Aust. J. Math. Anal. Appl.
Vol. **22** (2025), No. 2, Art. 5, 11 pp.
AJMAA

NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF FRACTIONAL INTEGRAL OPERATORS ON HYPERGROUPS

ARI WARDAYANI, MUTIA NUR ESTRI, WIHANTORO, IDHA SIHWANINGRUM

Received 25 November, 2024; accepted 5 August, 2025; published 26 September, 2025.

FACULTY OF MATHEMATICS AND NATURAL SCIENCES, JENDERAL SOEDIRMAN UNIVERSITY, INDONESIA.
<ari.wardayani@unsoed.ac.id
mutia.estri@unsoed.ac.id
wihantoro@unsoed.ac.id
idha.sihwaningrum@unsoed.ac.id>

ABSTRACT. We prove in this paper the necessary and sufficient conditions for the boundedness of the fractional integral operators on Lebesgue spaces over commutative hypergroups. The necessity proofs take into account the Haar measure, meanwhile the sufficiency proofs employ the maximal operators.

Key words and phrases: Commutative Hypergroup; Fractional Integral Operator; Lebesgue Spaces; Maximal Operator.

2010 *Mathematics Subject Classification.* Primary 42B20, 26A33. Secondary 47B38, 47G10.

ISSN (electronic): 1449-5910

The authors thanks to Jenderal Soedirman University for providing the research grant No: 26.756 /UN23.35.5/PT.01/II/2024.

1. INTRODUCTION

Fractional integral operator, I_α (for $0 < \alpha < n$), is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

This operator is also known as the Riesz potential, which is an extension of the solution of Poisson equation. In Physics, the solution of Poisson equation arises for example as a potential field caused by a given electric charge. It is known from the result of Hardy and Littlewood [7] as well as Sobolev [15] that the fractional integral operator is bounded from the Lebesgue spaces (over the Euclidean spaces) L^p to L^q (for $1 < p < q < \infty$) if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Nowadays, we can see an extensive study of such boundedness, for examples in [2], [4], [9], [10], [11], [12], [13], and [14]. In hypergroup version, Hajibayov [6] proved that the fractional integral operator

$$\begin{aligned} R_\alpha f(x) &= \left(\rho(e,r)^{\alpha-n} * f\right)(x) \\ &= \int_K T^x \rho(e,r)^{\alpha-n} f(y^\sim)\, d\mu(y) \\ &= \int_K \rho(e,r)^{\alpha-n} T^x f(y^\sim)\, d\mu(y) \end{aligned}$$

is bounded from Lebesgue spaces over commutative hypergroups $L^p(K,\mu)$ to $L^q(K,\mu)$ when $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (for $1 < p < q < \infty$). The proof of this boundedness emplyos the boundedness of maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{\mu B(e,2r)} \int T^x |f(y^\sim)| d\mu(y)$$

on the spaces under consideration and involves the condition of upper Ahlfors n-reguler by identity

$$\mu(B(e,r)) \le Cr^n \tag{1.1}$$

for some positive constant C which is independent of $r > 0$. The detail explanation of hypergroup commutative can be found in [1] and [8]. In the definition of fractional integral and maximal operators, T^x (for $x \in K$) denotes the generalized translation operator in which

$$T^x f(y) := f(\delta_x * \delta_y) = \int_K f\, d(\delta_x * \delta_y).$$

Every commutative hypergroup possesses a Haar measure (denoted by μ), that is μ satisfies

$$\int_K \int_K f(\delta_x * \delta_y)\, d\mu(y) = \int_K f(y)\, d\mu(y)$$

for every $x \in K$ and every $f \in C_c(K)$ (see [5]). Hence, we have

$$\int_K T^x f(y)\, d\mu(y) = \int_K f(y)\, d\mu(y). \tag{1.2}$$

The result of Hardy-Littlewood-Sobolev provided only the sufficient condition for the boundedness of fractional integral I_α on Lebesgue spaces. The necessary and sufficient conditions for the boundedness of the fractional integral operators I^*_α, defined by

$$I^*_\alpha f(x) := \int_X \frac{f(y)}{\rho(x,y)^{1-\alpha}}\, d\mu(y), \quad x \in \mathbb{X},$$

on Lebesgue spaces over metric measure space $(X,\mu) := (X,\rho,\mu)$ were then provided by [3] in the following theorem.

Theorem 1.1. *Let $1 < p < q < \infty$ and $0 < \alpha < 1$. The operator I^*_α is bounded from $L^p(X,\mu)$ to $L^q(X,\mu)$ if and only if there is a constant $C > 0$ such that*

$$\mu(B(e,r)) \leq Cr^s, \qquad s = \frac{pq(1-\alpha)}{pq+p-q}.$$

Having the result in [6], that is the boundedness of the fractional integral operator R_α in the Lebesgue spaces over commutative hypergroups, in this paper we will extend the results of [3] in hypergroup version.

2. MAIN RESULTS

We state our first result in the following theorem, which is hypergroup version of Theorem 1.1.

Theorem 2.1. *Let $1 < p < q < \infty$, $0 < \alpha < n$, and*

$$s = \frac{pq(n-\alpha)}{pq+p-q}$$

is positive. The operator R_α is a bounded operator from $L^p(K,\mu)$ to $L^q(K,\mu)$ if and only if there is a positive constant C such that

$$\mu(B(e,r)) \leq Cr^s.$$

Proof. (*Necessity*). Since

$$\begin{aligned}
\|\chi_{B(e,r)}\|_{L^p(K,\mu)} &= \left(\int_K |\chi_{B(e,r)}(y)|^p \, d\mu(y)\right)^{\frac{1}{p}} \\
&= \left(\int_{B(e,r)} d\mu(y)\right)^{\frac{1}{p}} \\
&= (\mu(B(e,r)))^{\frac{1}{p}} \\
&< \infty,
\end{aligned}$$

then $\chi_B \in L^p(K,\mu)$. From [6], we know that the fractional integral operator R_α is bounded from $L^p(K,\mu)$ to $L^q(K,\mu)$. As a consequence,

$$\|R_\alpha \chi_{B(e,r)}\|_{L^q(K,\mu)} \leq C\|\chi_{B(e,r)}\|_{L^p(K,\mu)} = \mu(B_{(e,r)})^{1/p}. \tag{2.1}$$

If $\rho(e,r) < r$, then equation (1.2) gives us

$$\begin{aligned}
\frac{\mu(B(e,r))}{r^{n-\alpha}} &= \int_{B(e,r)} \frac{d\mu(y)}{r^{n-\alpha}} \\
&\leq \int_{B(e,r)} \frac{d\mu(y)}{\rho(e,r)^{1-\alpha}} \\
&= \int_K \frac{\chi_{B(e,r)\ d\mu(y)}}{\rho(e,r)^{n-\alpha}} \\
&= \int_K \frac{T^x\chi_{B(e,r)\ d\mu(y)}}{\rho(e,r)^{n-\alpha}} \\
&= CR_\alpha \chi_{B(e,r)}(y)
\end{aligned}$$

or

$$r^{\alpha-n}\mu(B(e,r)) \leq CR_\alpha \chi_{B(e,r)}(y). \tag{2.2}$$

Having equations (2.1) and (2.2), we get

$$\begin{aligned} r^{\alpha-n}\mu(B(e,r))^{1+1/q} &= r^{\alpha-n}\mu(B(e,r))\mu(B(e,r))^{1/q} \\ &= r^{\alpha-n}\mu(B(e,r))\left(\int_{B(e,r)} d\mu(y)\right)^{1/q} \\ &= \left(\int_{B(e,r)} \left(r^{\alpha-n}\mu(B(e,r))\right)^q \; d\mu(y)\right)^{1/q} \\ &\leq C\left(\int_K |R_\alpha \chi_{B(e,r)}(y)|^q \, d\mu(y)\right)^{1/q} \\ &\leq \|R_\alpha \chi_B\|_{L^q(K,\mu)} \\ &\leq C\|\chi_{B(e,r)}\|_{L^p(K,\mu)} \\ &= C\mu(B(e,r))^{1/p}. \end{aligned}$$

As a consequence,

$$\mu(B(e,r))^{1+\frac{1}{q}-\frac{1}{p}} \leq Cr^{n-\alpha}.$$

Since

$$\begin{aligned} \frac{n-\alpha}{1+\frac{1}{q}-\frac{1}{p}} &= \frac{n-\alpha}{(pq+p-q)/pq} \\ &= \frac{(n-\alpha)pq}{pq+p-q} \\ &= s, \end{aligned}$$

we then have

$$\mu(B(e,r)) \leq Cr^{(n-\alpha)/(n+\frac{1}{q}-\frac{1}{p})} = Cr^s.$$

(*Sufficiency*.) To prove the sufficiency of the theorem, we write

$$\begin{aligned} R_\alpha f(x) &= \int_{B(e,r)} \rho(e,r)^{\alpha-n} T^x f(y^\sim) \, d\mu(y) + \int_{K\setminus B(e,r)} \rho(e,r)^{\alpha-n} T^x f(y^\sim) \, d\mu(y) \\ &= U_1 + U_2. \end{aligned}$$

By applying equation (2.1) and the assumption that $s > 0$, then the estimate of U_1 is given by

$$\begin{aligned}
|U_1| &\leq \int_{B(e,r)} \rho(e,r)^{\alpha-n} |T^x f(y^\sim)| d\mu(y) \\
&\leq \sum_{k=-\infty}^{-1} \int_{2^k r \leq \rho(e,r) \leq 2^{k+1} r} \rho(e,r)^{\alpha-n} T^x |f(y^\sim)| \, d\mu(y) \\
&\leq \sum_{k=-\infty}^{-1} \left(2^k r\right)^{\alpha-n} \int_{\rho(e,r) \leq 2^{k+1} r} T^x |f(y^\sim)| d\mu(y) \\
&= \sum_{k=-\infty}^{-1} \left(2^k r\right)^{\alpha-n} \int_{B(e,2^{k+1} r)} T^x |f(y^\sim)| \, d\mu(y) \\
&\leq \sum_{k=-\infty}^{-1} \left(2^k r\right)^{\alpha-n} \mu(B(e, 2^{k+2} r)) Mf(x) \\
&= CMf(x) \sum_{k=-\infty}^{-1} 2^{k(\alpha-n)+(k+2)s} r^{\alpha-n+s} \\
&= CMf(x) r^{\alpha-1+s} \sum_{k=-\infty}^{-1} \left(2^{k(\alpha-n+s)}\right) \\
&= C r^{\alpha-n+s} Mf(x).
\end{aligned}$$

Note that

$$\sum_{k=-\infty}^{-1} \left(2^{k(\alpha-n+s)}\right)$$

is convergence. This is because of

$$s = \frac{pq(n-\alpha)}{pq+p-q} > 0$$

gives us $pq > 0$, $n-\alpha > 0$, and $pq+p-q > 0$; and as $\alpha-n < 0$, $p-q < 0$, and $pq+p-q > 0$, we have

$$\begin{aligned}
\alpha - n + s &= \alpha - n + \frac{pq(n-\alpha)}{pq+p-q} \\
&= \frac{(\alpha-n)(pq+p-q) + pq(n-\alpha)}{pq+p-q} \\
&= \frac{(\alpha-n)(p-q)}{pq+p-q} \\
&> 0.
\end{aligned}$$

By applying Hölder inequality, we get the estimate for U_2, that is

$$\begin{aligned}
|U_2| &\leq \int_{K\setminus B(e,r)} \rho(e,r)^{\alpha-n} |T^x f(y^{\sim})|\, d\mu(y) \\
&\leq \sum_{k=1}^{\infty} \int_{2^k r \leq \rho(e,r) \leq 2^{k+1} r} \rho(e,r)^{\alpha-n} |T^x f(y^{\sim})|\, d\mu(y) \\
&\leq \sum_{k=1}^{\infty} \left(2^k r\right)^{\alpha-n} \int_{B(e,2^{k+1}r)} |T^x f(y^{\sim})|\, d\mu(y) \\
&\leq \sum_{k=0}^{\infty} \left(2^k r\right)^{\alpha-n} \left(\int_{B(e,2^{k+1}r)} |T^x f(y^{\sim})|^p d\mu(y)\right)^{1/p} \left(\int_{B(e,2^{k+1}r)} d\mu(y)\right)^{1/p'} \\
&\leq C \sum_{k=0}^{\infty} \left(2^k r\right)^{\alpha-n} \mu(B(e,2^{k+1}r))^{1/p'} \left(\int_{B(e,2^{k+1}r)} |T^x f(y^{\sim})|^p d\mu(y)\right)^{1/p}.
\end{aligned}$$

Here, $\frac{1}{p'} = 1 - \frac{1}{p}$. Note that

$$\sum_{k=0}^{\infty} \left(2^{k(\alpha-n+s/p')}\right)$$

is convergence since $\alpha - n < 0$, $p > 0$, and $pq + p - q > 0$ providing us with

$$\begin{aligned}
\alpha - n + s/p' &= \alpha - n + \frac{pq(n-\alpha)}{pq+p-q}\left(1 - \frac{1}{p}\right) \\
&= \frac{(\alpha-n)(pq+p-q) + pq(n-\alpha) - q(n-\alpha)}{pq+p-q} \\
&= \frac{-(n-\alpha)pq + (n-\alpha)pq + (n-\alpha)q - (n-\alpha)q + (\alpha-n)p}{pq+p-q} \\
&= \frac{(\alpha-n)p}{pq+p-q} \\
&< 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|U_2| &\leq C\|f\|_{L^p(K,\mu)} \sum_{k=0}^{\infty} \left(2^k r\right)^{\alpha-n} \left(2^{k+1} r\right)^{s/p'} \\
&= C\|f\|_{L^p(K,\mu)} \sum_{k=0}^{\infty} 2^{k(\alpha-n)} 2^{(k+1)s/p'} r^{(\alpha-1)+s/p'} \\
&= C\|f\|_{L^p(K,\mu)} \sum_{k=0}^{\infty} 2^{k\left(\alpha-n+\frac{s}{p'}\right)} r^{k(\alpha-n)+\frac{s}{p'}} \\
&= C r^{(\alpha-n)+\frac{s}{p'}} \|f\|_{L^p(K,\mu)}.
\end{aligned}$$

Now, we will choose

$$r^{\alpha-n+s} Mf(x) = r^{(\alpha-n)+\frac{s}{p'}} \|f\|_{L^p(K,\mu)}$$

such that

$$r^{s-\frac{s}{p'}} = \frac{\|f\|_{L^p(K,\mu)}}{Mf(x)}$$
$$r = \left(\frac{\|f\|_{L^p(K,\mu)}}{Mf(x)}\right)^{\frac{p}{s}}.$$

Hence, the estimates U_1 and U_2 yield

$$\begin{aligned}|R_\alpha f(x)| &\leq C\left(r^{\alpha-n+s}Mf(x) + r^{(\alpha-n)+\frac{s}{p'}}\|f\|_{L^p(K,\mu)}\right)\\ &= C\|f\|_{L^p(K,\mu)}^{\frac{(\alpha-n+s)p}{s}} Mf(x)^{1-\frac{(\alpha-n+s)p}{s}}\\ &= C\|f\|_{L^p(K,\mu)}^{1-\frac{p}{q}} Mf(x)^{\frac{p}{q}}\end{aligned}$$

as we have

$$\begin{aligned}1-\frac{(\alpha-n+s)p}{s} &= 1-\left(\frac{\alpha-n}{s}+1\right)p\\ &= 1-p+p\left(\frac{n-\alpha}{s}\right)\\ &= 1-p+p\left(\frac{n-\alpha}{pq(n-\alpha)/(pq+p-q)}\right)\\ &= 1-p+p\left(\frac{(n-\alpha)(pq+p-q)}{pq(n-\alpha)}\right)\\ &= 1-p+p\left(\frac{pq+p-q}{pq}\right)\\ &= \frac{p}{q}.\end{aligned}$$

From the assumption of the boundedness of maximal operator M on $L^p(K,\mu)$, we find the inequality

$$\begin{aligned}\left(\int_X |R_\alpha f(x)|^q\right)^{\frac{1}{q}} &\leq C\left(\int_X \left(\|f\|_{L^p(K,\mu)}^{1-\frac{p}{q}} Mf(x)^{\frac{p}{q}}\right)^q \, d\mu(y)\right)^{\frac{1}{q}}\\ &= C\|f\|_{L^p(K,\mu)}^{1-\frac{p}{q}}\left(\left(\int_X Mf(x)^p d\mu(y)\right)^{\frac{1}{p}}\right)^{\frac{p}{q}}\\ &\leq C\|f\|_{L^p(K,\mu)}^{1-\frac{p}{q}}\|Mf\|_{L^p(K,\mu)}^{\frac{p}{q}}\\ &\leq C\|f\|_{L^p(K,\mu)}^{1-\frac{p}{q}}\|f\|_{L^p(K,\mu)}^{\frac{p}{q}}\\ &= C\|f\|_{L^p(K,\mu)}.\end{aligned}$$

Hence, the desired inequality

$$\|R_\alpha f\|_{L^q(K,\mu)} \leq C\|f\|_{L^p(K,\mu)}.$$

follows. ∎

Next, we state our second theorem.

Theorem 2.2. *Let $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha}$. The operator R_α is bounded from $L^p(K,\mu)$ to $L^q(K,\mu)$ if and only if*

$$\mu(B(e,r)) \leq Cr^n \tag{2.3}$$

for some positive constant C which is independent of r.

Proof. When $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha}$, we have

$$\begin{aligned} s &= \frac{pq(n-\alpha)}{pq+p-q} \\ &= \frac{p\left(\frac{np}{n-\alpha p}\right)(n-\alpha)}{p\left(\frac{np}{n-\alpha p}\right)+p-\left(\frac{np}{n-\alpha p}\right)} \\ &= \frac{np(n-\alpha)}{np-\alpha p} \\ &= n. \end{aligned}$$

Then, by applying Theorem 2.1, we are done. ∎

Now, we will state our next result.

Theorem 2.3. *Let the commutative hypergroup K is equipped with measures μ and ν. Let $1 < p < q < \infty$ and $1 < \alpha < n$. Suppose there are positive constants C_1 and C_2 such that the measure μ satisfies*

$$C_1 r^n \leq \mu(B(e,r)) \leq C_2 r^n. \tag{2.4}$$

If the operator R_α is a bounded operator from $L^p(K,\mu)$ to $L^q(K,\nu)$, then there is a positive constant C such that

$$\nu(B(e,r)) \leq C\mu(B(e,r))^{q\left(\frac{1}{p}-\frac{\alpha}{n}\right)}.$$

Proof. We have

$$\|\chi_{B(e,r)}\|_{L^p(K,\mu)} = \mu(B(e,r))^{1/p} < \infty,$$

which means that $\chi_B \in L^p(K,\mu)$. Since the operator R_α is a bounded operator from $L^p(K,\mu)$ to $L^q(K,\nu)$, then

$$\|R_\alpha \chi_{B(e,r)}\|_{L^q(K,\nu)} \leq C\|\chi_{B(e,r)}\|_{L^p(K,\mu)} \leq C\mu(B(e,r))^{1/p}. \tag{2.5}$$

If $\rho(e,r) < r$, we have

$$\begin{aligned} \frac{\mu(B(e,r))}{r^{n-\alpha}} &= \int_{B(e,r)} \frac{d\mu(y)}{r^{n-\alpha}} \\ &\leq \int_{B(e,r)} \frac{d\mu(y)}{\rho(e,r)^{1-\alpha}} \\ &= \int_X \frac{\chi_{B(e,r)\ d\mu(y)}}{\rho(e,r)^{n-\alpha}} \\ &= \int_X \frac{T^x\chi_{B(e,r)\ d\mu(y)}}{\rho(e,r)^{n-\alpha}} \\ &= CR_\alpha\chi_{B(e,r)}(x) \end{aligned}$$

or alternatively

$$r^{\alpha-n}\mu(B(e,r)) \leq CR_\alpha\chi_{B(e,r)}(x).$$

The condition in equation (2.4) enable us to get

$$r^\alpha = \frac{r^\alpha r^n}{r^n} \le C\frac{r^\alpha \mu(B(e,r))}{r^n} \le CR_\alpha \chi_{B(e,r)}(y).$$

The last inequality and equation (2.5) then give us

$$\begin{aligned} r^\alpha \nu(B(e,r))^{1/q} &= r^\alpha \left(\int_{B(e,r)} d\nu(y)\right)^{1/q} \\ &= \left(\int_{B(e,r)} |r^\alpha|^q \, d\mu(y)\right)^{1/q} \\ &\le C\left(\int_K |R_\alpha \chi_{B(e,r)}(y)|^q \, d\nu(y)\right)^{1/q} \\ &\le \|R_\alpha \chi_B\|_{L^q(K,\nu)} \\ &\le C\|\chi_{B(e,r)}\|_{L^p(K,\mu)} \\ &= C\mu(B(e,r))^{\frac{1}{p}}. \end{aligned}$$

By applying once more equation (2.4), we get

$$\begin{aligned} \mu(B(e,r))^{\frac{\alpha}{n}} \nu(B(e,r))^{\frac{1}{q}} &= C\,(r^n)^{\frac{\alpha}{n}}\, \nu(B(e,r))^{\frac{1}{q}} \\ &= Cr^\alpha \nu(B(e,r))^{\frac{1}{q}} \\ &= C\mu(B(e,r))^{\frac{1}{p}}. \end{aligned}$$

Therefore,

$$\nu(B(e,r))^{\frac{1}{q}} \le C\mu(B(e,r))^{\frac{1}{p}-\frac{\alpha}{n}}$$

or

$$\nu(B(e,r)) \le C\mu(B(e,r))^{q\left(\frac{1}{p}-\frac{\alpha}{n}\right)}$$

as desired. ∎

The following theorem is a hypergroup version of the result in [3].

Theorem 2.4. *Let $1 < p < q < \infty$. Assume that the measures μ and ν satisfy*

(i) $\mu B(x,r) \le Cr^s$ *for some positive s;*
(ii) $\nu B(x,r) \le Cr^{(1-\alpha-\frac{s}{p'})q}$ *for $p' = \frac{p}{p-1}$, $0 < \alpha < 1 - \frac{s}{p'}$ and $C > 0$ is independent of $x \in X$ and $r > 0$.*

Then, R_α is bounded from $L^p(K,\mu)$ into $L^q(K,\nu)$.

Proof. Firstly, by applying condition (i), we find estimate

$$\begin{aligned} U(r) &= \int_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-1)p'} d\mu(y) \\ &= \sum_{k=0}^{\infty} \int_{2^k r \le \rho(e,y) \le 2^{k+1} r} \rho(e,y)^{(\alpha-1)p'} d\mu(y) \\ &\le \sum_{k=0}^{\infty} (2^k r)^{(\alpha-1)p'} \int_{B(e,2^{k+1}r)} d\mu(y) \\ &\le C' \sum_{k=0}^{\infty} (2^k r)^{(\alpha-1)p'} \mu(B(e,2^{k+1}r)) \\ &\le C r^{(\alpha-1)p'} \sum_{k=0}^{\infty} 2^{k(\alpha-1)p'} (2^{k+1} r)^s \\ &= C r^{(\alpha-1)p'+s} \sum_{k=0}^{\infty} 2^{k[(\alpha-1)p'+s]} \\ &\le C r^{(\alpha-1)p'+s} \end{aligned}$$

Note that $\sum_{k=0}^{\infty} 2^{k(\alpha-1)p'+s}$ is convergent since

$$\begin{aligned} 0 < \alpha < 1 - \frac{s}{p'} \\ p'(\alpha-1)+s < 0. \end{aligned}$$

Furthermore, condition (ii), that is $\nu B(x,r) \le C r^{(1-\alpha-\frac{s}{p'})q}$, gives us

$$\begin{aligned} [\nu B(e,2r)]\,[U(r)]^{\frac{q}{p'}} &\le [\nu B(e,2r)]\, r^{\left[(\alpha-1)p'+\frac{sp'}{p'}\right]\frac{q}{p'}} \\ &\le C r^{\left(1-\alpha-\frac{s}{p'}\right)q} r^{(\alpha-1)q+\frac{sq}{p'}} \\ &= C r^{\left(1-\alpha+\alpha-1-\frac{s}{p'}+\frac{s}{p'}\right)q} \\ &= C. \end{aligned}$$

By taking q_1, where $1 < q_1 < q$, we may also take p_1 in which $p_1 < p$, $1 < p_1 < q_1$ and

$$\left(1-\alpha-\frac{s}{{p_1}'}\right) q_1 = \left(1-\alpha-\frac{s}{p'}\right) q$$

Such p_1 is available for q_1 is close to q. We also could take p_2 and q_2 with $p_2 < q_2, 1 < p < p_2, 1 < q < q_2$ and

$$\left(1-\alpha-\frac{s}{{p_2}'}\right) q_2 = \left(1-\alpha-\frac{s}{p'}\right) q$$

This provide us with

$$\nu\left(\{x \in K : R_\alpha f(x) > \lambda\}\right) \le \frac{C}{\lambda^{-q_1}} \left(\int_K |f(x)|^{p_1} d\mu(x)\right)^{\frac{q_1}{p_1}} \tag{2.6}$$

and

$$\nu\left(\{x \in K : R_\alpha f(x) > \lambda\}\right) \le \frac{C}{\lambda^{-q_2}} \left(\int_K |f(x)|^{p_2} d\mu(x)\right)^{\frac{q_2}{p_1}} \tag{2.7}$$

By interpolating equation (2.6) and (2.7), R_α is bounded from $L^p(K,\mu)$ into $L^p(K,\nu)$. ∎

3. CONCLUDING REMARKS

Theorem 2.3 can be stated alternatively by taking into account the condition of upper Ahlfors n-reguler by identity (1.1).

REFERENCES

[1] W. R. BLOOM and H. HEYER, *Harmonic analysis of probability measures on hypergroups*, de Gruyter, (1995).

[2] F. CHIARENZA and M. FRASCA, Morrey spaces and Hardy-Littlewood maximal function, *Rend. Mat. Appl.* **7**, 273–279 (1987).

[3] D. E. EDMUNDS, V. KOKILASHVILI, and A. MESKHY, *Bounded and compact integral operator*, Springer Science+Business Media, (2002).

[4] J. GARCÍA-CUERVA J and A.E. GATTO, Boundedness properties of fractional integral operators associated to non-doubling measures, *Studia Math.*, **162**, 245–261 (2004).

[5] G. GIGANTE, Transference for hypergroups, *Collect. Math.* **52**(2), 127–155 (2001).

[6] M. G. HAJIBAYOV, Boundedness in Lebesgue spaces of Riesz potentials on commutative hypergroups, *Global Journal of Mathematical Analysis* **3(1)** 18–25 (2015).

[7] G.H. HARDY and J.E. LITTLEWOOD, Some properties of fractional integrals. I, *Math. Zeit.* **27**, 565–606 (1927).

[8] R. I. JEWETT, Spaces with an abstract convolution of measure, *Adv. Math.* **18**, 1–101 (1975).

[9] E. NAKAI, Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces, *Math. Nachr.* **166**, 95–103 (1994).

[10] Y. RAMADANA and H. GUNAWAN, Boundedness of the Hardy-Littlewood maximal operator, fractional integral operators, and Calderon-Zygmund operators on generalized weighted Morrey spaces, *Khayyam J. Math.* **9**(2), 7pp (2024).

[11] D. SALIM, M. T. HAKIKI, Y. SAWANO, D. I. HAKIM, and M. JAMALUDIN, Fractional operators with homogeneous kernel on the Calderon product of Morrey spaces, *Math. Nachr.* **297**(9), 7pp (2024).

[12] Y. SAWANO and S. SUGANO, Complex interpolation and the Adams theorem, *Potential Anal.* **54**(2), 299–305 (2021).

[13] I. SIHWANINGRUM, S. MARYANI and H. GUNAWAN, Extension of Hardy-Littlewood-Sobolev inequalities for Riesz potentials on hypergroups, *Mediterr. J. Math.* **17** pp. 203 (2020).

[14] I. SIHWANINGRUM, RENNY, Y. DASRIL, Some inequalities for the generalized Riesz potential on the generalized Morrey spaces over hypergroups, *Austral. J. Math. Anal. Appl.* **21**(2), Art. 5, 9 pp (2024).

[15] S. L. SOBOLEV, On a theorem in functional analysis (Russian), *Mat. Sob.* **46**, 471–497 (1938), [English translation in *Amer. Math. Soc. Transl. ser. 2,* **34**, 39–68 (1963)].