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ON THE BOUNDEDNESS OF THE DISCRETE HILBERT TRANSFORM: AN ELEMENTARY PROOF

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ABSTRACT. In this short note, we present an elementary proof of the boundedness of the discrete Hilbert transform on $\ell_p(\mathbb{Z})$ spaces for $1 < p < \infty$. Our approach relies solely on Hölder's inequality, avoiding more sophisticated tools from harmonic analysis. This offers a simplified and accessible pathway to understanding a classical result in operator theory.

Key words and phrases: Hilbert transform, Lebesgue sequence spaces.

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1. INTRODUCTION

The Hilbert transform is a central object in harmonic analysis and has profound applications across mathematics and engineering, especially in signal processing and the theory of singular integrals. Its discrete analogue, the *discrete Hilbert transform* (DHT), serves as a fundamental example of a singular integral operator acting on sequences, and plays a crucial role in discrete analysis and related fields.

Introduced in early studies on singular operators, the DHT is defined, for a sequence $f = \{f(n)\}_{n\in\mathbb{Z}}$, by

$$Hf(n) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{f(m)}{n-m}.$$

This operator preserves many of the essential features of its continuous counterpart, including its boundedness on classical sequence spaces $\ell_p(\mathbb{Z})$ for $1 < p < \infty$. Such boundedness is a cornerstone result that guarantees stability and regularity in the analysis of sequences and discrete signals.

Standard proofs of the boundedness of the DHT often rely on sophisticated tools such as Fourier analysis, interpolation theorems, or the theory of singular integrals. In contrast, the aim of this note is to offer a completely elementary proof of this fact using only Hölder's inequality and basic properties of ℓ_p spaces.

Our approach is inspired by a pedagogical interest in making classical results in harmonic analysis more accessible to a broader audience, particularly students and researchers with minimal background in advanced functional analysis. This proof illustrates how fundamental inequalities and clever estimations can yield powerful results in operator theory.

We believe that this elementary perspective sheds light on the inner workings of the DHT and may serve as a stepping stone for further explorations in both pure and applied contexts.

We begin recalling the definition of Lebesgue "little" $\ell_p = \ell_p(\mathbb{Z})$ spaces.

Definition 1.1. For $1 \leq p < \infty$ the Lebesgue sequence space (also known as discrete Lebesgue space), is the set of all sequence of real numbers $f = \{f(n)\}_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{Z}} |f(n)|^p < \infty.$$

The set of all such sequences is denoted by $\ell_p(\mathbb{Z})$, which endowed with the norm

$$\|f\|_{\ell_p(\mathbb{Z})} = \|\{f(n)\}_{n\in\mathbb{Z}}\|_{\ell_p(\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}} |f(n)|^p\right)^{1/p},$$

turns out to be a Banach space.

The following result, known as *Hölder inequality*, will be the main tool in proving our main result. We establish it without proof, but the interested reader may find its proof and related topics in [1, Lemma 2.3], [3], [5] and [6, Chapter 9].

Theorem 1.1 (Hölder inequality)**.** *Let p and q be real numbers such that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p(\mathbb{Z})$ and $g \in l_q(\mathbb{Z})$, then*

$$\sum_{n\in\mathbb{Z}} |f(n)g(n)| \leq \left(\sum_{n\in\mathbb{Z}} |f(n)|^p\right)^{\frac{1}{p}} \left(\sum_{n\in\mathbb{Z}} |g(n)|^q\right)^{\frac{1}{q}}.$$

For more details on $\ell_p(\mathbb{Z})$, see [1].

David Hilbert (1862-1943) introduced the Hilbert transform in 1905, which includes the discrete Hilbert transform as a special case, and it is defined as an operator H that acts on the sequence $f = \{f(n)\}_{n\in\mathbb{Z}}$, and it is given by

$$Hf(n) = \sum_{\substack{m\in\mathbb{Z} \\ m\neq n}} \frac{f(m)}{n-m}. \tag{1.1}$$

For more on the Hilbert transform, the reader may check references [2, 4] and [7].

2. MAIN RESULT

Although Hilbert transform and its properties have been widely studied, this is not the case for the DHT. In this regard, it is our goal to present an elementary proof for the boundedness of DHT, which only involves Hölder inequality. The result reads as follows.

Theorem 2.1. *The Hilbert transform* (1.1) *is a bounded linear operator on* $\ell_p(\mathbb{Z})$ *for* $1 < p < \infty$. *Moreover, the following inequality*

$$\|Hf\|_{\ell_p(\mathbb{Z})} \leq 2\|f\|_{\ell_p(\mathbb{Z})}, \tag{2.1}$$

holds, for all $f \in \ell_p(\mathbb{Z})$.

Proof. Let $f = \{f(n)\}_{n\in\mathbb{Z}}$ be a sequence in $\ell_p(\mathbb{Z})$. Then, according to (1.1) and by Hölder inequality, we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |Hf(n)|^p &= \sum_{n=-\infty}^{\infty} \left| \sum_{n\neq m} \frac{f(m)}{n-m} \right|^p \\
&= \sum_{n=-\infty}^{\infty} \left| \sum_{n\neq m} \frac{f(m)}{n-m} \right| \left| \sum_{n\neq m} \frac{f(m)}{n-m} \right|^{p-1} \\
&\leq \sum_{n=-\infty}^{\infty} \sum_{n\neq m} \left| \frac{f(m)}{n-m} \right| \left| \sum_{n\neq m} \frac{f(m)}{n-m} \right|^{p-1} \\
&= \sum_{n\neq m} \frac{1}{|n-m|} \sum_{n=-\infty}^{\infty} |f(m)| \left| \sum_{n\neq m} \frac{f(m)}{n-m} \right|^{p-1} \\
&= \sum_{k=1}^{\infty} \sum_{2^{k-1}\leq|n-m|<2^k} \frac{1}{|n-m|} \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} \left(\sum_{n=-\infty}^{\infty} \left| \sum_{n\neq m} \frac{f(m)}{|n-m|} \right|^p \right)^{\frac{1}{q}} \\
&\leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} \left(\sum_{n=-\infty}^{\infty} |Hf(n)|^p \right)^{\frac{1}{q}}.
\end{aligned}$$

Finally

$$\left(\sum_{n=-\infty}^{\infty} |Hf(n)|^p \right)^{1-\frac{1}{q}} \leq 2 \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}},$$

hence

$$\left(\sum_{n=-\infty}^{\infty} |Hf(n)|^p \right)^{\frac{1}{p}} \leq 2 \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}},$$

which proves (2.1). ∎

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