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ON THE CONSTRUCTION OF DYADIC WAVELET FRAMES IN LOCAL FIELDS OF POSITIVE CHARACTERISTIC

VISHAL DHAWAN, KHALIL AHMAD, SATYAPRIYA, AND RAJ KUMAR

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DEPARTMENT OF MATHEMATICS, KIRORI MAL COLLEGE, UNIVERSITY OF DELHI, DELHI, INDIA.
dhawanmathskmc@gmail.com

DEPARTMENT OF MATHEMATICS, AL-FALAH UNIVERSITY, FARIDABAD, HARYANA, INDIA
kahmad49@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI, INDIA.
kmc.satyapriya@gmail.com

DEPARTMENT OF MATHEMATICS, KIRORI MAL COLLEGE, UNIVERSITY OF DELHI, DELHI, INDIA.
rajkmc@gmail.com

ABSTRACT. This paper explores the construction of dyadic wavelet frames in $L^2(K)$, where K is a local field with positive characteristic. Using frame multiresolution analysis (FMRA), we establish a systematic method for generating wavelet frames within this setting. While conventional results indicate that two functions are necessary for constructing wavelet frames, we demonstrate that under specific conditions, a single function is sufficient. By leveraging properties of local fields, we provide a detailed characterization of the refinement equation and necessary frame conditions. These results enhance the theoretical understanding of wavelet frames and open new directions for applications in harmonic analysis and signal representation over non-Archimedean fields.

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1. INTRODUCTION

1.1. **Historical Background.** Wavelets and multiresolution analysis have emerged as essential tools in signal processing, driven by the need for enhanced efficiency and precision. Unlike Fourier analysis, which struggles with localized and non-stationary signals due to its reliance on sinusoidal elements, wavelets offer a more adaptable and localized signal representation. In the conventional multiresolution analysis (MRA) framework, a set of scaling functions and wavelet functions establish an orthonormal basis for the signal space. Notable contributions from researchers like Y. Meyer [19], Mallat [18], C. Chui [9], and I. Daubechies [11] have significantly advanced both the theoretical and practical aspects of orthonormal wavelet bases over the past decade.

The construction of orthonormal wavelet bases hinges on the notion of Multiresolution Analysis (MRA), which entails a hierarchy of nested approximation subspaces denoted as V_j. These subspaces are generated by a scaling function ϕ belonging to the space V_0. By applying dilation and translation operations, represented by $\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$ for $j, k \in \mathbb{Z}$, an orthonormal basis for V_j is formed. Adapting MRA and wavelet theory to local fields with positive characteristics, such as field K, necessitates tailoring the framework to accommodate the unique properties of these fields. Incorporating a prime element of the field becomes crucial in shaping the foundations of MRA for such contexts, facilitating the analysis of signals within the framework of locally compact Abelian groups.

While orthonormal bases offer a solid foundation, they have limitations in representing certain signal types effectively. Frame multiresolution analysis addresses these limitations by utilizing frames, which introduce redundancy and flexibility into the signal representation process. Originating in the late 1990s and early 2000s, this approach aims to overcome the constraints of traditional wavelet-based methods, especially in representing complex signal structures. Current research in this area focuses on developing efficient algorithms for frame decomposition, reconstruction, and designing optimized frames, with applications spanning various domains such as image processing, audio signal analysis, and data compression.

A recent advancement by Shah and Abdullah [26] extended the concept of multiresolution analysis (MRA) to local fields with positive characteristics, diverging from the traditional Euclidean space framework. In this extension, the translation set operating on the scaling function to generate the subspace V_0 expands beyond a group structure, encompassing both $\mathcal{L}$ and translations of $\mathcal{L}$, where $\mathcal{L} = \{u(n) : n \in \mathbb{N}\}$ represents distinct coset representations of the unit disc $\mathfrak{D}$ within K^+. Ahmad and Sheikh [1] pioneered the concept of non-uniform wavelet frames in non-Archimedean local fields, providing a comprehensive characterization of tight nonuniform wavelet frames within these fields. Expanding on this groundwork, the concept of non-uniform, non-stationary wavelets and associated multiresolution analysis in local fields was introduced. Additionally, Shah [24] proposed frame multiresolution analysis (FMRA) on local fields, extending MRA principles to accommodate fields with positive characteristics.

The paper is organized as follows: Section 1 offers background information and a concise literature review. Section 2 introduces preliminaries and establishes necessary notations. In Section 3, we present a detailed construction method for a wavelet frame derived from a dyadic frame multiresolution analysis in $L^2(K)$. We also discuss the conditions under which a single function ψ can generate a wavelet frame for $L^2(K)$.

2. NOTATIONS AND PRELIMINARIES

2.1. **A Background about Local Fields.** [2, 3, 6, 12, 27, 24, 26, 25] In this section, we introduce the notation for local fields, which will be used consistently throughout the paper. A local field, denoted by K, is both an algebraic field and a topological space that satisfies the following essential properties: it is locally compact, complete, totally disconnected, and non-discrete. The additive and multiplicative groups associated with K are denoted by K^+ and K^*, respectively. A Haar measure on K^+, represented by dx, can be chosen such that for any nonzero element $\alpha \in K$, the measure transforms as $d(\alpha x) = |\alpha| dx$. Here, $|\alpha|$ is the absolute value (or valuation) of α, with the convention $|0| = 0$.

The absolute value function $|\cdot|$ satisfies the following fundamental properties:

(1) $|x| \geq 0$ with equality if and only if $x = 0$;
(2) $|xy| = |x| \cdot |y|$ for all $x, y \in K$;
(3) $|x + y| \leq \max(|x|, |y|)$, known as the ultrametric inequality.

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the ring of integers of K, which serves as the unique maximal compact subring of K. Closely related is the subset $\mathfrak{P} = \{x \in K : |x| < 1\}$, known as the prime ideal of K, which is the unique maximal ideal in $\mathfrak{D}$ and exhibits both principal and prime properties.

Due to the total disconnectedness of K, the absolute values $|x|$ as x varies over K form a discrete set, often expressed as $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Consequently, there exists an element within $\mathfrak{P}$ having the largest absolute value. We designate $\mathfrak{p}$ as a fixed element attaining this maximal absolute value within $\mathfrak{P}$, referring to it as a prime element of K. As an ideal in $\mathfrak{D}$, the prime ideal satisfies $\mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

It can be shown that $\mathfrak{D}$ is both compact and open, which in turn implies that $\mathfrak{P}$ shares these properties. As a result, the residue field $\mathfrak{D}/\mathfrak{P}$ forms a finite field isomorphic to $GF(q)$, where $q = p^c$ for some prime p and some positive integer c. A rigorous proof of this assertion can be found in [20].

Remark 2.1. Since this paper focuses on dyadic wavelet frames, we adopt the convention $p = q = 2$ for convenience in all subsequent discussions.

For each integer k, we define the fractional ideals $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$. Each $\mathfrak{P}^k$ is both compact and open, forming a subgroup of K^+ [20]. Moreover, any element $x \in \mathfrak{P}^k$ can be uniquely expressed as

$$x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell,$$

where c_ℓ are coset representatives of the quotient group $\mathfrak{D}/\mathfrak{P}$.

2.1.1. ***Fourier Analysis on Local Fields***. Consider a measurable subset E of K, where its measure $|E|$ is defined as the integral of its characteristic function $\chi_E(x)$ with respect to the normalized Haar measure dx on K, ensuring that $|\mathfrak{D}| = 1$.
From fundamental observations, it follows that $|\mathfrak{P}| = q - 1$ and $|\mathfrak{p}| = q - 1$, where q is given by p^c. For a comprehensive discussion on these notions, one may refer to [20].
A crucial aspect of local fields is the existence of a nontrivial, unitary, and continuous character Υ on K^+. Notably, K^+ is self-dual (see [20]).
We now consider a specific character Υ on K^+, which remains trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. Such a character is constructed by choosing an arbitrary nontrivial character and appropriately scaling it. This process is particularly pertinent for local fields with positive characteristic.
For $y \in K$, we define the character shift $\Upsilon_y(x) = \Upsilon(yx)$ for $x \in K$.
The Fourier transform of a function $f \in L^1(K)$ is given by:

$$\widehat{f}(\omega) = \int_K f(x)\overline{\Upsilon_\omega(x)}\, dx$$

Alternatively, this can be rewritten as:

$$\widehat{f}(\omega) = \int_K f(x)\Upsilon(-\omega x)\, dx$$

This formulation mirrors classical Fourier analysis on the real line but is adapted to the locally compact, non-Archimedean nature of the field K.
To extend this definition to functions in $L^2(K)$, we introduce the characteristic functions Φ_k for $k \in \mathbb{Z}$, where Φ_k represents the characteristic function of $\mathfrak{P}^k$.

Definition 2.1. For $f \in L^2(K)$, let $f_k = f\Phi_{-k}$. Then, the Fourier transform is defined as:

$$\widehat{f}(\omega) = \lim_{k\to\infty} \int_{|x|\leq q^k} f(x)\overline{\Upsilon_\omega(x)}\, dx,$$

where the limit is taken in the $L^2(K)$ sense.

A fundamental result regarding this transform is encapsulated in the following theorem (Theorem 2.3 in [20]):

Theorem 2.1. *The Fourier transform is a unitary operator on* $L^2(K)$.

2.1.2. *Operators on* $L^2(K)$. Wavelet theory extensively utilizes translation, modulation, and dilation operators. We define these fundamental operators in the space $L^2(K)$:

(1) **Translation Operator:** To construct a translation set for $L^2(K)$, we consider the coset representatives of the quotient group $K^+/\mathfrak{D}$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and suppose $\{u(n) : n \in \mathbb{N}_0\}$ forms a set of distinct coset representatives. The translation operator $T : L^2(K) \to L^2(K)$ is then given by:

$$(T_n f)(x) = f(x - u(n)), \ n \in \mathbb{N}_0.$$

(2) **Modulation Operator:** For a given $y \in K$, we define the modulation operator as:

$$(E_y f)(x) = (\Upsilon_y f)(x) = \Upsilon(xy) f(x), \ x \in K.$$

(3) **Dilation Operator:** The dilation operator acts on $L^2(K)$ as:

$$(Df)(x) = q^{1/2} f(\mathfrak{p}^{-1} x).$$

Considering the set $\{u(n)\}_{n=0}^{\infty}$ as a complete collection of unique coset representatives of $\mathfrak{D}$ within K^+, the set of characters $\{\Upsilon_{u(n)}\}_{n=0}^{\infty}$ constitutes an exhaustive list of distinct characters on $\mathfrak{D}$. As demonstrated in [20], we establish the following result:

Lemma 2.2. [27] *Let* $\{u(n)\}_{n=0}^{\infty}$ *be a complete set of coset representatives of* $\mathfrak{D}$ *in* K^+. *Then,* $\{\Upsilon_n\}_{n=0}^{\infty}$ *forms a complete orthonormal system on* $\mathfrak{D}$*, where* $\Upsilon_n = \Upsilon_{u(n)}$ *for all* $n \in \mathbb{N}_0$.

For $f \in L^1(\mathfrak{D})$, the Fourier coefficients are defined as:

$$\widehat{f}(u(n)) = \int_{\mathfrak{D}} f(x) \overline{\Upsilon_{u(n)}(x)} \, dx.$$

The corresponding Fourier series expansion is given by:

$$\sum_{n=0}^{\infty} \widehat{f}(u(n)) \Upsilon_{u(n)}(x),$$

and satisfies the standard L^2-orthogonality relation:

$$\int_{\mathfrak{D}} |f(x)|^2 \, dx = \sum_{n=0}^{\infty} |\widehat{f}(u(n))|^2.$$

Lastly, quotient group structures play a significant role in this framework. Using previous results (see [2, 3, 6, 12, 27]), we recall:

$$\mathfrak{P}^{-1}/\mathfrak{P} \cong \mathfrak{D}/\mathfrak{P} \cong GF(q).$$

Setting $\{u(n) : 0 \leq n \leq q-1\}$ as coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$, we conclude:

$$\mathfrak{P}^{-1}/\mathfrak{D} = \{\mathfrak{D}, u(1) + \mathfrak{D}\}, \quad \mathfrak{D}/\mathfrak{P} = \{\mathfrak{P}, u(1)\mathfrak{p} + \mathfrak{D}\}.$$

These structural results provide a foundation for further analysis in the upcoming sections.

2.2. **Frames and Their Fundamental Properties.** We now explore the concept of frames within an arbitrary, separable Hilbert space $\mathcal{H}$ and highlight some of their key properties. For a more detailed exposition, the reader is referred to [7].

Definition 2.2. Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathbb{I}$ be a countable index set. A sequence $\{f_\beta\}_{\beta\in\mathbb{I}}$ is called a *frame* for $\mathcal{H}$ if there exist positive constants A and B such that:

$$A\|f\|^2 \leq \sum_{\beta\in\mathbb{I}} |\langle f, f_\beta\rangle|^2 \leq B\|f\|^2, \quad \forall\, f \in \mathcal{H}.$$

Here, the constants A and B are known as the *frame bounds*, with A serving as the *lower bound* and B as the *upper bound*. A frame is deemed *exact* if the removal of any single element renders it no longer a frame. If $A = B$, the frame is called *tight*, and when $A = B = 1$, it is referred to as a *Parseval frame*.

In wavelet theory, we frequently encounter families generated by translations of a single function. Hence, it becomes crucial to determine the conditions under which a collection of the form $\{T_k\phi : k \in \mathbb{N}_0\}$, where $\phi \in L^2(K)$, constitutes a frame sequence. To facilitate this analysis, we introduce a function Φ, which represents a complex-valued function on K, defined as:

$$\Phi(\xi) = \sum_{n\in\mathbb{N}_0} |\widehat{\phi}(\xi + u(n))|^2. \tag{2.1}$$

It is evident that Φ is K-integrally periodic and belongs to $L^1(\mathfrak{D})$. For further details, we refer to [24] and related literature. With this formulation in place, we present a crucial lemma that establishes bounds for the function Φ. This result, which generalizes a theorem by Benedetto and Li [5], demonstrates that the frame properties of $\{T_k\phi : k \in \mathbb{N}_0\}$ can be fully characterized through Φ.

Lemma 2.3. *Let $\phi \in L^2(K)$ be given. Then $\{T_n\phi : n \in \mathbb{N}_0\}$ forms a frame sequence with bounds A and B if and only if*

$$A \leq \Phi(\xi) \leq B, \quad \forall\, \xi \in K\backslash\mathcal{N},$$

where $\mathcal{N}$ denotes the null set of Φ, given by

$$\mathcal{N} = \{\xi \in K : \Phi(\xi) = 0\}.$$

2.3. **Frame Multiresolution Analysis on $L^2(K)$.** This section presents the formal definition of Frame Multiresolution Analysis (FMRA) on Locally Compact Abelian (LCA) groups. The concept of FMRA was initially introduced for $G = \mathbb{R}$ by J. J. Benedetto and S. Li in their seminal work [5]. The definition provided here serves as a generalized version of their original framework.

Definition 2.3. [24] A Frame Multiresolution Analysis (FMRA) for $L^2(K)$ consists of a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ along with a function $\phi \in V_0$, satisfying the following conditions:

(i) The subspaces are nested:

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots .$$

(ii) The union of all subspaces is dense in $L^2(K)$ and their intersection is trivial:

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K), \quad \text{and} \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

(iii) Each subspace is a scaled version of V_0:

$$V_j = D^j V_0.$$

(iv) The subspaces are translation-invariant:

$$f \in V_0 \implies T_\lambda f \in V_0, \quad \forall \lambda \in \Lambda.$$

(v) The set $\{T_k\phi : k \in \mathbb{N}_0\}$ forms a frame for V_0.

The function ϕ that generates an FMRA is known as the *scaling function*. The subspaces V_j are referred to as *approximation spaces* or *multiresolution subspaces*.
A classical Multiresolution Analysis (MRA) differs from an FMRA in that condition (v) requires an orthonormal basis rather than a frame. Notably, condition (v) ensures that:

$$\overline{\text{span}}\{T_k\phi : k \in \mathbb{N}_0\} = V_0.$$

If ϕ generates an FMRA, we obtain the structural relation:

$$V_j = D^j\Big(\overline{\text{span}}\{T_k\phi : k \in \mathbb{N}_0\}\Big) = \overline{\text{span}}\{D^jT_k\phi : k \in \mathbb{N}_0\}, \quad j \in \mathbb{Z}. \tag{2.2}$$

An FMRA is classified as *exact* if the frame for V_0 is exact; otherwise, it is termed *non-exact*. The necessary conditions for a function ϕ to generate an FMRA can be derived by modifying classical MRA principles.
For an in-depth discussion on constructing frame multiresolutions on local fields, we refer the reader to [24]. The following theorem concisely presents the fundamental conditions for a function ϕ to generate an FMRA for $L^2(K)$.

Theorem 2.4. [24] *A function $\phi \in L^2(K)$ generates a Frame Multiresolution Analysis (FMRA) if it satisfies the following criteria:*

(i) *The subspaces $\{V_j : j \in \mathbb{Z}\}$ are defined as in Equation* (2.2)*.*

(ii) *There exists a K-integral periodic function $m_0 \in L^\infty(\mathfrak{D})$ such that*

$$\widehat{\phi}(\mathfrak{p}^{-1}(\xi)) = m_0(\xi)\widehat{\phi}(\xi). \tag{2.3}$$

(iii) *The sequence $\{T_k\phi : k \in \mathbb{N}_0\}$ forms a frame sequence.*

(iv) $|\widehat{\phi}| \neq 0$ *on a neighborhood of* $0 \in K$.

Equation (2.3) is known as the *refinement equation*, and a function ϕ satisfying this equation is termed *refinable*. The function m_0, appearing in (2.3), is referred to as the *two-scale symbol* or *refinement mask*. It satisfies the key relation:

$$\Phi(\mathfrak{p}^{-1}(\xi)) = |m_0(\xi)|^2\Phi(\xi) + |m_0(\xi + \mathfrak{p}u(1))|^2\Phi(\xi + \mathfrak{p}u(1)). \tag{2.4}$$

Our principal objective is to construct a wavelet frame using the given FMRA. Throughout this discussion, we assume that ϕ generates an FMRA satisfying all conditions of Theorem 2.4 and follows dyadic dilations.
To achieve this, we decompose $L^2(K)$ into simpler components, analogous to classical MRA. Let W_j denote the orthogonal complement of V_j in V_{j+1}, leading to the decomposition:

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

The existence of functions in $L^2(K)$ whose translations form a frame for W_0 is central to constructing a wavelet frame. The following lemma formalizes this principle [7, Chapter 17], [24].

Lemma 2.5. [24] *If* $\phi \in L^2(K)$ *generates an FMRA, then:*

(i) $W_j = D^j W_0$ *for all* $j \in \mathbb{Z}$.

(ii) If $\psi_1, \psi_2, \ldots, \psi_n \in W_0$ *form a frame for* W_0*, then* $\{D^jT_k\psi_i\}$ *forms a frame for* W_j*, and* $\{D^jT_k\psi_i\}$ *for all* j *constitutes a frame for* $L^2(K)$ *with identical bounds.*

Lemma 2.5 indicates that our objective is simplified to the construction of functions $\psi_1, \psi_2, \ldots, \psi_n$ in $L^2(K)$ such that the family $\{T_\lambda\psi_i : \lambda \in \Lambda, 1 \leq i \leq n\}$ forms a frame for W_0. Consequently, it becomes crucial for us to provide a characterization of the space W_0.

Lemma 2.6. [24] *Assume that* $\phi \in L^2(K)$ *generates an FMRA of dydic dilation with two-scale symbol* $m_0 \in L^\infty(\mathfrak{D})$*. If, for any* K*-integral periodic function* $F \in L^2(\mathfrak{D})$*, we define* $f \in V_1$ *by*

$$\widehat{f}\big(\mathfrak{p}^{-1}(\xi)\big) = F(\xi)\widehat{\phi}(\xi), \tag{2.5}$$

then $f \in W_0$ *if and only if*

$$\Big(F\overline{m_0}\Phi\Big)(\xi) + \Big(F\overline{m_0}\Phi\Big)(\xi + \mathfrak{p}u(1)) = 0 \tag{2.6}$$

hold true for a.e. $\xi \in K$.

3. DYADIC WAVELET FRAMES

In previous studies on Multiresolution Analysis (MRA) and Frame Multiresolution Analysis (FMRA), it has been demonstrated that when an FMRA is generated by dyadic dilations, only two functions are sufficient to construct a frame for the space W_0. For an in-depth exploration of FMRA with dyadic dilations in the case $G = \mathbb{R}$, one may refer to [14]. Similarly, [28] delves into the generalized MRA structure on Euclidean spaces. For discussions on dyadic wavelet frames on locally compact Abelian groups, [16] provides valuable insights. Additionally, [17] explores Dyadic Riesz bases originating from a Riesz MRA.

Remark 3.1. Drawing inspiration from previous research and assuming that the function ϕ generates an FMRA with dyadic dilations, our goal here is to construct two functions ψ_1 and ψ_2 such that the family

$$\big\{T_k\psi_i : k \in \mathbb{N}_0,\ i = 1, 2\big\} \tag{3.1}$$

forms a frame for W_0. We break down this process into two distinct steps:

- First, we aim to demonstrate the existence of two functions $\psi_1, \psi_2 \in W_0$ such that their translates generate W_0, i.e.,

  $$W_0 = \overline{\text{span}}\{T_k\psi_i : k \in \mathbb{N}_0,\ i = 1, 2\}.$$

  Additionally, we will provide an explicit expression for these two functions.
- Subsequently, we will establish that the family consisting of translates of functions ψ_1 and ψ_2, obtained in the previous step, indeed constitutes a frame for W_0.

This systematic approach allows us to construct a wavelet frame for $L^2(K)$ using only two functions, thereby streamlining the analysis process.

The first task can be simplified considerably. We will provide an alternative characterization for the family $\big\{T_k\psi_i : k \in \mathbb{N},\ i = 1, 2\big\}$ to generate the space W_0. In this alternative approach, we establish a sufficient condition that reduces our task to merely checking the solvability of a system of linear equations. These insights are encapsulated in the following theorem.

Theorem 3.1. *Assume that $\phi \in L^2(K)$ generates an FMRA of dydic dilation and let for some K-integral periodic $m_1, m_2 \in L^\infty(\mathfrak{D})$, the functions $\psi_1, \psi_2 \in V_1$ be defined by:*

$$\widehat{\psi_1}(\mathfrak{p}^{-1}(\xi)) = m_1(\xi)\widehat{\phi}(\xi) \quad \textit{and} \quad \widehat{\psi_2}(\mathfrak{p}^{-1}(\xi)) = m_2(\xi)\widehat{\phi}(\xi).$$

If there exist K-integral periodic functions G_0, G_1 and $G_2 \in L^\infty(\mathfrak{D})$ such that the equations

$$\sum_{j=0}^{1}\Big(\overline{m_0}m_i\Phi\Big)(\xi + \mathfrak{p}u(j)) = 0,\ i = 1, 2 \tag{3.2}$$

$$\sum_{i=0}^{2}\left(m_i\Phi G_i\right)(\xi) = \Phi(\xi) \tag{3.3}$$

$$\sum_{i=0}^{2}(m_i\Phi)(\xi + \mathfrak{p}u(1))G_i(\xi) = 0 \tag{3.4}$$

are satisfied for a.e. $\xi \in K$, then we have $W_0 = \overline{\textit{span}}\big\{T_n\psi_i : n \in \mathbb{N}_0,\ i = 1, 2\big\}$.

Proof. Equation (3.2) along with Lemma 2.6 implies that both ψ_1 and ψ_2 belong to W_0. Furthermore, as W_0 is a closed and translation-invariant subspace of $L^2(K)$, we have the following implication:

$$\overline{\text{span}}\{T_n\psi_i : n \in \mathbb{N}_0, i = 1, 2\} \subseteq W_0. \tag{3.5}$$

For any $\xi \in K$ and for any $\ell \in \mathbb{N}_0$, an easy manipulation of the equations (3.3) and (3.4) yields

$$\frac{1}{2}\widehat{\phi}(\mathfrak{p}(\xi))\,\overline{\Upsilon_\ell(\mathfrak{p}(\xi))} = \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(n)+\ell}\,\Upsilon_n(\xi)\widehat{\phi}(\xi) + \sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}\,\Xi_n(\xi)\widehat{\psi}_i(\xi);$$

i.e. we have

$$\frac{1}{\sqrt{2}}D^{-1}\mathcal{E}_{-\ell}\widehat{\phi} = \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(n)+\ell}\,\Upsilon_n\widehat{\phi} + \sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}\,\Upsilon_n\widehat{\psi}_i.$$

Taking the inverse Fourier transform of the above equation, we obtain

$$DT_\ell\phi = \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(n)+\ell}\,T_{-n}\phi + \sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}\,T_{-n}\psi_i. \tag{3.6}$$

Since $\psi_1, \psi_2 \in W_0$ and since $\phi \in V_0$ generates an FMRA, therefore we get

$$\sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}\,T_{-n}\psi_i \in W_0 \text{ and } \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(n)+\ell}\,T_{-n}\phi \in V_0.$$

Now let $f \in W_0$ and let $\epsilon > 0$ be arbitrary. Since $\{DT_n\phi\}_{n\in\mathbb{N}_0}$ is a frame for V_1, therefore there exists a finite set $\mathbb{N}_\epsilon \subset \mathbb{N}_{\not\!\!\prime}$ and a finite sequence $\{b_\ell\}_{\ell\in\mathbb{N}_\epsilon}$ such that

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell DT_\ell\phi - f\right\|^2 < \epsilon.$$

A substitution from (3.6) now gives us

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell\left(\sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(n)+\ell}\,T_{-n}\phi + \sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}\,T_{-n}\psi_i\right) - f\right\|^2 < \epsilon.$$

Now we use orthogonality of the two terms appearing on the right hand side of equation (3.6), to get

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(n)+\ell}T_{-n}\phi\right\|^2 + \left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell \sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}T_{-n}\psi_i - f\right\|^2 < \epsilon.$$

This implies that

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell \sum_{i=1}^{2}\sum_{n\in\mathbb{N}_0} g^i_{\mathfrak{p}(n)+\ell}T_{-n}\psi_i - f\right\|^2 < \epsilon;$$

and from this, we conclude that $f \in \overline{\text{span}}\{T_n\psi_i : n \in \mathbb{N}_0, i = 1, 2\}$. We now have the reverse inclusion in the expression (3.5) and thus we can write

$$W_0 = \overline{\text{span}}\{T_n\psi_i : n \in \mathbb{N}_0, i = 1, 2\}.$$

This completes the proof. ∎

Using the suficient condition given in the above lemma, we now proceed to prove our first aim, i.e. we find the two functions ψ_1, ψ_2 generating the space W_0.

Theorem 3.2. *Let K be a local field and let $\phi \in L^2(K)$ generates an FMRA of dydic dilation. Then there always exist two functions $\psi_1, \psi_2 \in W_0$ such that*

$$W_0 = \overline{span}\{T_n\psi_i : n \in \mathbb{N}_0,\ i = 1, 2\}.$$

Proof. Utilizing Lemma 3.1, it suffices to demonstrate that equations (3.2), (3.3), and (3.4) are satisfied almost everywhere on K. It's noteworthy that each term appearing in these equations is K-integral periodic, thereby reducing the requirement to establish their satisfaction almost everywhere on $\mathfrak{D}$.

To streamline our computations further, we partition the set $\mathfrak{D}$ into four disjoint parts as follows:

$$\begin{aligned}
\mathfrak{D}^{(0)} &= \{\xi \in \mathfrak{D} : \Phi(\xi) = 0, \Phi(\xi + \mathfrak{p}u(1)) = 0\} \\
\mathfrak{D}^{(1)} &= \{\xi \in \mathfrak{D} : \Phi(\xi) > 0, \Phi(\xi + \mathfrak{p}u(1)) = 0\} \\
\mathfrak{D}^{(2)} &= \{\xi \in \mathfrak{D} : \Phi(\xi) = 0, \Phi(\xi + \mathfrak{p}u(1)) > 0\} \\
\mathfrak{D}^{(12)} &= \{\xi \in \mathfrak{D} : \Phi(\xi) > 0, \Phi(\xi + \mathfrak{p}u(1)) > 0\}
\end{aligned}$$

Note that

$$\mathfrak{D}^{(0)} = T_{\mathfrak{p}u(1)}\mathfrak{D}^{(0)},\ \mathfrak{D}^{(12)} = T_{\mathfrak{p}u(1)}\mathfrak{D}^{(12)},\ \mathfrak{D}^{(1)} = T_{\mathfrak{p}u(1)}\mathfrak{D}^{(2)} \text{ and } \mathfrak{D}^{(2)} = T_{\mathfrak{p}u(1)}\mathfrak{D}^{(1)}.$$

On $\mathfrak{D}^{(0)}$, we can define m_1, m_2, G_0, G_1 and G_2 to be arbitrary bounded K- integral periodic functions. In particular, we may set them equal to 0.

Equation (3.4) always holds on $\mathfrak{D}^{(1)}$. We can further choose $m_2 = 0$ on $\mathfrak{D}^{(1)}$ so that (3.2) holds for $i = 2$. Also, from (2.4), there are constants $a, b > 0$ such that either $m_0 = 0$ or $a \le |m_0| \le b$ on $\mathfrak{D}^{(1)}$. If $m_0 = 0$, then (3.2) holds for $i = 1$ and (3.3) only forces

$$m_1(\xi)G_1(\xi) = 1;$$

which is easy to achieve. One choice is to take

$$m_1(\xi) = G_1(\xi) = 1.$$

If $a \le |m_0| \le b$, then, for $i = 1$, (3.2) forces $m_1(\xi) = 0$ so that (3.3) only requires

$$m_0(\xi)G_0(\xi) = 1,$$

which can be accomplished by taking

$$G_0(\xi) = \frac{1}{m_0(\xi)}.$$

Clearly, this G_0 satisfies all the required properties. Also, in both the subcases, any K-integral periodic function in $L^\infty(\mathfrak{D})$ can work as G_2. To avoid any complex calculations, we choose $G_2 = 0$ on $\mathfrak{D}^{(1)}$.
Equation (3.3) is always satisfied on $\mathfrak{D}^{(2)}$. Similarly to the previous case, we choose $m_2(\xi + \mathfrak{p}u(1)) = 0$ to ensure compatibility with the definition of m_2 on $\mathfrak{D}^{(1)}$ and to guarantee that equation (3.2) holds for $i = 2$. Further, from (2.4), there are constants $a, b > 0$ such that either $m_0(\xi + \mathfrak{p}u(1)) = 0$ or $a \leq m_0(\xi + \mathfrak{p}u(1)) \leq b$ for all $\xi \in \mathfrak{D}^{(2)}$. If $0 < a \leq m_0(\xi + \mathfrak{p}u(1)) \leq b$, then, for $i = 1$, (3.2) forces

$$m_1(\xi + \mathfrak{p}u(1)) = 0;$$

which is compatible with the definition of m_1 on $\mathfrak{D}^{(1)}$. Further, in this case, (3.4) reduces to

$$m_0(\xi + \mathfrak{p}u(1))G_0(\xi) = 0,$$

which can be accomplished by taking $G_0 = 0$.
If $m_0(\xi + \mathfrak{p}u(1)) = 0$, then (3.2) holds for $i = 1$ and (3.4) forces

$$m_1(\xi + \mathfrak{p}u(1))G_1(\xi) = 0;$$

which may be achieved by taking $G_1(\xi) = 0$. Also, the choice $G_2 = 0$ can be made for both the above subcases. The justification for this choice is similar to the previous case.
For the final case, we further divide the set $\mathfrak{D}^{(12)}$ into two subparts:

$$\begin{aligned}\mathfrak{D}_1^{(12)} =& \{\xi \in \mathfrak{D}^{12} : \Phi(\mathfrak{p}^{-1}(\xi)) = 0\}\\ \mathfrak{D}_2^{(12)} =& \{\xi \in \mathfrak{D}^{12} : \Phi(\mathfrak{p}^{-1}(\xi)) > 0\}\end{aligned}$$

If $\xi \in \mathfrak{D}_1^{(12)}$, then (2.4) forces $m_0(\xi) = m_0(\xi + \mathfrak{p}u(1)) = 0$. This means that (3.2) gets trivially satisfied for both $i = 1, 2$. Also, any K-integral periodic function in $L^\infty(\mathfrak{D})$ can work as G_0. In particular, we let $G_0(\xi) = 0$. Now, we are left with two linear equations in four variables. We make the choice $m_1(\xi) = 1$ and $m_2(\xi) = \Xi_{u(1)}(\xi)$. Then we get $G_1(\xi) = \frac{1}{2}$ and $G_2(\xi) = \frac{1}{2}\,\overline{\Upsilon_{u(1)}(\xi)}$. Thus, we are able to find m_1, m_2, G_0, G_1 and G_2 satisfying the equations (3.2), (3.3) and (3.4).
Lastly, let $\xi \in \mathfrak{D}_2^{(12)}$. Equation (2.4) implies that

$$\frac{A}{B} \leq |m_0(\xi)|^2 + |m_0(\xi + \mathfrak{p}u(1))|^2 \leq \frac{B}{A};$$

where A and B respectively denote the lower and upper frame bound for the frame $\{T_n\phi : n \in \mathbb{N}_0\}$.
Now set $m_1(\xi) = (\overline{m_0}\Phi)(\xi + \mathfrak{p}u(1))\ \Upsilon_{u(1)}(\xi)$ and $m_2(\xi) = 0$. Observe that, with these choices of m_1 and m_2, (3.2) always holds for both $i = 1, 2$. The remaining equations, (3.3) and (3.4), now constitute a system of two linear equations in two unknowns, G_0 and G_1. The determinant

Δ of this system is given by:

$$\Delta(\xi) = \begin{vmatrix} m_0(\xi)\Phi(\xi) & m_1(\xi)\Phi(\xi) \\ (m_0\Phi)(\xi + \mathfrak{p}u(1)) & (m_1\Phi)(\xi + \mathfrak{p}u(1)) \end{vmatrix}$$

It is easy to note that

$$|\Delta(\xi)| \geq \frac{A^4}{B} > 0. \tag{3.7}$$

This implies that G_0 and G_1, occurring in (3.3) and (3.4), are unique and using Cramer's rule, we can write

$$G_0(\xi) = \tfrac{\Phi(\xi)(m_1\Phi)(\xi+\mathfrak{p}u(1))}{\Delta(\xi)} \quad \text{and} \quad G_1(\xi) = -\tfrac{\Phi(\xi)(m_0\Phi)(\xi+\mathfrak{p}u(1))}{\Delta(\xi)}.$$

Further (3.7) and Lemma 2.3 implies that

$$|G_0(\xi)| \leq \tfrac{B^3}{A^4}||m_1||_\infty \quad \text{and} \quad |G_1(\xi)| \leq \tfrac{B^3}{A^4}||m_0||_\infty.$$

This implies that both G_0 and G_1 are K-integral periodic functions in $L^\infty(\mathfrak{D})$. As for G_2, we can make an arbitrary choice within the space $L^\infty(\mathfrak{D})$, and here we opt for $G_2 = 0$.
Combining all cases, we can summarize as follows:

$$m_1(\xi) = \begin{cases} (\overline{m_0}\Phi)(\xi + \mathfrak{p}u(1))\ \Upsilon_{u(1)}(\xi) & , \xi \in \mathfrak{D}_2^{(12)} \\ 1 & , \xi \in \mathfrak{D}_1^{(12)} \\ 1 & , \xi \in \mathfrak{D}^{(1)} \text{ and } m_0(\xi) = 0 \\ 0 & , \text{ otherwise} \end{cases} \tag{3.8}$$

$$m_2(\xi) = \begin{cases} \Upsilon_{u(1)}(\xi) & , \xi \in \mathfrak{D}_1^{(12)} \\ 0 & , \text{ otherwise} \end{cases} \tag{3.9}$$

Thus we conclude that the equations (3.2),(3.3) and (3.4) are satisfied on $\mathfrak{D}$ and hence on K. The proof is now complete. ∎

Our next objective is to construct a frame for W_0. To achieve this, we present a result that extends Lemma 2.3 and deals with a multiwavelet frame structure instead of a frame generated by a single element. The proof of this result can be found in [21].

Lemma 3.3. *Let $f_1, f_2, \cdots f_n \in L^2(K)$ and let $L(\xi)$ denote the $n \times n$ matrix*

$$L(\xi) = \left[\sum_{k\in\mathbb{N}_0} \widehat{f}_i(\xi + u(k))\overline{\widehat{f}_j(\xi + u(k))}\right]_{1\leq i,j\leq n}.$$

If $\widetilde{\mathcal{N}}$ denote the set

$$\widetilde{\mathcal{N}} = \left\{\xi \in \mathfrak{D} : \sum_{k\in\mathbb{N}_0} \left|\widehat{f}_i(x + u(k))\right|^2 > 0 \textit{ for some } 1 \leq i \leq n\right\};$$

then the family $\{T_k f_i : k \in \mathbb{N}_0,\ 1 \leq i \leq n\}$ *is a frame for its closed linear span if and only if the following two conditions hold:*

(i) *The largest eigenvalue of the matrix* $L(\xi)$ *is essentially bounded on* $\mathfrak{D}$*.*
(ii) *The smallest nonzero eigenvalue of* $L(\xi)$ *is bounded away from zero on the set* $\widetilde{\mathcal{N}}$*.*

Now, after giving some notations analogous to (2.1), we finally show that the functions ψ_1 and ψ_2 obtained in Theorem 3.2 can now be used to construct a frame for the space W_0.

$$\Psi_1(\xi) = \sum_{k\in\mathbb{N}_0} |\widehat{\psi_1}(\xi + u(k))|^2 \quad \text{and} \quad \Psi_2(\xi) = \sum_{k\in\mathbb{N}_0} |\widehat{\psi_2}(\xi + u(k))|^2.$$

Theorem 3.4. *Consider a local field* K *and suppose* $\phi \in L^2(K)$ *generates an FMRA with dyadic dilation. Then, there always exist two functions* $\psi_1, \psi_2 \in W_0$ *such that the family* $\{T_k\psi_i : k \in \mathbb{N}_0,\ i = 1, 2\}$ *generates a frame for its closed linear span.*

Proof. Theorem 3.2 tells us that there always two functions in W_0 such that

$$W_0 = \overline{\text{span}}\{T_k\psi_i : k \in \mathbb{N}_0,\ i = 1, 2\}.$$

Moreover, these functions ψ_1 and ψ_2 are explicitly given by

$$\widehat{\psi_1}(\mathfrak{p}^{-1}(\xi)) = m_1(\xi)\widehat{\phi}(\xi) \quad \text{and} \quad \widehat{\psi_2}(\mathfrak{p}^{-1}(\xi)) = m_1(\xi)\widehat{\phi}(\xi);$$

where m_1 and m_2 are given by (3.8) and (3.9) respectively. We will now show that the family $\{T_k\psi_i : k \in \mathbb{N}_0,\ i = 1, 2\}$, generated using these two functions ψ_1 and ψ_2 of Theorem 3.2, is a frame for W_0. For this, we intend to use Lemma 3.3, with a slight modification. Keeping in mind the definitions of the functions ψ_1 and ψ_2, we find it easier to work with the matrix $L(\mathfrak{p}^{-1}(\xi))$ instead of $L(\xi)$, and thus, the two conditions in Lemma 3.3 are now to be proved only for the set $\widetilde{\mathfrak{D}} = \mathfrak{p}(\mathfrak{D})$. To simplify our computations, we further partition the set $\widetilde{\mathfrak{D}}$ into four disjoint parts as follows:

$$\begin{aligned}
\widetilde{\mathfrak{D}}^{(0)} &= \{\xi \in \widetilde{\mathfrak{D}} : \Phi(\xi) = 0, \Phi(\xi + \mathfrak{p}u(1)) = 0\} \\
\widetilde{\mathfrak{D}}^{(1)} &= \{\xi \in \widetilde{\mathfrak{D}} : \Phi(\xi) > 0, \Phi(\xi + \mathfrak{p}u(1)) = 0\} \\
\widetilde{\mathfrak{D}}^{(2)} &= \{\xi \in \widetilde{\mathfrak{D}} : \Phi(\xi) = 0, \Phi(\xi + \mathfrak{p}u(1)) > 0\} \\
\widetilde{\mathfrak{D}}^{(12)} &= \{\xi \in \widetilde{\mathfrak{D}} : \Phi(\xi) > 0, \Phi(\xi + \mathfrak{p}u(1)) > 0\}
\end{aligned}$$

Note that this division closely resembles the one made in Theorem 3.2. Furthermore, we have a similar relationship between the subsets $\widetilde{\mathfrak{D}}^{(0)}$, $\widetilde{\mathfrak{D}}^{(1)}$, $\widetilde{\mathfrak{D}}^{(2)}$, and $\widetilde{\mathfrak{D}}^{(12)}$ as we did in Theorem 3.2. With these functions ψ_1 and ψ_2, we observe that the matrix $L(\mathfrak{p}^{-1}(\xi))$ has the following representation:

$$L(\mathfrak{p}^{-1}(\xi)) = \begin{bmatrix} (|m_1|^2\Phi)(\xi) + T_{\mathfrak{p}u(1)}(|m_1|^2\Phi)(\xi) & (m_1\overline{m_2}\Phi)(\xi) + T_{\mathfrak{p}u(1)}(m_1\overline{m_2}\Phi)(\xi) \\ (\overline{m_1}m_2\Phi)(\xi) + T_{\mathfrak{p}u(1)}(\overline{m_1}m_2\Phi)(\xi) & (|m_2|^2\Phi)(\xi) + T_{\mathfrak{p}u(1)}(|m_2|^2\Phi)(\xi) \end{bmatrix}.$$

Let $\theta_+(\xi)$ and $\Theta(\xi)$ respectively denote the smallest nonzero eigenvalue and the largest eigenvalue of the matrix $L(\mathfrak{p}^{-1}(\xi))$.
We note that $L(\mathfrak{p}^{-1}(\xi))$ is a zero matrix when any of the following conditions hold:

- $\circ$ $\xi \in \widetilde{\mathfrak{D}}^{(0)}$;
- $\circ$ $\xi \in \widetilde{\mathfrak{D}}^{(1)}$ and $m_0(\xi) \neq 0$;
- $\circ$ $\xi \in \widetilde{\mathfrak{D}}^{(2)}$ and $m_0(\xi + \mathfrak{p}u(1)) \neq 0$.

We need not prove anything for the cases mentioned above and so we will now investigate the remaining cases one by one.
If $\xi \in \widetilde{\mathfrak{D}}^{(1)}$ and $m_0(\xi) = 0$, then the matrix $L(\mathfrak{p}^{-1}(\xi))$ becomes

$$L(\mathfrak{p}^{-1}(\xi)) = \begin{bmatrix} \Phi(\xi) & 0 \\ 0 & 0 \end{bmatrix}$$

This implies that $\theta_+(\xi) = \Theta(\xi) = \Phi(\xi)$. Furthermore, Lemma 2.3 establishes the relation $A \leq \Phi(\xi) \leq B$. Hence, the assertions of Lemma 3.3 are satisfied in this case.
The case where $\xi \in \widetilde{\mathfrak{D}}^{(2)}$ and $m_0(\xi + \mathfrak{p}u(1)) = 0$ can be addressed similarly.
Now, only the case where $\xi \in \widetilde{\mathfrak{D}}^{(12)}$ remains. We further divide this set into two disjoint parts as follows:

$$\begin{aligned} \widetilde{\mathfrak{D}}_1^{(12)} =& \{\xi \in \widetilde{\mathfrak{D}}^{12} : \Phi(\mathfrak{p}^{-1}(\xi)) = 0\} \\ \widetilde{\mathfrak{D}}_2^{(12)} =& \{\xi \in \widetilde{\mathfrak{D}}^{12} : \Phi(\mathfrak{p}^{-1}(\xi)) > 0\}. \end{aligned}$$

If $\xi \in \widetilde{\mathfrak{D}}_2^{(12)}$, then the matrix $L(\mathfrak{p}^{-1}(\xi))$ has the representation:

$$L(\mathfrak{p}^{-1}(\xi)) = \begin{bmatrix} \Psi_1(\mathfrak{p}^{-1}(\xi)) & 0 \\ 0 & 0 \end{bmatrix};$$

where $\Psi_1(\mathfrak{p}^{-1}(\xi)) = |m_1(\xi)|^2\Phi(\xi) + |m_1(\xi + \mathfrak{p}u(1))|^2\Phi(\xi + \mathfrak{p}u(1))$. It is easy to note $\Psi_1(\mathfrak{p}^{-1}(\xi)) > 0$ and thus we have $\theta_+(\xi) = \Theta(\xi) = \Psi_1(\mathfrak{p}^{-1}(\xi))$. An easy calculation further gives us that

$$\frac{A^4}{B} \leq \Psi_1(\mathfrak{p}^{-1}(\xi)) \leq \frac{B^4}{A};$$

and thus the assertions of the Lemma 3.2 are proved.
Finally, let $\xi \in \widetilde{\mathfrak{D}}_1^{(12)}$. Then the matrix $L(\mathfrak{p}^{-1}(\xi))$ has the representation:

$$L(\mathfrak{p}^{-1}(\xi)) = \begin{bmatrix} \Phi(\xi) + \Phi(\xi + \mathfrak{p}u(1)) & \overline{\Upsilon_{u(1)}(\xi)}\,(\Phi(\xi) - \Phi(\xi + \mathfrak{p}u(1))) \\ \Upsilon_{u(1)}(\xi)\,(\Phi(\xi) - \Phi(\xi + \mathfrak{p}u(1))) & \Phi(\xi) + \Phi(\xi + \mathfrak{p}u(1)) \end{bmatrix}$$

After straightforward calculations, we find that the two eigenvalues corresponding to the matrix above are $2\Phi(\xi)$ and $2\Phi(\xi + \mathfrak{p}u(1))$. Furthermore, we have:

$$2A \leq 2\Phi(\xi) \leq 2B$$

and

$$2A \leq 2\Phi(\xi + \mathfrak{p}u(1)) \leq 2B.$$

Hence, it is evident that the assertions of Lemma 3.2 are also satisfied for this case. By combining all cases, we conclude from Lemma 3.3 that the family of translates $\{T_k\psi_i : k \in \mathbb{N}_0,\ i = 1, 2\}$ forms a frame for W_0. ∎

3.1. **The case of Single Generator.** In the context of locally compact Abelian groups G, it has been noted that for classical MRA with dyadic dilation, an orthonormal basis for $L^2(G)$ can always be formed from dilations and translations of a single function [10]. A similar result holds for Riesz wavelet bases arising from Riesz MRA of dyadic dilations [17]. However, in the case of frames, this may not hold true [16].
Extending this discussion to local fields, similar expectations arise, particularly considering Theorem 3.1, where two non-trivial functions are needed to generate a frame for $L^2(K)$. Notably, the structure of the function ψ_2 from Theorem 3.2 and prior works by J. J. Benedetto, S. Li [5, 4], and R. Kumar, Satyapriya, F. A. Shah [16, 17] suggest that the measure associated with the set

$$\Omega = \{\xi \in \mathfrak{D} : \Phi(\mathfrak{p}^{-1}(\xi)) = 0,\ \Phi(\xi) > 0,\ \Phi(\xi + \mathfrak{p}u(1)) > 0\} \tag{3.10}$$

will play a significant role here. Let's address the case when $\mu_K(\Omega) > 0$. We start by showing in the following lemma that the set W_0 is nontrivial in this scenario. It's noteworthy that this set Ω is equivalent to the set $\mathfrak{D}_1^{(12)}$ defined in the proof of Theorem 3.2.

Lemma 3.5. *Assume that $\phi \in L^2(K)$ generates an FMRA of dydic dilation with two-scale symbol $m_0 \in L^\infty(\mathfrak{D})$. Assume that the set Ω, defined in* (3.10)*, has a positive measure and define functions K-integral periodic functions $P_1, P_2 \in L^2(\mathfrak{D})$ by*

$$P_1(\xi) = \chi_\Omega(\xi) \tag{3.11}$$

$$P_2(\xi) = \chi_{\Omega\cap(\mathfrak{p}\mathfrak{D})}(\xi) - \chi_{\Omega\cap(\mathfrak{p}u(1)+\mathfrak{p}\mathfrak{D})}(\xi) \tag{3.12}$$

Then the functions f_1, f_2 defined by

$$\widehat{f_1}(\mathfrak{p}^{-1}(\xi)) = P_1(\xi)\widehat{\phi}(\xi),$$
$$\widehat{f_2}(\mathfrak{p}^{-1}(\xi)) = P_2(\xi)\widehat{\phi}(\xi);$$

belong to W_0.

Proof. We can express P_1 and P_2 in terms of sequences $\{g_n^1\}_{n\in\mathbb{N}_0}$ and $\{g_n^2\}_{n\in\mathbb{N}_0}$ in $l^2(\mathbb{N}_0)$ as:

$$P_1(\xi) = \sum_{n\in\mathbb{N}_0} g_n^1 \Upsilon_{u(1)}(\xi),$$
$$P_2(\xi) = \sum_{n\in\mathbb{N}_0} g_\lambda^2 \Upsilon_{u(1)}(\xi).$$

For any $t \in K$, f_1 and f_2 can be expressed as:

$$f_1(t) = 2 \sum_{n \in \mathbb{N}_0} g^1_{-n} \phi(\mathfrak{p}^{-1}(t) - u(n)),$$

$$f_2(t) = 2 \sum_{n \in \mathbb{N}_0} g^2_{-n} \phi(\mathfrak{p}^{-1}(t) - u(n)).$$

It's evident that f_1 and f_2 belong to V_1. Moreover, they aren't identically zero on $\mathfrak{D}$ due to the lower frame bound A.

Utilizing Lemma 2.6, we aim to show f_1 and f_2 also belong to W_0. Thus, we need to demonstrate:

$$P_1 \overline{m_0} \Phi + T_{\mathfrak{p}u(1)}(P_1 \overline{m_0} \Phi) = 0,$$

$$P_2 \overline{m_0} \Phi + T_{\mathfrak{p}u(1)}(P_2 \overline{m_0} \Phi) = 0.$$

Since $P_1(\xi) = P_2(\xi) = 0$ when $\xi \notin \Omega$, both equations hold trivially in this case. We only need to verify them on Ω.

For $\xi \in \Omega$, combining the definitions of Ω and Φ, we have $m_0(\xi) = m_0(\xi + \mathfrak{p}u(1)) = 0$. Thus, both equations hold on Ω, completing the proof. ∎

Drawing from the insights provided in the preceding lemma, we now establish a theorem demonstrating that if $\mu_K(\Omega) > 0$, then there cannot exist a function $\psi \in L^2(K)$ capable of generating the space W_0.

Theorem 3.6. *Given that $\phi \in L^2(K)$ generates an FMRA of dyadic dilation with a two-scale symbol $m_0 \in L^\infty(\mathfrak{D})$, if the set Ω, as defined in* (3.10)*, has a measure $\mu_K(\Omega) > 0$, then it is impossible to find a function $\psi \in W_0$ such that the family $\{T_k\psi : k \in \mathbb{N}_0\}$ forms a frame for W_0.*

Proof. Suppose there exists a function $\psi \in W_0$ such that the family $\{T_n\psi\}_{n\in\mathbb{N}_0}$ forms a frame for W_0. Then, we can express f_1 and f_2 as:

$$f_1(t) = \sum_{n \in \mathbb{N}_0} g^1_n \psi(t - u(n)),$$

$$f_2(t) = \sum_{n \in \mathbb{N}_0} g^2_n \psi(t - u(n)); \quad t \in K.$$

For any $\xi \in K$, we have:

$$\widehat{f_1}(\xi) = C_1(\xi)\widehat{\psi}(\xi),$$

$$\widehat{f_2}(\xi) = C_2(\xi)\widehat{\psi}(\xi),$$

where $C_i(\xi) = \sum_{n\in\mathbb{N}_0} g^i_n \Upsilon_{u(n)}(\xi)$, $i = 1, 2$. Clearly, C_1 and C_2 are K-integral periodic functions in $L^2(\mathfrak{D})$.

Since $\psi \in W_0 \subset V_1$ and $\{DT_k\psi : k \in \mathbb{N}_0\}$ is a frame for V_1, we have:

$$\psi(t) = \sum_{n \in \mathbb{N}_0} h_n \psi(\mathfrak{p}^{-1}(t) - u(n)); \quad t \in K,$$

where $\{h_n\}_{n\in\mathbb{N}_0} \in l^2(\mathbb{N}_0)$. Consequently, for any $\xi \in K$, we can write:

$$\widehat{\psi}(\mathfrak{p}^{-1}(\xi)) = F(\xi)\widehat{\phi}(\xi),$$

with $F(\xi) = \frac{1}{2}\sum_{n\in\mathbb{N}_0} h_n \overline{\Upsilon_n(\xi)}$, a K-integral periodic function in $L^2(\mathfrak{D})$.
Substituting these into the expressions for $\widehat{f_1}$ and $\widehat{f_2}$, we get:

$$\widehat{f_1}(\mathfrak{p}^{-1}(\xi)) = C_1(\mathfrak{p}^{-1}(\xi))F(\xi)\widehat{\phi}(\xi),$$
$$\widehat{f_2}(\mathfrak{p}^{-1}(\xi)) = C_2(\mathfrak{p}^{-1}(\xi))F(\xi)\widehat{\phi}(\xi); \quad \xi \in K.$$

Now, for $\xi \in \Omega$, $\Phi(\xi) > 0$, implying there exists an $\ell \in \mathbb{N}_0$ such that $\widehat{\phi}(\xi + u(\ell)) \neq 0$. Thus, for this ℓ:

$$\widehat{f_1}(\mathfrak{p}^{-1}(\xi + u(\ell))) = C_1(\mathfrak{p}^{-1}(\xi))F(\xi)\widehat{\phi}(\xi + u(\ell)),$$
$$\widehat{f_2}(\mathfrak{p}^{-1}(\xi + u(\ell))) = C_2(\mathfrak{p}^{-1}(\xi))F(\xi)\widehat{\phi}(\xi + u(\ell)).$$

Comparing with the definitions of f_1 and f_2, we obtain:

$$C_1(\mathfrak{p}^{-1}(\xi))F(\xi) = P_1(\xi),$$
$$C_2(\mathfrak{p}^{-1}(\xi))F(\xi) = P_2(\xi).$$

For $\xi \in \Omega \cap (\mathfrak{p}\mathfrak{D})$, we have:

$$P_1(\xi) = 1, \quad P_1(\xi + \mathfrak{p}u(1)) = 1,$$
$$P_2(\xi) = 1, \quad P_2(\xi + \mathfrak{p}u(1)) = -1.$$

These values, when substituted back, lead to:

$$0 \neq F(\xi) = F(\xi - \mathfrak{p}u(1)),$$
$$0 \neq F(\xi) = -F(\xi - \mathfrak{p}u(1)).$$

These contradictions show that $\{T_k\psi : k \in \mathbb{N}_0\}$ cannot generate W_0 and hence cannot be a frame for W_0, thus completing the proof of the theorem. ∎

Now it remains to investigate the condition where $\mu_K(\Omega) = 0$. Before that, we state a sufficiency lemma, analogous to Theorem 3.1, which gives us the sufficient conditions under which a family of translates of a single function ψ spans the space W_0. We skip the proof of this lemma to avoid repetitiveness.

Theorem 3.7. *Assume that $\phi \in L^2(K)$ generates an FMRA of dydic dilation and let for some K-integral periodic $m \in L^\infty(\mathfrak{D})$, the function $\psi \in V_1$ be defined by:*

$$\widehat{\psi}(\mathfrak{p}^{-1}(\xi)) = m(\xi)\widehat{\phi}(\xi)$$

If there exist K-integral periodic functions G_0 and $G_1 \in L^\infty(\mathfrak{D})$ such that the equations

$$\left(\overline{m_0} m\Phi\right)(\xi) + \left(\overline{m_0} m\Phi\right)(\xi + \mathfrak{p}u(1)) = 0 \tag{3.13}$$

$$\left(m_0 \Phi G_0\right)(\xi) + \left(m\Phi G_1\right)(\xi) = \Phi(\xi) \tag{3.14}$$

$$(m_0\Phi)(\xi + \mathfrak{p}u(1))G_0(\xi) + (m\Phi)(\xi + \mathfrak{p}u(1))G_1(\xi) = 0 \tag{3.15}$$

are satisfied for a.e. $\xi \in K$, then we have $W_0 = \overline{span}\{T_n\psi_i : n \in \mathbb{N}_0,\ i = 1, 2\}$.

With an approach similar to Theorem 3.2, we can find the function m on $\mathfrak{D}$ and then extend it K-integral periodically to whole of K. With these observations at hand, we now present a theorem which asserts that if $\mu_K(\Omega) = 0$, then one function is enough to generate a frame for W_0.

Theorem 3.8. *If $\phi \in L^2(K)$ generates an FMRA, and Ω is defined as in* (3.10)*, then if Ω has measure zero, there exists a function $\psi \in W_0$ such that the family $\{T_k\psi : k \in \mathbb{N}_0\}$ forms a frame for W_0. Consequently, the family $\{D^jT_k\psi : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ constitutes a frame for $L^2(K)$.*

Proof. Observe that the function ψ_2, obtained in Theorem 3.2, is non zero only when $\xi \in \mathfrak{D}_1^{(12)}$. Also, we observe that $\Omega = \mathfrak{D}_1^{(12)}$. Since, by our assumption, Ω is a null set, therefore, a similar approach, as used in Theorem 3.2, tells us that the functions m, G_0 and G_1 can be chosen as follows:

$$m(\xi) = \begin{cases} (\overline{m_0}\Phi)(\xi + \mathfrak{p}u(1))\ \chi_{u(1)}(\xi) & , \xi \in \mathcal{S}^{(12)} \\ 1 & , \xi \in \mathfrak{D}^{(1)} \text{ and } m_0(\xi) = 0 \\ 0 & , \text{ otherwise} \end{cases} \tag{3.16}$$

$$G_0(\xi) = \begin{cases} \frac{\Phi(\xi)(m\Phi)(\xi + \mathfrak{p}u(1))}{\Delta(\xi)} & , \xi \in \mathfrak{D}^{(12)} \\ \frac{1}{m_0(\xi)} & , \xi \in \mathfrak{D}^{(1)} \text{ and } m_0(\xi) \neq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$G_1(\xi) = \begin{cases} -\frac{\Phi(\xi)(m_0\Phi)(\xi + \mathfrak{p}u(1))}{\Delta(\xi)} & , \xi \in \mathfrak{D}^{(12)} \\ 1 & , \xi \in \mathfrak{D}^{(1)} \text{ and } m_0(\gamma) = 0 \\ 0 & , \text{ otherwise} \end{cases}$$

This concludes the proof. ∎

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