Aust. J. Math. Anal. Appl.
Vol. **22** (2025), No. 2, Art. 8, 11 pp.
AJMAA

A POSTERIORI ERROR ANALYSIS FOR A POLLUTION MODEL IN A BOUNDED DOMAIN OF THE ATMOSPHERE

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Received 17 February, 2025; accepted 12 August, 2025; published 26 September, 2025.

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ABSTRACT. This study conducts an a posteriori error analysis for a mathematical model of atmospheric pollution in a bounded domain. The finite element method is employed to approximate solutions to convection-diffusion-reaction equations, commonly used to model pollutant transport and transformation. The analysis focuses on deriving reliable and efficient error indicators for both temporal and spatial discretizations. Theoretical results establish upper and lower bounds for the discretization errors, ensuring optimal mesh refinement. Numerical simulations, supported by graphical representations, validate the theoretical findings by demonstrating the convergence of error indicators. These results confirm the effectiveness of the finite element method for solving atmospheric pollution models and highlight the importance of adaptive techniques for improving numerical accuracy.

Key words and phrases: Partial differential equations; Variational methods; Atmosphere pollution; Numerical analysis; Finite element method; A posteriori error.

2010 *Mathematics Subject Classification.* Primary 65N30, 65N15; Secondary 76D05, 86A10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Atmospheric pollution is a critical issue with significant environmental and health impacts, including climate change, species extinction, ozone layer depletion, and ice cap melting. Mathematical models, particularly convection-diffusion-reaction equations, are used to predict pollutant concentrations in the atmosphere, known as immission concentrations. These models account for transport, diffusion, and chemical reactions of pollutants, with applications in air quality prediction and ecological studies. Since analytical solutions are often unavailable, numerical methods such as finite differences and finite elements are essential.

Finite difference methods have been widely applied to discretize convection-diffusion-reaction equations, particularly for complex geometries and nonlinear boundary conditions [23, 24, 28]. Adaptive meshing techniques improve solution accuracy by refining grids in critical regions [24]. Finite element methods, including Galerkin and Petrov-Galerkin approaches, offer robust solutions for convection-diffusion problems, addressing convergence and stability challenges [16, 21, 22]. Combined methods leverage the strengths of both approaches [27]. Applications include flame modeling [15], ecological migration [18], and industrial processes [17, 19].

A posteriori error analysis is crucial for assessing numerical accuracy without exact solutions. It provides reliable error bounds and guides adaptive mesh refinement [12, 29, 30, 34]. This work extends prior studies [1, 4] by proving theoretical convergence results for the finite element method and conducting a posteriori error analysis for a pollution model. Numerical simulations validate these findings.

The paper is organized as follows: Section 2 introduces the model and its discretizations. Section 3 constructs and proves the reliability and efficiency of error indicators. Section 4 presents numerical results, and Section 5 concludes.

2. THE CONTINUOUS, SPACE SEMI-DISCRETE AND FULL DISCRETISATION PROBLEMS

2.1. **Notation.** Given a separable Banach space X, provided with the norm $\|\cdot\|_X$, for any $t \in [0,T]$, we denote by $L^2(0,t,X)$ the space of measurable functions v from $(0,t)$ to X such that:

$$\|v\|_{L^2(0,t,X)} = \left(\int_0^T \|v(\cdot,s)\|_X^2\, ds\right)^{1/2}.$$

We define also the space $C(0,t,X)$ of continuous functions v mapping $[0,t]$ to X.

Let $(\cdot,\cdot)$ stand for the inner product on $L^2(\Omega)$ or $L^2(\Omega)^d$, and, by extension, for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, with Ω a domain in $\mathbb{R}^d$, $d \in \{1,2,3\}$.

2.2. **Model.** Let $T > 0$, $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ a bounded domain with a sufficiently regular boundary. Let $f \in L^2(0,T;H^{-1}(\Omega))$, $u_0 \in L^2(\Omega)$.

Consider the model: find a function u such that:

$$\begin{cases} \frac{\partial u}{\partial t} + \vec{E}\cdot(\alpha\nabla u) + \sigma u = \mu\Delta u + f & \text{on } [0,T]\times\Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\cdot,0) = u_0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

where $\vec{E}$ is a vector in $\mathbb{R}^d$ with all elements equal to 1, α is the air velocity, $\sigma > 0$ is the pollutant deterioration rate, $\mu > 0$ is the diffusion coefficient, and $f(t;x)$ is the pollution source intensity.

The space $H^1(\Omega)$ is equipped with the norm of any $v \in H^1(\Omega)$ defined by:

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla v|^2 + \int_\Omega |v|^2\right)^{1/2}.$$

2.3. **Weak formulation.**

Theorem 2.1. *The weak formulation of the problem 2.1 is: find a function $u \in L^2(\Omega)$ such that:*

$$\begin{cases} a_t(u,v) = \ell(v), & \forall v \in H_0^1(\Omega), \\ u(x,0) = u_0(x), & \forall x \in \Omega, \end{cases} \tag{2.2}$$

where the bilinear and linear forms are defined by:

$$a_t(u,v) = \int_\Omega \frac{\partial u(x,t)}{\partial t} v(x) + \int_\Omega \vec{E} \cdot (\alpha \nabla u(x,t)) v(x) + \mu \int_\Omega \nabla u(x,t) \cdot \nabla v(x) + \sigma \int_\Omega u(x,t) v(x),$$

$$\ell(v) = \int_\Omega f(x,t) v(x).$$

The problem 2.2 has a unique solution in $L^2(0,T;H_0^1(\Omega)) \cap C(0,T;H_0^1(\Omega)) \cap H_0^1(\Omega)$ with $\frac{\partial u}{\partial t} \in L^2(0,T;H_0^1(\Omega))$.

Proof. See [4]. ∎

2.4. **Discretisation.**

2.4.1. *Time discretisation.* We define a sequence $(t_0, t_1, \dots, t_N)$ of reals such that $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$, and denote the time step by $\tau_p = t_p - t_{p-1}$, with $p \in \{1, 2, \dots, N\}$. The semi-discretisation of problem 2.2, obtained by applying the backward Euler scheme, is formulated as follows: given u_0, find a sequence $(u_p)_{1 \leq p \leq N}$ in $H_0^1(\Omega)$ such as:

$$a_p(u_p, v) = \int_\Omega u_{p-1}(x) v(x)\, dx + \tau_p \int_\Omega f(x,t_p) v(x)\, dx, \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

where

$$a_p(u_p, v) = \int_\Omega u_p(x) v(x)\, dx + \tau_p \int_\Omega \vec{E} \cdot (\alpha \nabla u_p(x)) v(x)\, dx + \tau_p \sigma \int_\Omega u_p(x) v(x)\, dx.$$

2.4.2. *Space discretisation.* For any $p \in \{1, 2, \dots, N\}$ and $h > 0$, let $\mathscr{T}_{ph}$ be a triangulation of Ω such that $h = \max_{K \in \mathscr{T}_{ph}} h_K$, with h_K the diameter of element K, and $\bar{\Omega}$ being the union of all elements in $\mathscr{T}_{ph}$. We approximate the continuous space $V = L^2(0,T,H_0^1(\Omega)) \cap C(0,T,H^2(\Omega) \cap H_0^1(\Omega))$ by a P_1-Lagrange finite element space V_{ph} defined as follows:

$$V_{ph} = \{v_{ph} \in C(\bar{\Omega}) : \forall K \in \mathscr{T}_{ph}, v_{ph} \in P_1(K)\}.$$

In turn, the full discretisation is: given an approximation $\tilde{u}_0$ of u_0, find a sequence $(u_{ph})_{1 \leq p \leq N}$ in V_{ph} such that:

$$a_{ph}(u_{ph}, v_{ph}) = \int_\Omega u^{p-1}(x) v_{ph}(x)\, dx + \tau_{ph} \int_\Omega f(x, t_{ph}) v_{ph}(x)\, dx, \quad \forall v_{ph} \in V_{ph}, \tag{2.4}$$

where

$$a_{ph}(u_{ph}, v_{ph}) = \int_\Omega u_{ph}(x) v_{ph}(x)\, dx + \tau_{ph} \int_\Omega \vec{E} \cdot (\alpha \nabla u_{ph}(x)) v_{ph}(x)\, dx$$
$$+ \tau_{ph} \mu \int_\Omega \nabla u_{ph}(x) \cdot \nabla v_{ph}(x)\, dx + \tau_{ph} \sigma \int_\Omega u_{ph}(x) v_{ph}(x)\, dx.$$

According to theorem 1.7 [1], there exists a linear operator $L : H_0^1(\Omega) \to V_{ph}$ such that the following estimation holds.

$$\|v - \mathscr{L}v\|_{L^2(K)} \lesssim h_K \|v\|_{H_0^1(\tilde{K})}$$
$$\|v - \mathscr{L}v\|_{L^2(\partial K)} \lesssim h_K^{1/2} \|v\|_{H_0^1(\tilde{K})}$$

where $\tilde{K}$ is the patch of the element K, i.e., the set of all elements sharing an edge/face with the element K.

3. A POSTERIORI ERROR ANALYSIS

In this section, we perform an a-posteriori error analysis. Widely used in the literature, the a-posteriori estimation uses the data of the problem, without involving the exact solution, to control the error. In other words, the a-posteriori error analysis aims to bound the approximation error by quantities that don t require the exact solution, such as the domain, the model parameters, etc. The error is bounded by two quantities named respectively the upper bound and the lower bound. As the problem we are dealing with is time-dependent, we investigate the analysis in time, then in space, and then combine them to infer the global error. At each of these stages, we determine the upper bound and the lower bound of the error. Before we will neeed the following operators and functions :

3.1. **A-posteriori error estimation in time.** The upper bound and the lower bound in an a-posteriori analysis are computed from quantities named local indicators. They are built using the residual equation, which is deduced from the variational formulation of the problem. To simplify the following analysis, we introduce some notation.

Given a sequence $(v_p)_{0\leq p\leq N}$ in $H_0^1(\Omega) \oplus V_{ph}$, we denote its Lagrange interpolant, which is affine on each interval $[t_{p-1}, t_p]$ and equal to v_p at t_p, i.e., defined by:

$$\forall t \in [t_{p-1}, t_p], \quad v_p(x,t) = \frac{t_p - t}{\tau_p} v_{p-1} + \frac{t - t_{p-1}}{\tau_p} v_p. \tag{3.1}$$

Further, we denote by $\pi_\tau v$ the step function of the sequence $(v_p)_{0\leq p\leq N}$, constant and equal to $v(t_p)$ on each interval (t_{p-1}, t_p).

The time discretisation error is denoted by $e_\tau = u - u_\tau$. This error satisfies the residual equation (3.2). Here, we deduce the residual equation by using the definition of the bilinear form a_t, and the fact that u and u_τ are respectively solutions of the problems (2.1) and (2.2).

$$\begin{aligned} a_t(e_\tau, v) = &\int_\Omega (f(x,t) - \pi_\tau f(x,t))\, v(x) + \int_\Omega \alpha \vec{E} \cdot \nabla (u_p(x) - u_\tau(x,t))\, v(x) \\ &+ \mu \int_\Omega \nabla (u_p(x) - u_\tau(x,t)) \cdot \nabla v(x)\, dx + \sigma \int_\Omega (u_p(x) - u_\tau(x,t))\, v(x). \end{aligned} \tag{3.2}$$

We can notice that for any $t \in (t_{p-1}, t_p)$, one has:

$$u_p(x) - u_\tau(x,t) = \frac{t_p - t}{\tau_p}(u_p(x) - u_{p-1}(x)) \tag{3.3}$$

Inspired by [2, 3, 5], from the residual equation, we define the time local indicators as follows:

$$\tau_p = \left(\|\nabla(u_{ph} - u_{p-1,h})\|^2\, dx + \mu \|\vec{E} \cdot \nabla(u_{ph} - u_{p-1,h})\|^2\, dx + \sigma \int_K |u_{ph} - u_{p-1,h}|^2\, dx \right)^{1/2}.$$

In the remaining analysis, we will frequently use the following inequality. Given two numbers a and b, one has:

$$ab \leq \frac{a^2 + b^2}{2}. \tag{3.4}$$

Lemma 3.1. *For any* $v \in H_0^1(\Omega)$*, one has:* $\int_\Omega \vec{E} \cdot \nabla v(x) v(x)\, dx = \frac{1}{2} \int_{\partial\Omega} v(x)^2 \vec{E} \cdot n(x)\, d\epsilon$.

Proof. We consider the case $d = 2$. The same procedure holds in the case $d = 3$. According to Theorem 2.8 of [7], one has :

$$\int_\Omega \vec{E} \cdot \nabla v(x)\, dx = -\int_\Omega (v(x), v(x)) \cdot \nabla v(x)\, dx + \int_{\partial\Omega} v(x)(v(x), v(x))\, d\epsilon$$
$$= -\int_\Omega \vec{E} \cdot \nabla v(x) v(x)\, dx + \int_{\partial\Omega} v(x)^2 \vec{E} \cdot n(x)\, d\epsilon$$

where $d\epsilon$ is a positive measure defined on the boundary $\partial\Omega$.
Then one can deduce that:

$$\int_\Omega \vec{E} \cdot \nabla v(x) v(x)\, dx = \frac{1}{2} \int_{\partial\Omega} v(x)^2 \vec{E} \cdot n(x)\, d\epsilon.$$

∎

Theorem 3.2 (Upper bound)**.** *For any* $n \in \{0, 1, \ldots, N\}$*, we have:*

$$\|e_\tau(t)\|^2 + \alpha \int_0^{t_n} \int_{\partial\Omega} e_\tau(x,t)^2 \vec{E} \cdot n(x)\, d\epsilon - \alpha \|e_\tau(t)\|^2 - \mu \|\nabla e_\tau(t)\|^2 \leq \sum_{p=1}^n \eta_{\tau p}^2 + \frac{1}{\sigma} \|f - \pi_\tau f\|^2_{L^2(0,t_n;H^{-1}(\Omega))}$$
$$+ \sum_{p=1}^n \tau_p \left(\|\vec{E} \cdot \nabla(u_{p-1} - u_{p-1,h})\|^2 + \mu \|\nabla(u_{p-1} - u_{p-1,h})\|^2 + \sigma \|u_{p-1} - u_{p-1,h}\|^2 \right) \tag{3.5}$$
$$+ \sum_{p=1}^n \tau_p \left(\|\vec{E} \cdot \nabla(u_p - u_{p,h})\|^2 + \mu \|\nabla(u_p - u_{p,h})\|^2 + \sigma \|u_p - u_{p,h}\|^2 \right).$$

Proof. The time discretization error e_τ satisfies equation 3.2. Taking $v = e_\tau(\cdot, t)$, and using the Cauchy-Schwarz inequality, for $t \in [t_{p-1}, t_p]$, we get:

$$a_t(e_\tau, e_\tau(\cdot,t)) \leq \|f(t) - \pi_\tau f(t)\| \|e_\tau(t)\| + \phi \|\vec{E} \cdot \nabla(u_p - u_\tau(t))\| \|e_\tau(t)\| + \mu \|\nabla(u_p - u_\tau(t))\| \|\nabla e_\tau(t)\|$$
$$+ \sigma \|u_p - u_\tau(t)\| \|e_\tau(t)\|.$$

We express the right-hand side of this estimation as a sum through the use of inequality 3.4. Also, we refer to the definition of the bilinear form a_t. Then, by rearranging each term, one gets:

(3.6)
$$\partial_t \|e_\tau(t)\|^2 + \alpha \int_{\partial\Omega} e_\tau(t)^2 \vec{E} \cdot n(x)\, d\epsilon - \alpha \|e_\tau(t)\|^2 - \mu \|\nabla e_\tau(t)\|^2 \lesssim \frac{1}{\sigma} \|f(t) - \pi_\tau f(t)\|^2$$
$$+ \alpha \|\vec{E} \cdot \nabla(u_p - u_\tau(t))\|$$
$$+ \mu \|\nabla(u_p - u_\tau(t))\|^2 + \sigma \|u_p - u_\tau(t)\|^2.$$

For each $p \in \{0, 1, \ldots, n\}$, we integrate the previous inequality over the interval (t_{p-1}, t_p), and we use equation 3.3 in the last two terms, which yields to the integral $\int_{t_{p-1}}^{t_p} \frac{t_p - t}{\tau_p}\, dt = \frac{\tau_p}{3}$ since the quantity $u_p - u_{p-1}$ doesn't depend on time.

(3.7)
$$\|e_\tau(t)\|^2 + \alpha \int_0^{t_n} \int_{\partial\Omega} e_\tau(x,t)^2 \vec{E} \cdot n(x)\, d\epsilon - \alpha \|e_\tau(t)\|^2 + \mu \|\nabla e_\tau(t)\|^2 \lesssim \frac{1}{\sigma} \|f - \pi_\tau f\|^2_{L^2(0,t_n,H^{-1}(\Omega))}$$
$$+ \alpha \sum_{p=1}^n \tau_p \|\vec{E} \cdot \nabla(u_p - u_{p-1})\|^2 + \mu \sum_{p=1}^n \tau_p \|\nabla(u_p - u_{p-1})\|^2 + \sigma \sum_{p=1}^n \tau_p \|u_p - u_{p-1}\|.$$

In order to introduce the quantities u_{ph} and $u_{p-1,h}$, we apply the triangular inequality in the last two terms on the right-hand side. Then, we square each of these inequalities and express the right-hand side as the sum of the squares of their terms, even if it means repeatedly using inequality 3.4. Thus, we obtain:

$$\|\vec{E}\cdot\nabla(u_p-u_{p-1})\|^2 \leq \|\vec{E}\cdot\vec{(}u_{ph}-u_{p-1,h})\|^2+\|\vec{E}\cdot\nabla(u_p-u_{ph})\|^2+\|\vec{E}\cdot\nabla(u_{p-1,h}-u_{p-1})\|^2$$
$$\|\nabla(u_p-u_{p-1})\|^2 \leq \|\nabla(u_{ph}-u_{p-1,h})\|^2+\|\nabla(u_p-u_{ph})\|^2+\|\nabla(u_{p-1,h}-u_{p-1})\|^2$$
$$\|u_p-u_{p-1}\|^2 \leq \|u_{ph}-u_{p-1,h}\|^2+\|u_p-u_{ph}\|^2+\|u_{p-1,h}-u_{p-1}\|^2.$$

∎

Corollary 3.3. *For any $n\in\{0,1,\dots,N\}$, we have:*

$$\left\|\frac{\partial e_\tau}{\partial t}\right\|^2_{L^2(0,t_n;,H^{-1}(\Omega))} \lesssim \sum_{p=1}^{n}\eta^2_{\tau p}+\frac{1}{\sigma}\|f-\bar{f}\|^2_{L^2(0,t_n,H^{-1}(\Omega))}$$
$$(3.8)\qquad +\sum_{p=1}^{n}\tau_p\left(\mu\|\vec{E}\cdot\nabla(u_{p-1}-u_{p-1,h})\|^2+\mu\|u_{p-1}-u_{p-1,h}\|^2+\sigma\|u_{p-1}-u_{p-1,h}\|\right)$$
$$+\sum_{p=1}^{n}\tau_p\left(\alpha\|\vec{E}\cdot\nabla(u_p-u_{ph})\|^2+\mu\|u_p-u_{ph}\|^2+\sigma\|u_p-u_{ph}\right)+.$$

Proof. By definition, we know that:

$$\left\|\frac{\partial e_\tau(t)}{\partial t}\right\|_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)}\frac{\left(\frac{\partial e_\tau(t)}{\partial t},v\right)}{\nabla v}.$$

From the residual equation (6), we deduce the following:

$$\left(\frac{\partial e_\tau(t)}{\partial t},v\right) \leq (f(t)-\pi_\tau f(t),v)+\alpha(\vec{E}\nabla(u_p-u_\tau(t)),v)+\mu(\nabla(u_p-u_\tau(t)),\nabla v)$$
$$+(u_p-u_\tau(t),v)+\alpha(\vec{E}\nabla e_\tau(t),v)+\mu(\nabla e_\tau(t),\nabla v)+\sigma(e_\tau(t),v)$$

Except for the first term on the right-hand side, where we make use of the continuity of a linear form, we apply the Cauchy-Schwarz inequality and the Poincaré inequality. These lead for $t\in(t_{p-1},t_p)$ to the following bound:

$$\left\|\frac{\partial e_\tau(t)}{\partial t}\right\|_{H^{-1}(\Omega)} \lesssim \|f(t)-\pi_\tau f(t)\|_{H^{-1}(\Omega)}+\phi\|\vec{E}\nabla(u_p-u_\tau(t))\|+\mu\|\nabla(u_p-u_\tau(t))\|+\sigma\|u_p-u_\tau(t)\|$$
$$+\alpha\|\vec{E}\nabla e_\tau(t)\|+\mu\|\nabla e_\tau(t)\|+\sigma\|e_\tau(t)\|.$$

We can upper bound the right-hand side of this estimation using the same procedure as in the proof of Theorem 2. ∎

Now, let s find an upper bound for the local estimator. To this end, we use the triangle inequality and inequality 3.4. One has:

$$\eta^2_{\tau p} \lesssim \alpha\tau_p\|\vec{E}\nabla(u_{ph}-u_p)\|^2_K+\alpha\tau_p\|\vec{E}\nabla(u_p-u_{p-1})\|^2_K+\alpha\tau_p\|\vec{E}\nabla(u_{p-1}-u_{p-1h})\|^2_K$$
$$+\mu\tau_p\|\nabla(u_{ph}-u_p)\|^2_K+\mu\tau_p\|\nabla(u_p-u_{p-1})\|^2_K+\mu\tau_p\|\nabla(u_{p-1}-u_{p-1,h})\|^2_K$$
$$+\sigma\tau_p\|u_{ph}-u_p\|^2_K+\sigma\tau_p\|u_p-u_{p-1}\|^2_K+\sigma\tau_p\|u_{p-1}-u_{p-1,h}\|^2_K.$$

In this inequality, we just need to estimate the terms $\alpha\tau_p\|\vec{E}\nabla(u_p - u_{p-1})\|_K^2 + \mu\tau_p\|\nabla(u_p - u_{p-1})\|_K^2 + \sigma\tau_p\|u_p - u_{p-1}\|_K^2$. For this purpose, we use the residual equation (6) and replace v by $u_p - u_\tau(t)$. Then, we transform the right-hand side into a sum with inequality (8) and obtain:

$$\int_\Omega \alpha\vec{E}\cdot\nabla(u_p - u_\tau(t)) \times (u_p - u_\tau(t)) + \mu\|\nabla(u_p - u_\tau(t))\|^2 + \sigma\|u_p - u_\tau(t)\|^2 \lesssim \frac{1}{2}\|\partial_t e_\tau(t)\|^2$$

$$+\frac{1}{2}\|u_p - u_\tau(t)\|^2 + \frac{1}{2}\alpha\|\vec{E}\nabla e_\tau(t)\|^2 + \frac{1}{2}\alpha\|u_p - u_\tau(t)\|^2 + \frac{1}{2}\mu\|\nabla e_\tau(t)\|^2$$

$$+\frac{1}{2}\mu\|\nabla(u_p - u_\tau(t))\|^2 + \frac{1}{2}\sigma\|e_\tau(t)\|^2 + \frac{1}{2}\sigma\|f(t) - \pi_\tau f(t)\|^2 + \frac{1}{2}\sigma\|u_p - u_\tau(t)\|^2.$$

3.2. **A-posteriori error estimation in space.** We recall that u_p and u_{ph} represent respectively the solution of the problem 2.3 and 2.4. Then, we define the space approximation error by $e_{ph} = u_p - u_{ph}$. We deduce the residual equation by using the definition of the bilinear form a_p, and the fact that u_p and u_{ph} are respectively the solution of the problems 2.3 and 2.4. The approximation error $e_{p,h}$ satisfies equation 3.9:

The space approximation error is $e_{ph} = u_p - u_{ph}$, satisfying:

(3.9)
$$a_p(e_{ph}, v) = \tau_p \sum_{K\in\mathscr{T}_{ph}} \int_K r_{ph}(x)v(x)\,dx + \tau_p \sum_{K\in\mathscr{T}_{ph}} \int_{\partial K\setminus\partial\Omega} G_{ph}(x)v(x)\,d\epsilon, \quad \forall v \in H_0^1(\Omega),$$

where $r_{ph} = f(t_p) - \frac{u_{ph} - u_{p-1,h}}{\tau_p} - \alpha\vec{E}\cdot\nabla u_{ph} - \sigma u_{ph}$.

Any interior edge γ is shared by two triangles K and K'. According to the orientation, the value of the normal derivative $\frac{\partial u_{ph}}{\partial n}$ along γ varies. We define the jump discontinuity of the normal flux on the edge γ as follows:

$$\left[\frac{\partial u_{ph}}{\partial n}\right] = \mathbf{n}_K \cdot (\nabla u_{ph})_K - \mathbf{n}_{K'} \cdot (\nabla u_{ph})_{K'}.$$

From equation 3.9, we expand the last term on the right-hand side and regroup its terms using the jump discontinuity to obtain the space residual equation below 3.10:

(3.10)
$$a_p(e_{p,h}, v) = \tau_p \sum_{K\in\mathscr{T}_{p,h}} \int_K r_{ph}(x)v(x)\,dx + \tau_p \sum_{K\in\mathscr{T}_{ph}} \int_{\partial K\setminus\partial\Omega} G_{ph}(x)v(x)\,d\epsilon, \quad \forall v \in H_0^1(\Omega),$$

where G_{ph} is the residual defined by $G_{ph} = -\left[\frac{\partial u_{p,h}}{\partial n}\right]$.

Consequently, one can deduce the Galerkin orthogonality relation given for any $v_{ph} \in V_{ph}$ by:

$$a_p(e_{ph}, v_{ph}) = 0.$$

Theorem 3.4. *For any $p \in \{1, 2, \ldots, N\}$, the space approximation error satisfies:*

$$\|e_{ph}\|_{H_0^1(\Omega)} \le \tau_p \sum_{K\in\mathscr{T}_{ph}} \left(h_K^2\|\nabla r_{ph}\|_K^2 + h_K^{1/2}\|G_{ph}\|_{\partial K\setminus\partial\Omega}^2\right). \tag{3.11}$$

Proof. Let $v \in H^1(\Omega)$. The interpolant $\mathscr{L}v$ of v belongs to V_{ph}, and by the Galerkin orthogonality relation, we have:

$$a_p(e_{ph}, v) = a_p(e_{ph}, v - \mathscr{L}v).$$

We apply the continuous and discrete versions of the Cauchy-Schwarz inequality to obtain:

$$a_p(e_{ph},v) \leq \tau_p \left(\sum_{K\in T_{ph}} \|\nabla r_{ph}\|_K^2 \|v-\mathscr{L}v\|_K^2 + \|G_{ph}\|^2_{\partial K\setminus\partial\Omega} \|v-\mathscr{L}v\|^2_{\partial K\setminus\partial\Omega} \right)^{1/2}. \tag{3.12}$$

Using the estimate (5), we obtain:

$$a_p(e_{ph},v) \lesssim \tau_p \sum_{K\in\mathscr{T}_{ph}} \left(h_K^2 \|\nabla r_{ph}\|_K^2 + h_K^{1/2} \|G_{ph}\|^2_{\partial K\setminus\partial\Omega} \right)^{1/2} \|v\|_\Omega.$$

Replacing v with e_{ph} and using the coercivity of the bilinear form a_p, we get:

$$\|e_{p,h}\|^2_{H_0^1(\Omega)} \lesssim \tau_p \sum_{K\in\mathscr{T}_{ph}} \left(h_K^2 \|\nabla r_{p,h}\|_K^2 + h_K^{1/2} \|G_{p,h}\|^2_{\partial K\setminus\partial\Omega} \right).$$

∎

Usually, the exact residuals $G_{p,h}$ and $r_{p,h}$ are replaced by an approximated residual element respectively [6]. A way of doing that consists in projecting the residual $G_{p,h}$ and $r_{p,h}$ onto finite-dimensional spaces. In the literature, it is common to use the mean values of each of these residuals as projections. We introduce $\bar{G}_{p,h}$ and $\bar{r}_{ph}$ as the corresponding approximations of the residuals G_{ph} and r_{ph}:

$$\bar{r}_{ph} = \frac{1}{|\Omega|} \int_\Omega r_{ph}(x)\,dx,$$
$$\bar{G}_{ph} = \frac{1}{|\Omega|} \int_\Omega G_{ph}(x)\,dx.$$

However, the estimation 3.11 suggests that we define the local indicators $(\eta_K)_{K\in\mathscr{T}_{ph}}$ as follows:

$$\eta_K = \left(h_K^2 \|\bar{r}_{ph}\|_K^2 + h_K^{1/2} \|\bar{G}_{ph}\|_K^2 \right)^{1/2}.$$

Then, we define also the residual terms $(\xi_K)_{K\in\mathscr{T}_{ph}}$ as:

$$\xi_K = \left(h_K^2 \|\bar{r}_{ph} - r_{ph}\|_K^2 + h_K^{1/2} \|\bar{G}_{ph} - G_{ph}\|_K^2 \right)^{1/2}.$$

With these notations, we apply the triangular inequality to the estimation 3.11 and use the inequality 3.4 to obtain the upper bound of the approximation by the following corollary.

Corollary 3.5 (Global upper bound)**.** *For any $p \in \{1,2,\ldots,N\}$, one has:*

$$\|e_{ph}\|_{H_0^1(\Omega)} \lesssim \tau_p \sum_{K\in\mathscr{T}_{ph}} (\eta_{tp}^2 + \eta_K^2). \tag{3.13}$$

4. NUMERICAL RESULTS

As for our numerical results, we proceeded in three steps. Firstly, we simulated the indicators by taking different time steps. We found that as time increases, we have an optimal convergence of our constructed indicators. This allows us to conclude the equivalence of our indicators with the error. In a second step, we proceeded at chosen times to the calculation of the spatial indicators by acting on the discretization step. Our numerical results also give us an optimal convergence. Finally, we compared the two error indicators to the theoretical error, which confirms the theoretical results found. Then we put, $\alpha = 1, \sigma = 1, \mu = 0.5$ and f defined by :

$$f(x,y,t) = \left[\frac{\pi}{2} \left(\alpha_1 \cos\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}y\right) + \alpha_2 \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}y\right) \right) \right] e^{-\left(\frac{\pi^2}{2}\mu+\sigma\right)t}.$$

$$r_{ph} = f(t_p) - \frac{u_{ph} - u_{p-1,h}}{\tau_p} - \alpha\, \mathbf{E} \cdot \nabla u_{ph} + \mu \Delta u_{ph} - \sigma u_{ph}.$$

The formular of local indicator depending to the time we use is defined as foffow :

$$\eta_t^p = \left(\alpha\tau_p \|\nabla(u_{p,h} - u_{p-1,h})\|_K^2 + \mu\|\mathbf{E}\cdot\nabla(u_{p,h} - u_{p-1,h})\|_K^2 + \sigma\tau_p\|u_{p,h} - u_{p-1,h}\|_K^2\right)^{\frac{1}{2}}$$

We recall that the space error indicator is noted : η_K and is calculated on each element of the mesh. Thus, at each instant, we calculate the norm L^2 of the vector containing the value of the error indicator in space for all the elements of the mesh. We will also do the same thing for the time error indicator, noted η_{tp}.

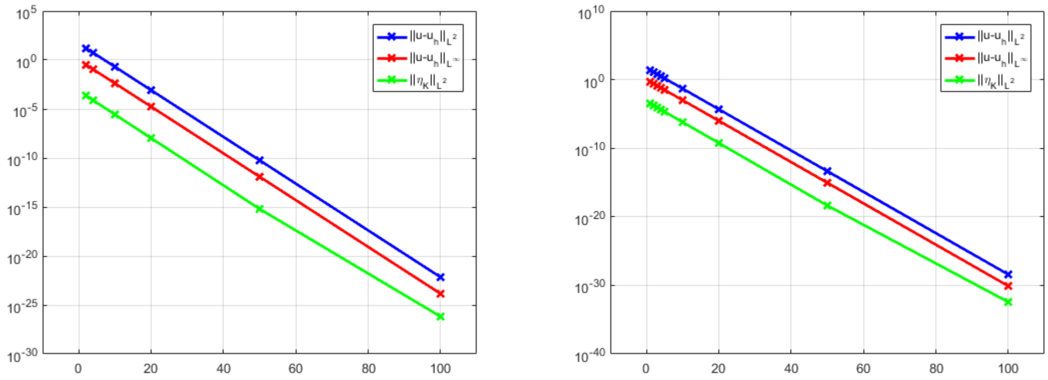

Figure 1:

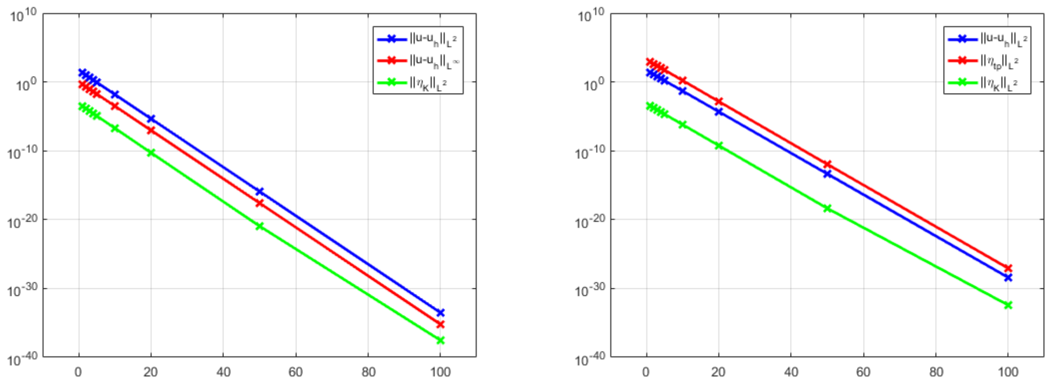

Figure 2:

Analyzing Figures 1, 2 we notice that when h tends to 0, $||u - u_h||_{L^2}$ and $||u - u_h||_{L^\infty}$ tend towards 0 and therefore the numerical solution u_h tends towards the exact solution u of the problem. Therefore, the numerical method used is convergent in time. We also note that $||\eta_K||_{L^2}$ tends to 0 when h tends to 0; this means that the temporal error influences the spatial error. We also see that $||\eta_K||_{L^2}$ is less than $||\eta_t p||_{L^2}$. Which means that the error due to spatial discretization is better controlled than that due to temporal discretization.

5. CONCLUSION

In this research work, proven convergence using error indicators of the pollution model in a domain bounded by the atmosphere. the method is globally convergent, but the convergence is dominated by the temporal discretization error. This implies that, even if the method globally converges to the exact solution, the speed of this convergence is mainly determined by the temporal error and therefore, improving the temporal precision can improve the overall convergence.

DECLARATIONS

Conflict of Interest. The authors declare that they have no conflict of interest for the publication of this paper.

Authors' Contributions. All authors read and approved the final manuscript.

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