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ON THE EXISTENCE OF SOLUTIONS FOR FUNCTIONAL INTEGRAL INCLUSIONS

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ABSTRACT. In this paper, we study the sufficient conditions for the existence of compact sets of solutions for a class of functional integral inclusions, that are based on a fixed point theorem of Dhage [8], we also give the existence of integrable solutions by using the nonlinear alternative of Leray-Schauder [9]. An illustrative examples and applications are given in the end of this paper.

Key words and phrases: Fixed point theorems; Functional inclusions; Integrable solutions; Set of solutions.

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1. INTRODUCTION

The functional integral equations of various types play a very important role in numerous mathematical research areas. An interesting feature of functional integral equations is its role in the study of many problems of functional differential Equations see for instance [2], [3] and [13]-[18].

In the functional equation case, there are many articles that discusses the existence of integrable solutions. For example, in [17] and [19] the authors discuss the existence of integrable solutions for functional equations of type

$$x(t) = g(t) + f(t, \int_0^t k(t,s)x(\theta(s))ds)$$

and

$$x(t) = f_1(t, x(\theta_1(t))) + f_2(t, \int_0^{\theta_2(t)} k(t,s)f_3(s, x(\theta_3(s)))ds)$$

In the functional inclusions case, there are many authors discusses the existence of some type of solutions see for instance [1], [11], [12] and [14].

In [5] and [6] the author discuss the existence of solutions of the functional integral inclusion

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s)F(s, x(\eta(s)))ds.$$

He also proved (see [7]) the existence of solutions of the functional integral inclusion

$$x(t) \in p(t) + \int_0^{\varrho(t)} k_1(t,s)F(s, x(\theta(s)))ds + \int_0^{\sigma(t)} k_2(t,s)G(s, x(\eta(s)))ds.$$

In [13] the authors discuss the existence of integrable solution for nonlinear functional integral inclusion of type

$$x(t) \in p(t) + F(t, I^\alpha f(t, x(\varphi(t)))).$$

In the present paper, we investigate the existence of integrable solutions for the functional integral inclusion

$$\begin{aligned} x(t) \in F(t, &\int_0^{\sigma_1(t)} k_1(t,s)f_1(s, x(\theta_1(s)))ds) \\ + G(t, &\int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x(\theta_2(s)))ds),\ t \in [0,1] = I \end{aligned} \tag{1.1}$$

and some properties of the set of solutions, where $F : I \times \mathbb{R} \to P(\mathbb{R})$ and $G : I \times \mathbb{R} \to P(\mathbb{R})$ are multivalued functions and $k_1,\ k_2,\ f_1,\ g_1,\ \theta_1,\ \theta_2,\ \sigma_1, \sigma_2$ are functions satisfying special hypotheses.

As an example we take the following functional inclusion

$$x(t) \in F(t, I^\alpha f_1(t, x(\theta_1(t)))) + G(t, I^\beta g_1(t, x(\theta_2(t)))),$$

and as an application we give the existence of solutions for the following nonlocal differential inclusion

$$x'(t) \in F(t, D^\gamma x(\theta_1(t))) + G(t, D^\delta x(\theta_2(t)))$$

$$x(0) + \sum_{i=1}^{n} x(t_i) = c$$

2. PRELIMINARIES

In this section, we present some definitions, notations and theorems which will be used in this paper (see [8], [10], [16], [22] and [23]).
Let $L^1(I) = L^1(I, \Re)$ be the Banach space of all Lebesgue integrable function on I.
Let $P_{cp}(Y)$ be the family of nonempty compact subsets of Y.
Let $P_{cl,cv}(Y)$ be the family of nonempty closed and convex subsets of Y.
Let $P_{cl,bd}(Y)$ be the family of nonempty closed and bounded subsets of Y.
Let $P_{cp,cv}(Y)$ be the family of nonempty compact and convex subsets of Y.
Let (X, d) be a metric space and let $A \subseteq X,\ x \in X$ and $d(x, A) = \inf\{d(a, x);\ a \in A\}$.

Definition 2.1. [4, 16] For any $A, B \in P_{cl,bd}(X)$, the Hausdorff distance is defined by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\right\}.$$

Now we give the definitions of Carathéodory-Lipschitz multivalued maps and single-valued maps.

Definition 2.2. [4, 16] A multivalued mapping $F : I\times\mathbb{R} \to P_{cl,bd}(\mathbb{R})$ is Carathéodory-Lipschitz if the following conditions hold

(a) there exists $L > 0$ such that

$$H_d(F(t, x), F(t, y)) \leq L|x - y|$$

for each $t \in I$ and all $x, y \in \Re$,

(b) $t \mapsto F(t, x)$ is measurable for all $x \in \mathbb{R}$.

Definition 2.3. [4, 16] A single-valued mapping $f : I \times \mathbb{R} \to \mathbb{R}$ is Carathéodory-Lipschitz if the following conditions hold

(a) there exists $L > 0$ such that

$$|f(t, x) - f(t, y)| \leq L|x - y|$$

for each $t \in I$ and all $x, y \in \mathbb{R}$,

(b) $t \mapsto f(t, x)$ is measurable for all $x \in \mathbb{R}$.

Let E be a Banach Space.

Definition 2.4. [8] A continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, where $\psi(0) = 0$ is called $D-$function.

Definition 2.5. [8] A multivalued function $Q : E \to P_{cl,cv}(E)$ is said to be nonlinear $D-$contraction if there is a $D-$function ψ such that

$$H_d(Q(x), Q(y)) \leq \psi(d(x, y))$$

for all $x,\ y\ \in E$, where $\psi(r) < r$.

Remark 2.1. A simple example of $D-$function is $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\psi(r) = kr,\ k > 0,$$

and a simple example of nonlinear $D-$contraction is a contraction multivalued function.

Also, we give the auxiliary theorems that we need in the sequel.

Theorem 2.1. [4, 16] *Let $(I, \beta(I))$ be a measurable space, $X \subseteq \Re$ an interval, (Y, d) a Polish space (i.e., complete and separable), and*

$$F : I \times X \to P_{cp}(Y)$$

be a Carathéodory-Lipschitz.
Then F admits a Carathéodory-Lipschitz selection $f : I \times X \to Y$ (Where the Lipschitz constant of f is less or equal then the Lipschitz constant of F).

Theorem 2.2. [20] *Let (T, τ, μ) be a complete finite measure space, Y be a separable Banach space and Z be a complete, separable, metric space. Let*

$$F : T \times Z \to P(Y)$$

be a closed, convex (Possibly empty-)valued correspondence such that

(i) *$F(.,.)$ is measurable with respect to the $\sigma-$algebra $\tau \otimes B(Z)$,*
(ii) *for each $t \in T$, $F(t, .)$ is lower semicontinuous.*

Then there exists a Carathéodory-type selection for F. Moreover this selection is jointly measurable.

Later, we also shall use the theorems cited below,
For an arbitrary function $x \in L^1$ let us put

$$(K_i x)(t) = \int_0^t k_i(t, s)x(s)ds,\ t \in \Re_+,\ i = 1, 2.$$

The operators K_i defined in such a way is the well known linear Volterra integral operators.

Theorem 2.3. [21, 24] *If the Volterra integral operators K_i transform the space L^1 into itself then they are continuous.*

Theorem 2.4. [8] *Let X be a closed, convex and bounded subset of a Banach space E and let $S, T : X \to P_{cp,cv}(E)$ be two multivalued functions such that*

- *(a) S is a nonlinear $D-$contraction,*
- *(b) T is compact and closed,*
- *(c) $Sx + Tx \subset X$ for all $x \in X$.*

Then the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in E.

Theorem 2.5. [10] *(Kolmogorov compactness criterion) Let $\Omega \subseteq L^p(I, \Re),\ 1 \le p < \infty$, if*

- *(i) Ω is bounded in $L^p(I, \mathbb{R})$,*
- *(ii) $x_h \to x$ as $h \to 0$, uniformly with respect to $x \in \Omega$,*

then Ω is relatively compact in $L^p(I, \mathbb{R})$, where

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)ds.$$

Theorem 2.6. [9]*(nonlinear alternative of Leray-Schauder type)*
Let U be an open subset of convex set D in a Banach space X. Assume $0 \in U$ and let $H : \overline{U} \to D$ be a compact and continuous operator, then either

- *(a1) H has a fixed point in $\overline{U}$, or*
- *(a2) there exists $\gamma \in (0, 1)$ and $x \in \partial U$ such that $x = \gamma Hx$.*

3. THE MAIN RESULTS

In this section, we state and prove our main results.

Theorem 3.1. *Assume that the following conditions hold*

- *(C1) the functions* $K_1,\ K_2$ *map* L^1 *into itself,*
- *(C2) the mltivalued function* $F : I \times \Re \to P_{cp,cv}(\mathbb{R})$ *is Carathéodory-C-Lipschitz,*
- *(C3) the multivalued function* $G : I \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ *is measurable with respect to the* $\sigma-$*algebra* $\tau \otimes B(\mathbb{R})$ *and for each* $t \in I$*,* $G(t, .)$ *is lower semicontinuous,*
- *(C4) there exists a function* $a_G \in L^1(I, \mathbb{R}_+)$ *and* $b_G > 0$ *such that*
$$\|G(t, x)\| = \sup\{|v|,\ v \in G\} \leq a_G(t) + b_G|x|,\ a.e,\ t \in I$$
for all $x \in \mathbb{R}$*.*
- *(C5) the function* f_1 *is measurable in the first variable and* c_1-*Lipschitz in the second variable.*
- *(C6) the function* g_1 *is carathéodory and there exist* $a_{g_1} \in L^1(I, \mathbb{R}_+)$ *and* $b_{g_1} > 0$ *such that*
$$\|g_1(t, x)\| \leq a_{g_1}(t) + b_{g_1}|x|,\ a.e,\ t \in I$$
for all $x \in \mathbb{R}$*.*
- *(C7)* $\sigma_1,\ \sigma_2\ :\ I \to I$ *are continuous functions and* $\theta_1,\ \theta_2 : (0, 1) \to (0, 1)$ *are absolutely continuous and there exists a constants* $M_1,\ M_2 > 0$ *such that* $\theta_1'(t) \geq M_1,\ \theta_2'(t) \geq M_2,\ a.e.\ t \in (0, 1)$*.*
- *(C8) suppose that there exists a real number* $r > 0$ *such that*
$$\frac{\int_0^1 \|F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds)\|_{H_d}dt + \|a_G\| + b_G|K_2|\|a_{g_1}\| + b_G b_{g_1}|K_2|\frac{r}{M_2}}{1 - \frac{Cc_1|K_1|}{M_1}} < r,$$
$$\frac{Cc_1|K_1|}{M_1} < 1.$$

Then, problem 1.1 has integrable solutions and the set of solutions is compact in $L^1(I, \Re)$*.*

Proof. Since $F(t, .)$ is C-Lipchitz, then $\forall a \in F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, x(\theta_1(s)))ds)$,
$$|a| \leq \frac{Cc_1|K_1}{M_1}|\|x\| + \|F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds))\|_{H_d},\ t \in I$$
indeed, we have
$$\begin{aligned}
\|F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, x(\theta_1(s)))ds)\|_{H_d} &= H_d(F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, x(\theta_1(s)))ds), 0) \\
&\leq H_d(F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, x(\theta_1)(s))ds), F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds)) \\
&\quad + H_d(F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds), 0) \\
&\leq H_d(F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, x(\theta_1(s)))ds), F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds)) \\
&\quad + \|F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds)\|_{H_d} \\
&\leq \frac{Cc_1|K_1|}{M_1}\|x\| + \|F(t, \int_0^{\sigma_1(t)} k_1(t, s)f_1(s, 0)ds)\|_{H_d},\ \forall t \in I,
\end{aligned}$$

hence for any $a \in F(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)$,

$$|a| \leq \frac{Cc_1|K_1|}{M_1}\|x\| + \|F(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s,0)ds)\|_{H_d}.$$

Let B_r be the open ball of center 0 and radius r, (i.e.,$B_r = \{x \in L^1;\ \|x\|_1 < r\}$).
Now, we define $A : \overline{B_r} \to P(L^1(I, \mathbb{R}))$ as follows

$$Ax = \{u \in L^1(I);\ u(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)\}$$

where f is Carathéodory-C-Lipschitz selection of F, and $B : B_r \to P(L^1(I, \mathbb{R}))$ by

$$Bx = \{v \in L^1(I);\ v(t) = g(t, \int_0^{\sigma_2(t)} k_2(t,s) g_1(s, x(\theta_2(s)))ds)\}$$

where g is Carathéodory selection of G.
Hence the problem (1) is equivalent to the operator inclusion

$$x(t) \in (Ax)(t) + (Bx)(t).$$

First we prove that A has convex, compact values and is a contraction.
Step 1. Let $u_2,\ u_3 \in Ax,\ x \in B_r$, then there are $f_2,\ f_3 \in F$ such that

$$u_2(t) = f_2(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)$$

and

$$u_3(t) = f_3(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)$$

and let $\lambda \in [0,1]$,

$$\lambda u_2(t) + (1-\lambda)u_3(t) =$$

$$\lambda f_2(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds) + (1-\lambda) f_3(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds),$$

since F has convex values, then

$$\lambda f_2(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds) + (1-\lambda) f_3(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds) \in$$

$$F(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds),$$

therefore $\lambda u_2 + (1-\lambda)u_3 \in Ax$, hence Ax is convex for each $x \in B_r$.
Step 2. Let u_n be a sequence in $Ax,\ x \in B_r$, then there exists a sequence f_n such that

$$u_n(t) = f_n(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)$$

since F has compact values then there exists a subsequence denoted by f_n which is convergent to $f \in F$ (we take $u(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)$), this implies that Ax is compact for each $x \in B_r$.
Step 3. let $x,\ y \in B_r$, and $u_2 \in Ax$, then

$$u_2(t) = f_2(t, \int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\theta_1(s)))ds)$$

for some $f_2 \in F$, since

$$H(F(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,x(\theta_1(s)))ds), F(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,y(\theta_1(s)))ds))$$
$$\leq \frac{Cc_1|K_1|}{M_1}\|x-y\|_1,$$

we obtain that, there exists

$$w(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,y(\theta_1(s)))ds) \in F(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,y(\theta_1(s)))ds)$$

such that

$$\int_0^1 |f_2(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,x(\theta_1(s)))ds) - w(t)|dt \leq \frac{Cc_1|K_1|}{M_1}\|x-y\|_1$$

Thus the multi-valued operator U defined by

$$U(t) = S_{F(t,\int_0^{\sigma_1(t)} k_1(t,s)f_1(s,y(\theta_1(s)))ds)} \cap K(t)$$

where $K(t) = \{w| \int_0^1 |f_2(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,x(\theta_1(s)))ds) - w(t)|dt \leq \frac{Cc_1|K_1|}{M_1}\|x-y\|_1\}$, has nonempty values and is measurable, let u_3 be a measurable selection for U, (which exists by Kuratowski-Ryll-Nardzewski selection theorem),
then $u_3(t) \in F(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,y(\theta_1(s)))ds)$ ($u_3 \in Ay$) and

$$\int_0^1 |u_2(t) - u_3(t)|dt \leq \frac{Cc_1|K_1|}{M_1}\|x-y\|_1$$

hence, we obtain

$$\|u_2 - u_3\|_1 \leq \frac{Cc_1|K_1|}{M_1}\|x-y\|_1$$

From this and the analogous inequality obtained by interchanging the roles of x and y we get that

$$H_d(Ax, Ay) \leq \frac{Cc_1|K_1|}{M_1}\|x-y\|_1$$

for all $x, y \in B_r$, This shows that A is a multi-valued contraction since $\frac{Cc_1|K_1|}{M_1} < 1$.
Now, we show that B is compact, let Ω be bounded set in $L^1(I)$, let $y \in \Omega$, and let $v \in By$, then

$$v(t) = g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s,y(\theta_2(s)))ds)$$

this implies that

$$|v(t)| \leq a_G(t) + b_G|\int_0^{\sigma_2(t)} k_2(t,s)g_1(s,y(\theta_2(s)))ds|$$
$$\leq a_G(t) + b_G|\int_0^{\sigma_2(t)} k_2(t,s)[a_{g_1}(s) + b_{g_1}|y(\theta_2(s))|]ds|$$
$$\leq a_G(t) + b_G[|K_2|\|a_{g_1}\| + b_{g_1}|K_2|\int_0^1 |y(\theta_2(s))|ds]$$

therefore

$$\|v\|_1 \leq \|a_G\| + b_G|K_2|\|a_{g_1}\| + b_G b_{g_1}|K_2|\frac{\|y\|}{M}$$

hence $B\Omega$ is bounded in $L^1(I)$, it remains to show that $v_h \to v$ in $L^1(I)$ as $h \to 0$ uniformly with respect to $v \in B\Omega$, we have the following

$$\begin{aligned}\|v_h - v\|_1 = \int_0^1 |v_h(t) - v(t)|dt &= \int_0^1 |\frac{1}{h}\int_t^{t+h} v(s) - v(t)ds|dt \\ &\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |g(s, \int_0^{\sigma_2(s)} k_2(s,\tau)g_1(\tau, y(\theta_2(\tau)))d\tau) \\ &\quad - g(t, \int_0^{\sigma_2(t)} k_2(t,\eta)g_1(\eta, y(\theta_2(\eta)))d\eta)|dsdt\end{aligned}$$

Now $g \in L^1(I)$ and the function $t \mapsto \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, y(\theta_2(s)))ds$ is in $L^1(I)$, then (cf. [23])

$$\frac{1}{h}\int_t^{t+h} |g(s, \int_0^{\sigma_2(s)} k_2(s,\tau)g_1(\tau, y(\theta_2(\tau)))d\tau) - g(t, \int_0^{\sigma_2(t)} k_2(t,\eta)g_1(\eta, y(\theta_2(\eta)))d\eta)|ds \to 0$$

as $h \to 0$, therefore we deduce that $B\Omega$ is relatively compact.
Now, let $x_n \to x$ in $L^1(I)$, and $v_n \in Bx_n$ such that $v_n \to v$ in $L^1(I)$.
We have $v_n \in Bx_n$, implies that

$$v_n(t) = g_n(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x_n(\theta_2(s)))ds)$$

where $g_n(.,.) \in G(.,.)$ since g_n and g_1 are continuous in the second variable and from the compactness of values of G and by the Lebesgue dominated convergence theorem we have

$$\lim_{n\to+\infty} v_n(t) = g_n(t, \int_0^{\sigma_2(t)} \lim_{n\to+\infty} k_2(t,s)g_1(s, x_n(\theta_2)(s))ds)$$

this implies that

$$v(t) = g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x(\theta_2(s)))ds) \in (Bx)(t)$$

where g is limit of g_n, hence B is closed.
Finally let $y \in Ax + Bx$, $x \in B_r$, then

$$|y(t)| \leq \frac{Cc_1K_1}{M_1}\|x\|_1 + \|F(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,0)ds)\|_{H_d}$$

$$+\|a_G\| + b_G|K_2|\|a_{g_1}\| + b_Gb_{g_1}|K_2|\frac{\|x\|}{M}$$

$$\leq \frac{Cc_1K_1}{M}r + \|F(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,0)ds)\|_{H_d} + \|a_G\| + b_G|K_2|\|a_{g_1}\| + b_Gb_{g_1}|K_2|\frac{r}{M} < r$$

$$\|y\|_1 < r$$

this implies that $Ax + Bx \subset B_r$ for all $x \in B_r$, this all conditions of Theorem 2.3. are satisfied then, problem 1.1 has integrable solutions and the set of solutions is compact in $L^1(I)$.

Example 3.1. *Let the following functional inclusion*

$$x(t) \in F(t, I^\alpha \frac{1}{6}x(t)) + G(t, I^\beta x(t)),\ t \in I = [0,1]$$

where

$$F : I \times S \longrightarrow P_{cl,cv}(\Re),$$

$S = [-1,1],\ F(t,x) = (t^2 + x^2)S$ *hence* F *is* $2-$*Lipschitz in the second variable we have* $(H_d(F(t,x), F(t,y)) \leq 2(\|x-y\|))$ *and is measurable in the first variable. For*

$$G : I \times \mathbb{R} \longrightarrow P(\mathbb{R}),$$

defined by

$$G(t,x) = \begin{cases} \{0\}\ if\ x = 0, \\ [0,1]\ otherwise, \end{cases}$$

G *is* $\ell \otimes B(\mathbb{R})$ *measurable and lower semicontinuous in the second variable, now we can apply the preceding theorem to this example,*

to see this it is sufficient to take

(1) $\sigma_1(t) = \sigma_2(t) = \theta_1(t) = \theta_2(t) = t,\ M_1 = M_2 = 1$,
(2) $|K_1| = \frac{1}{\Gamma(\alpha+1)},\ |K_2| = \frac{1}{\Gamma(\beta+1)}$,
(3) $f_1(t,x) = \frac{1}{6}x$ hence f_1 is measurable in the first variable and $\frac{1}{6}-$Lipschitz in the second variable,
(4) $a_G = 1,\ b_G = Const > 0,\ a_{g_1} = 0,\ b_{g_1} = 1$,

and hence we can find $r > 0$ such that the condition $C8$ is satisfied.
Now, we give another existence theorem for the problem 1.1, the proof is based on fixed point theorem of Leray Schauder.

Theorem 3.2. *Assume that the following conditions hold*

- *(C1) the functions* $K_1,\ K_2$ *maps* L^1 *into itself,*
- *(C2) the mltivalued function* $F : I \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ *is measurable with respect to the* $\sigma-$*algebra* $\tau \otimes B(\mathbb{R})$ *and for each* $t \in T$, $F(t,.)$ *is lower semicontinuous,*
- *(C3) there exists a function* $a_F \in L^1(I, \mathbb{R}_+)$ *and* $b_F > 0$ *such that*
$$\|F(t,x)\| = \sup\{|v|,\ v \in F\} \leq a_F(t) + b_F|x|,\ a.e,\ t \in I$$
for all $x \in \mathbb{R}$*.*
- *(C4) the multivalued function* $G : I \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ *is measurable with respect to the* $\sigma-$*algebra* $\tau \otimes B(\mathbb{R})$ *and for each* $t \in T$, $G(t,.)$ *is lower semicontinuous,*
- *(C5) there exists a function* $a_G \in L^1(I, \mathbb{R}_+)$ *and* $b_G > 0$ *such that*
$$\|G(t,x)\| = \sup\{|v|,\ v \in G\} \leq a_G(t) + b_G|x|,\ a.e,\ t \in I$$
for all $x \in \mathbb{R}$*.*
- *(C6) the function* f_1 *is carathéodory and there exists* $a_{f_1} \in L^1(I, \mathbb{R}_+)$ *and* $b_{f_1} > 0$ *such that*
$$\|f_1(t,x)\| \leq a_{f_1}(t) + b_{f_1}|x|,\ a.e,\ t \in I$$
for all $x \in \mathbb{R}$*.*
- *(C7) the function* g_1 *is carathéodory and there exists* $a_{g_1} \in L^1(I, \mathbb{R}_+)$ *and* $b_{g_1} > 0$ *such that*
$$\|g_1(t,x)\| \leq a_{g_1}(t) + b_{g_1}|x|,\ a.e,\ t \in I$$
for all $x \in \mathbb{R}$*.*
- *(C8)* $\sigma_1,\ \sigma_2\ :\ I \to I$ *are continuous functions and* $\theta_1,\ \theta_2 : (0,1) \to (0,1)$ *are absolutely continuous and there exist constants* $M_1,\ M_2 > 0$ *such that* $\theta'(t)_1 \geq M_1$ *and* $\theta'(t)_2 \geq M_2\ \forall t \in (0,1)$*.*
- *(C8) suppose that there exists a real number* $r > 0$ *such that*
$$\frac{\|a_F\| + b_F|K_1|\|a_{f_1}\| + \|a_G\| + b_G|K_2|\|a_{g_1}\|}{1 - (\frac{b_F b_{f_1}|K_1|}{M_1} + \frac{b_G b_{g_1}|K_2|}{M_2})} < r.$$

Then, problem 1.1 has an integrable solution.

Proof. We have from the conditions $(C2)$ and $(C4)$ there exist Carathéodory selections functions $f,\ g$ of the multivalued mappings $F,\ G$ respectively, hence the functional inclusion 1.1 is transformed into the following functional equation

$$x(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s), x(\theta_1(s))ds) + g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s), x(\theta_2(s))ds) \quad (2)$$

now we define the following operator

$$H : L^1(I,\mathbb{R}) \to L^1(I,\mathbb{R})$$

by

$$(Hx)(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s, x(\theta_1(s)))ds) + g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x(\theta_2(s)))ds)$$

Let $x_n \to x$ in $L^1(I,\mathbb{R})$, then

$$(Hx_n)(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s, x_n(\theta_1(s)))ds) + g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x_n(\theta_2(s)))ds)$$

hence

$$\lim_{n\to+\infty}(Hx_n)(t) = \lim_{n\to+\infty}[f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s, x_n(\theta_1(s)))ds)$$
$$+g(t, \int_0^{\sigma_1(t)} k_2(t,s)g_1(s, x_n(\theta_2(s)))ds)]$$

and by Lebesgue dominated theorem we get

$$\lim_{n\to+\infty}(Hx_n)(t) = f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s, x(\theta_1(s)))ds)$$
$$+g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x(\theta_2(s)))ds) = (Hx)(t)$$

therefore H is continuous.
Now, we will show that H is compact, let Ω be a bounded subset of $L^1(I,\mathbb{R})$ we will prove that $H\Omega$ is bounded in $L^1(I,\mathbb{R})$, to see this let $x \in \Omega$ (i.e., $\exists R > 0$, such that $\|x\| \le R$)

$$|(Hx)(t)| \le |f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s, x(\theta_1(s)))ds)| + |g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s, x(\theta_2(s)))ds)|$$

and from the condition $(C3)$ and $(C5) - (C7)$, we get

$$|(Hx)(t)| \le a_F(t) + b_F|\int_0^{\sigma_1(t)} k_1(t,s)[a_{f_1}(s) + b_{f_1}|x(\theta_1(s))|ds]|+$$
$$a_G(t) + b_G|\int_0^{\sigma_2(t)} k_2(t,s)[a_{g_1}(s) + b_{g_1}|x(\theta_2(s))|ds]|$$

hence

$$|(Hx)(t)| \le a_F(t) + b_F|K_1|[\|a_{f_1}\| + b_{f_1}\frac{\|x\|}{M_1}]| + a_G(t) + b_G|K_2|[\|a_{g_1}\| + b_{g_1}\frac{\|x\|}{M_2}]|$$

therefore

$$\|(Hx)(t)\| \le \|a_F\| + b_F|K_1|[\|a_{f_1}\| + b_{f_1}\frac{R}{M_1}]| + \|a_G\| + b_G|K_2|[\|a_{g_1}\| + b_{g_1}\frac{R}{M_2}]|$$

this implies that $H\Omega$ is bounded in $L^1(I,\mathbb{R})$.
Now we show that $(Hx)_h \to (Hx)$ in $L^1(I,\mathbb{R})$ as $h \to 0$ uniformly with respect to $(Hx) \in \Omega$, we have the following

$$\begin{aligned}
\|(Hx)_h - (Hx)\| = \int_0^1 |(Hx)_h(t) - (Hx)(t)|dt &= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Hx)(\tau)d\tau - (Hx)(t)|dt \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} ((Hx)(\tau) - (Hx)(t))d\tau|dt \\
&\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |f(\tau, \int_0^{\sigma_1(\tau)} k_1(\tau,s)f_1(s,x(\theta_1(s)))ds) \\
&\quad -f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,x(\theta_1(s)))ds)|d\tau dt \\
&\quad + \int_0^1 \frac{1}{h}\int_t^{t+h} |g(\tau, \int_0^{\sigma_2(\tau)} k_2(\tau,s)g_1(s,x(\theta_2(s)))ds) \\
&\quad -g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s,x(\theta_2(s)))ds)|d\tau dt
\end{aligned}$$

since $f,\ f_1,\ g,\ g_1 \in L^1(I,\Re)$, then (cf. [23])

$$\begin{aligned}
&\int_0^1 \frac{1}{h}\int_t^{t+h} |f(\tau, \int_0^{\sigma_1(\tau)} k_1(\tau,s)f_1(s,x(\theta_1(s)))ds) \\
&\quad -f(t, \int_0^{\sigma_1(t)} k_1(t,s)f_1(s,x(\theta_1(s)))ds)|d\tau dt \\
&\quad + \int_0^1 \frac{1}{h}\int_t^{t+h} |g(\tau, \int_0^{\sigma_2(\tau)} k_2(\tau,s)g_1(s,x(\theta_2(s)))ds) \\
&\quad -g(t, \int_0^{\sigma_2(t)} k_2(t,s)g_1(s,x(\theta_2(s)))ds)|d\tau dt \to 0
\end{aligned}$$

as

$$h \to 0$$

therefore we deduce that $H\Omega$ is relatively compact, that is, H is a compact operator.
Set $\overline{U} = \overline{B_r} = \{x \in L^1(I,\Re);\ \|x\| \leq r\}$ and $D = X = L^1(I,\Re)$. Then in the view of assumption $(C8)$ condition $(a2)$ of Theorem 2.6 does not hold, Theorem 2.6 implies that H has a fixed point, which is solution of the problem 1.1, this completes the proof.

Example 3.2. *As an application of theorem we take the following functional inclusion (we take* $k_1(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ k_2(t,s) = \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$ *and* $\sigma_1(t) = \sigma_2(t) = t$*)*

(3.1) $$x(t) \in F(t, I^\alpha f_1(t,x(\theta_1(t)))) + G(t, I^\beta g_1(t,x(\theta_2(t))))$$

which is used to get the solutions of the following differential inclusion

(3.2) $$x'(t) \in F(t, D^\gamma x(\theta_1(t)) + G(t, D^\delta x(\theta_2(t)),\ t \in [0,1]$$

with nonlocal condition

$$x(0) + \sum_{i=1}^{i=n} x(t_i) = c,\ 0 < t_1 < t_2 < ... < x_{t_n} \leq 1,$$

we have the following result

Theorem 3.3. *Assume that the following conditions hold*

- *(1)F is measurable$C-$Lipschitz,*
- *(2)G is measurable with respect to $\sigma-$algebra $\tau \otimes B(\mathbb{R})$ and for each $t \in T$, $G(t,.)$ is lower semicontinuous.,*
- *(3) there exist a function $a_G \in L^1(I, \mathbb{R}_+)$ and $b_G > 0$ such that*

$$\|G(t,x)\| = \sup\{|v|,\ v \in G\} \leq a_G(t) + b_G|x|),\ a.e,\ t \in I$$

- *(4) $\theta_1,\ \theta_2 : (0,1) \to (0,1)$ are absolutely continuous functions, and there exist $M_1,\ M_2 > 0$ such that $\theta_1'(t) > M_1$ and $\theta_2' > M_2$,*
- *(5)$\exists r > 0$ such that*

$$\frac{\int_0^1 \|F(s,0)\|_{H_d} ds + |a_G|_1 + b_G \frac{r}{\Gamma(\alpha+1)M_1}}{1 - \frac{C}{\Gamma(\beta+1)M_2}} < r.$$

Then the problem 3.1 has integrable solutions and the set of solutions is compact in L^1.

To get the solutions of 3.2, we assume,

$$y(t) = x'(t),$$

we get

$$x(t) = x(0) + \int_0^t y(s)ds = x(0) + I^1 y(t)$$

therefore

$$x(\theta(t)) = x(0) + \int_0^{\theta(t)} y(s)ds$$

hence

$$x(\theta(0)) = x(0) + \int_0^{\theta(0)} y(s)ds$$

this implies that

$$x(\theta(t)) = x(\theta(0)) + \int_{\theta(0)}^{\theta(t)} y(s)ds$$

and therefore

$$D^\gamma x(\theta(t)) = I^{1-\gamma}\theta'(t)y(\theta(t))$$

hence the problem (3) is transformed into the functional inclusion problem

$$y(t) \in F(t, I^{1-\gamma}\theta_1'(t)y(\theta_1(t))) + G(t, I^{1-\delta}\theta_2'(t)y(\theta_2(t))),$$

which is discussed earlier in theorem 3.1.

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