z) + \theta(z)\delta(z) \end{array}$$

Therefore, $\mathcal{B}_1$ is a generalized Jordan left (γ, θ)-derivation on $\mathcal{S}$ having Jordan left (γ, θ)-derivation δ. Applying the result from [6], we obtain the intended outcome. ∎

Implications of Theorem 3.1 lead to the following finding by considering $\gamma = \theta = I$:

Theorem 3.2. [4] *Let two integers $m, p \geq 1$ be fixed and $\mathcal{S}$ be a semiprime ring having $(m+p)!$-torsion freeness. If two mappings $\mathcal{B}_1, \delta : \mathcal{S} \to \mathcal{S}$ are additive and satisfying $2\mathcal{B}_1(y^{m+p}) = y^m\mathcal{B}_1(y^p) + y^p\delta(y^m) + y^p\mathcal{B}_1(y^m) + y^m\delta(y^p)$ for every y in $\mathcal{S}$. Then,*

(1) *we say δ a derivation of $\mathcal{S}$ and for each y, z in $\mathcal{S}$, $[\delta(y), z] = 0$.*
(2) *$\delta(\mathcal{S}) = Z(\mathcal{S})$,*
(3) *one give $\mathcal{S}$ commutative or the other give $\delta = 0$ on $\mathcal{S}$,*
(4) *$\mathcal{B}_1$ will be a generalized derivation of $\mathcal{S}$,*
(5) *$\mathcal{B}_1(y) = aq$ for some $m \in Q_l(\mathcal{S}_C)$ for all $y \in \mathcal{S}$.*

Author's contribution

All authors contributed equally to this publication.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors sincerely express gratitude to the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia, for funding this research work.

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