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NEW JACOBI ELLIPTIC FUNCTION WAVE SOLUTIONS FOR CONFORMABLE FRACTIONAL BENJAMIN-BONA-MAHONEY-BURGERS EQUATION

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ABSTRACT. In this paper, Jacobi elliptic function expansion method is applied to solve fractional Benjamin-Bona-Mahoney-Burgers equation with conformable derivative and power law nonlinearity. This method is straightforward, concise, effective and can be used for many other nonlinear evolution equations. Numerical solutions are given to illustrate the accuracy and validity of this method.

Key words and phrases: Jacobi elliptic function expansion method; fractional Benjamin-Bona-Mahoney-Burgers equation; conformable derivative.

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1. INTRODUCTION

Benjamin-Bona-Mahoney-Burgers is an important nonlinear partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. BBM-Burgers described unidirectional propagation of long waves in a certain nonlinear dispersive system [11], [12]. The fractional order of the BBM-Burgers equation is used to model the shallow water problems. The fractional BBM-Burger (BBM) equation [9], [12], can be written as :

$$D_t^{\alpha} u(x,t) + a u_x + b_1 u^n u_x + b_2 u^{2n} u_x + c u_{xx} + k u_{xxt} = 0 \tag{1.1}$$

Where a, b_1, b_2, c and k are nonzero constants. Kumar [13] used new fractional homotopy analysis transform method to time fractional BBM-Burgers equation and found the series solution of this equation. Song Fakhari et al. [16] found different type of solutions of the fractional BBM-Burgers equation by using homotopy analysis method. A large amount of papers have been published concerning the solution of the fractional differential equations in nonlinear phenomena.They proposed several methods to solve the fractional differential equations such as the fractional sub-equation method [2], [3], the first integral method [4], [7], the variational iteration method [5], [6] and the (G'/G)-expansion method [8], [12]. Most of these methods studied the exact solution of fractional equations using Jumarie's modified Riemann-Liouville derivative and the chain rule for fractional derivative. However Cheng-shi [15] gave two counterexamples and stated the opinion that Jumarie's modified Riemann-Liouville derivative with the two basic formulae of derivative are incorrect. Xiaohua Liu [10] gave one counterexample to show that formula of the chain rule is incorrect.

The aim of this work is to apply the Jacobi elliptic function expansion method for solving fractional BBM-B equation with conformable derivative and power law nonlinearity.

This study is organized as follows. In section 2, we give the literature pertaining to conformable fractional derivative. In section 3, steps of the Jacobi elliptic function expansion method are presented. In section 4, the method is applied to solve a solution of the space-time fractional Benjamin-Bona-Mahoney-Burgers equation. Finally, a conclusion is given in section 5.

2. BASIC TOOLS

We enlist some definitions and basic results of two forms that are the most popular in the fractional derivative. They are described in the sense of

(1) Riemann-Liouville definition. For
$\mu \in [n-1, n)$, the μ derivative of h is

$$D_b^{\mu} h(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_b^t \frac{h(x)}{(t-x)^{\mu-n+1}} dx$$

and

(2) Caputo definition. For $\mu \in [n-1, n)$, the μ derivative of h is

$$D_b^{\mu} h(t) = \frac{1}{\Gamma(n-\mu)} \int_b^t \frac{h^n(x)}{(t-x)^{\mu-n+1}} dx.$$

2.1. Conformable fractional derivative.

Khalil [14] defined the fractional derivative in the new sense of the following conformable fractional derivative.

For a given function $l : [0, \infty) \rightarrow \Re$, the conformable fractional derivative of l with α order is denoted by

$$C_\alpha l(t) = \frac{l(t+\epsilon t^{1-\alpha})}{\epsilon}$$

for all $t > 0, \alpha \in (0,1)$.

This newly defined fractional derivative is capable of satisfying some well known required properties.

2.2. **Theorem** [14]**.** let $\alpha \in (0,1]$, f, g $\alpha-$ differentiable at $t > 0$ and $a, b, c \in \mathbb{R}$ then

(1) $C_\alpha(af + bg) = aC_\alpha(f) + bC_\alpha(g)$, for all $a, b \in \mathbb{R}$.

(2) $C_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.

(3) $C_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.

(4) $C_\alpha(fg) = fC_\alpha(g) + gC_\alpha(f)$.

(5) $C_\alpha(\frac{f}{g}) = \frac{gC_\alpha(f) - fC_\alpha(g)}{g^2}$.

(6) if in addition, f is differentiable, then
$C_\alpha(f)(t) = t^{1-\alpha}\dfrac{df}{dt}(t)$.

the conformable fractional derivative of some functions are listed as follows:

(1) $C_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}$
(2) $C_\alpha(-x^{-\alpha}) = \frac{\alpha}{x^{2\alpha}}$
(3) $C_\alpha(sinh(bx)) = bx^{1-\alpha}cosh(bx)$
$C_\alpha(cosh(bx)) = bx^{1-\alpha}sinh(bx)$
(4) $C_\alpha(sech(bx)) = -bx^{1-\alpha}sech^2(bx)$
$C_\alpha(coth(bx)) = -bx^{1-\alpha}csch^2(bx)$

Conformable fractional differential operator satisfies some critical fundamental properties like the chain rule, Taylor series expansion and Laplace transform.

2.3. **Jacobi elliptic sine and cosine functions** [17]**.** In this section we introduce in details the derivatives of Jacobi elliptic functions. First, we consider the second order partial deferential equation (PDE)

$$\frac{\partial^2 \Xi}{\partial x \partial t} = \alpha \sin(\Xi). \tag{2.1}$$

Applying the linear transformation n $\mu = k(x - \lambda t)$ to the above PDE leads to the following ordinary differential equation (ODE)

$$\psi'' = \frac{-\alpha}{k^2\lambda} sin(\Xi). \tag{2.2}$$

which is equivalent to

$$[\frac{1}{2}\psi'] = \frac{-\alpha}{k^2\lambda} sin^2 \frac{1}{2}(\Xi) + c. \tag{2.3}$$

By forcing $c = 1$, $\frac{-\alpha}{k^2\lambda} = -m^2$ and $v = \frac{1}{2}(\Xi)$ we write (2.3) as

$$(v')^2 = 1 - m^2 \sin^2 v, \tag{2.4}$$

or

$$v' = \sqrt{1 - m^2 \sin^2 v}, \tag{2.5}$$

Separating the variables in (2.5) leads to the following integral

$$\int \frac{1}{\sqrt{1 - m^2 \sin^2 v}} dv = \int d\mu, \tag{2.6}$$

which is known as the legendre elliptic integral of first kind and $m \in (0, 1)$ is a parameter which is known as the modulus. Now, we define

$$u = u(t) \tag{2.7}$$

$$* = \int_0^{\Xi} \frac{1}{\sqrt{1 - m^2 \sin^2 y}} dy \tag{2.8}$$

$$* = \int_0^{t=siny} \frac{1}{\sqrt{(1 - x^2)(1 - m^2x^2)}} dx, \tag{2.9}$$

and we propose that $u = f(t) \Rightarrow t = f^{-1}(u) = sn(u)$. Where $sn(u)$ is called the Jacobi elliptic sine function. To define the Jacobi elliptic cosine function, we let

$$u(t) = \int_0^{\Xi} \frac{1}{\sqrt{1 - m^2(1 - \cos^2 y)}} dy \tag{2.10}$$

$$= \int_0^{t=\sin y} \frac{1}{\sqrt{(1 - x^2)(1 - m^2x^2)}} dx. \tag{2.11}$$

and we set $\sqrt{1 - t^2} = \cos y$ to get

$$u(t) = \int_0^{\sqrt{1-t^2}=\cos y} \frac{1}{\sqrt{1 - x^2(1 - m^2x^2)}} dx. \tag{2.12}$$

Therefore,
$u = f(\sqrt{1 - t^2}) \Rightarrow \sqrt{1 - t^2} = f^{-1}(u) = cn(u)$ and $cn(u)$ is called the Jacobi elliptic cosine function. Based on the above two definitions, we write the following argument

$$t = sn(u)$$

$$\sqrt{1 - t^2} = cn(u)$$

$$\sqrt{1 - m^2t^2} = dn(u)$$

Thus, we reach the identities, $cn^2(u) = 1 - sn^2(u)$ and $dn^2(u) = 1 - m^2sn^2(u)$. To know more about these functions, we study their derivatives.
We start with $sn(u)$,

$$u = \int \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}$$

(2.13)
$$du = \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}$$

Thus,

(2.14)
$$\frac{dt}{du} = \sqrt{(1-t^2)(1-m^2t^2)}$$

Now we are ready to introduce the following formulas

$$\begin{aligned}\frac{d}{du}(sn(u)) &= \frac{d}{du}(t)\\ &= \sqrt{(1-t^2)(1-m^2t^2)}\\ &= cn(u)dn(u)\end{aligned}$$

$$\begin{aligned}\frac{d}{du}(cn(u)) &= \frac{d}{du}(\sqrt{1-sn^2(u)})\\ &= -sn(u)dn(u)\end{aligned}$$

$$\begin{aligned}\frac{d}{du}(dn(u)) &= \frac{d}{du}(\sqrt{1-m^2sn^2(u)})\\ &= -m^2sn(u)cn(u)\end{aligned}$$

Also, we can observe the last equation,

$$\begin{aligned}\frac{d}{du}(cn(u)) &= -sn(u)cn(u)\\ &= -\sqrt{1-cn^2(u)}\sqrt{1-m^2sn^2(u)}\\ &= -\sqrt{1-cn^2(u)}\sqrt{1-m^2+m^2cn^2(u)}\end{aligned}$$

Therefore,

$$cn(u) = -\int_0^t \frac{dx}{\sqrt{(1-x^2)(1-m^2+m^2x^2)}}.$$

On one hand $cn(u)$, on the other hand this can be obtained by substituting $t = \cos y$ in the elliptic integral

(2.15)
$$\int_0^{\Xi} \frac{dy}{\sqrt{1-m^2\sin^2 y}} = \int_0^{\Xi} \frac{dy}{\sqrt{1-m^2(1-\cos^2 y)}}$$

Equivalently,

(2.16)
$$-\int_0^{t=\cos y} \frac{dx}{(\sqrt{1-x^2)(1-m^2+m^2x^2)}} = f(t)$$

Thus, $u = f(t) \Rightarrow t = f^{-1} = cn(u)$
Also $sn(u) = t = f^{-1}(u)$ where

$$u = u(t) = \int_0^t \frac{dx}{\sqrt{(1-x^2)(1-m^2x^2)}} \tag{2.17}$$

Now, if $m \to 1$, then the integral becomes

$$\int_0^t \frac{dx}{\sqrt{(1-x^2)}} = \tfrac{1}{2}(ln(|t+1|) - ln(|1-t|)) \tag{2.18}$$

$$= \tanh^{-1}(t). \tag{2.19}$$

and leads to:

$$sn(u) = \tanh(u)$$

$$cn(u) = \sqrt{1 - sn^2(u)} = \sqrt{1 - \tanh^2(u)} = sech(u)$$

$$dn(u) = \sqrt{1 - sn^2(u)} = \sqrt{1 - \tanh^2(u)} = sech(u)$$

3. Analysis of the Method

We take into consideration the following general nonlinear FDE of the type

$$R(u(x,t), C_{\alpha t}u(x,t), C_{\alpha}u(x,t), C_{2\alpha t}u(x,t), ...) = 0 \tag{3.1}$$

where $C_\alpha u(x), C_\alpha u(t)$ are conformable fractional derivative of $u = u(x,t)$ [14] in which $C_\alpha u(x) = \frac{D^\alpha u}{Dx^\alpha}$, u is an unknown function, R is a polynomial, $t \geq 0$ and $0 < \alpha \leq 1$.

Step 1: Using the traveling wave transformation:

$$u(x,t) = U(\gamma), \gamma = x - \beta\frac{t^\alpha}{\alpha}. \tag{3.2}$$

where β is a constant to be determined later.
This enables us to use the following changes
$\frac{\partial^\alpha}{\partial t^\alpha}(.) = -\beta\frac{\partial}{\partial\gamma}$, $\frac{\partial}{\partial x}(.) = \frac{\partial}{\partial\gamma}$, $\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}(.) = \beta^2\frac{\partial^2}{\partial\gamma^2}$,....
to allow us convert (3.2) into an ordinary differential equation of integer order in the form

$$S(U, U', U'', U''', ...) = 0 \tag{3.3}$$

where $U' = \frac{dU}{d\gamma}, U'' = \frac{d^2U}{d\gamma^2}, ...$ In order to construct more general periodic and solitary wave solution of (1.1) by employing the Jacobi elliptic function expansion method.

Step 2: Kumar and al [18] suppose that $U(\gamma)$ can be formulated as a finite series of Jacobi elliptic sine and cosine functions. The initial estimate of the solution are given below

$$U(\gamma) = \sum_{j=0}^{n} a_j sn^j(\gamma, m), \tag{3.4}$$

and

$$U(\gamma) = -\sum_{j=0}^{n} \delta_j cn^j(\gamma, m), \tag{3.5}$$

Where $n, a_j(j = 0, 1, ..., n), \delta_j(j = 0, 1, ..., n)$ are constants and m $(0 < m < 1)$ is called a modulus. Differentiating (3.4) and (3.5) with respect to γ , we obtain:

$$\frac{dU}{d\gamma} = \sum_{j=0}^{n} j a_j sn^{j-1}(\gamma, m) cn(\gamma, m) dn(\gamma, m), \tag{3.6}$$

and

$$\frac{dU}{d\gamma} = -\sum_{j=0}^{n} j \delta_j cn^{j-1}(\gamma, m) sn(\gamma, m) dn(\gamma, m), \tag{3.7}$$

where $dn(\gamma, m)$ presents the Jacobi elliptic function of third kind. The value of n is determined by balancing the nonlinear term and the highest derivative.

Step 3: Substituting (3.4) and (3.5) into (3.3) and equating the coefficients of all powers of $sn(\gamma, m), cn(\gamma, m), dn(\gamma, m)$ to zero, we get a system of algebraic equations for a_j and δ_j where $(j = 0, 1, 2, 3, ..., n)$.

Step 4: Solving the equations system in Step 3, and using the solutions of (3.3) we can construct a variety of exact solutions of (3.1).

4. APPLICATION

In this section, we apply the Jacobi elliptic function expansion method to fractional BBM-Burger (1.1), where the fractional derivative are the conformable fractional derivative with order α, we assume

$$U(\gamma) = u(x, t), \gamma = x - \beta \frac{t^\alpha}{\alpha} \tag{4.1}$$

This last converts the BBM-Burgers (1.1) equation with conformable derivative to an ordinary differential equation as follows

$$-\beta U_\gamma + aU_\gamma + b_1 U^n U_\gamma + b_2 U^{2n} U_\gamma + cU_{\gamma\gamma} - \lambda k U_{\gamma\gamma\gamma} = 0 \tag{4.2}$$

By setting the value of n = 1, the above ordinary differential equation takes the form

$$-\beta U_\gamma + aU_\gamma + b_1 U U_\gamma + b_2 U^2 U_\gamma + cU_{\gamma\gamma} - \lambda k U_{\gamma\gamma\gamma} = 0 \tag{4.3}$$

Using the Jacobi sine function and considering the homogeneous balance between u^2 and $u_{\gamma\gamma\gamma}$ we obtain $n = 1$. Therefore, the solution of (4.3) takes the form

$$U(\gamma) = a_0 + a_1 sn(\gamma, m) \tag{4.4}$$

Deriving $U(\gamma)$ we find

$$U_\gamma = a_1 cn(\gamma, m) dn(\gamma, m) \tag{4.5}$$

$$U_{\gamma\gamma} = -a_1 sn(\gamma, m) dn^2(\gamma, m) - a_1 m^2 sn(\gamma, m) cn^2(\gamma, m) \tag{4.6}$$

(4.7)
$$U_{\gamma\gamma\gamma} = -a_1 cn(\gamma, m) dn^3(\gamma, m) + 4a_1 m^2 sn^2(\gamma, m) dn(\gamma, m) cn(\gamma, m) - a_1 m^2 cn^3(\gamma, m) dn(\gamma, m).$$

Substituting the last equations in (4.3) and collecting various powers of $sn(\gamma, m)$, we get

$$sn^0cn^1dn^1 : -\beta a_1 + aa_1 + b_1a_0a_1 + b_2a_0^2a_1 = 0 \tag{4.8}$$

$$sn^1cn^1dn^1 : b_1a_1^2 + 2b_2a_0a_1^2 = 0 \tag{4.9}$$

$$sn^2cn^1dn^1 : -b_2a_1^3 + 4ka_1m^2 = 0 \tag{4.10}$$

By solving the above system of equations, we can determine the values of the coefficients as:

$a_0 = \frac{-b_1 \pm \sqrt{b_1^2 - 4(a-\beta)b_2}}{2b_2}$ and $a_1 = \pm\sqrt{\frac{4km^2}{b_2}}$

$$u(x,t) = U(\gamma) = \frac{-b_1 \pm \sqrt{b_1^2 - 4(a-\beta)b_2}}{2b_2} \pm \sqrt{\frac{4km^2}{b_2}}sn(\gamma, m), \gamma = x - \beta\frac{t^\alpha}{\alpha} \tag{4.11}$$

which is an exact periodic solution of the BBM-B equation with conformable derivative. For $m \to 1$,
$sn(\gamma, m) \to tanh(\gamma)$ and the above exact periodic solution is degenerated into a new form of solutions that can be written as:

$$u(x,t) = U(\gamma) = \frac{-b_1 \pm \sqrt{b_1^2 - 4(a-\beta)b_2}}{2b_2} \pm \sqrt{\frac{4k}{b_2}}tanh(\gamma), \gamma = x - \beta\frac{t^\alpha}{\alpha} \tag{4.12}$$

Fig.1 and Fig.2. show the exact solution of solitary waves, taking $\beta = 1$, $a = -2$, $b_1 = b_2 = 1$, $k = 2$, $c = 1$, and different values of α.

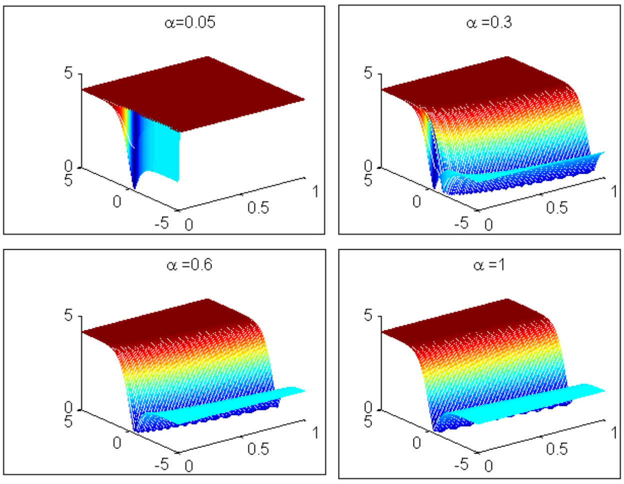

Figure 1: The solitary wave solution of BBM-B if $a_0 = \frac{-b_1 + \sqrt{b_1^2 - 4(a-\beta)b_2}}{2b_2}$

Using the Jacobi cosine function, therefore the solution of (4.3) has the form

$$U(\gamma) = \delta_0 + \delta_1 cn(\gamma, m) \tag{4.13}$$

Deriving $U(\gamma)$ we find

$$U_\gamma = -\delta_1 sn(\gamma, m)dn(\gamma, m) \tag{4.14}$$

$$U_{\gamma\gamma} = -\delta_1 cn(\gamma, m)dn^2(\gamma, m) + \delta_1 m^2 cn(\gamma, m)sn^2(\gamma, m) \tag{4.15}$$

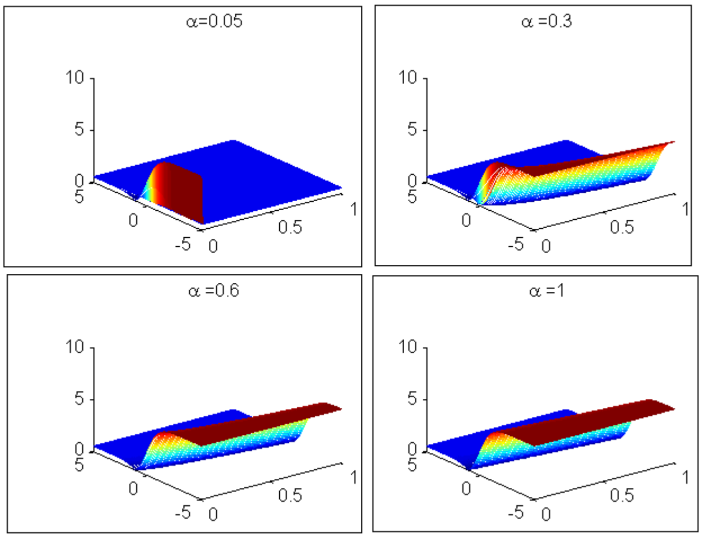

Figure 2: The solitary wave solution of BBM-B if $a_0 = \frac{-b_1 - \sqrt{b_1^2 - 4(a-\beta)b_2}}{2b_2}$

(4.16)
$U_{\gamma\gamma\gamma} = \delta_1 sn(\gamma, m) dn^3(\gamma, m) + 4\delta_1 m^2 cn^2(\gamma, m) dn(\gamma, m) sn(\gamma, m) - \delta_1 m^2 sn^3(\gamma, m) dn(\gamma, m).$

Substituting the last equations in (4.3) and collecting various powers of $sn(\gamma, m)$, we get

$$sn^1 cn^0 dn^1 : -\beta\delta_1 - a\delta_1 - b_1\delta_0\delta_1 - b_2\delta_0^2\delta_1 = 0 \tag{4.17}$$

$$sn^1 cn^1 dn^1 : -b_1\delta_1^2 - 2b_2\delta_0\delta_1^2 = 0 \tag{4.18}$$

$$sn^1 cn^2 dn^1 : -b_2\delta_1^3 + 4k\delta_1 m^2 = 0 \tag{4.19}$$

By solving the above system of equations, we can determine the values of the coefficients as:

$$\delta_0 = \frac{b_1 \pm \sqrt{b_1^2 - 4(\beta - a)b_2}}{-2b_2}$$

and

$$\delta_1 = \pm\sqrt{\frac{4km^2}{b_2}}$$

$$u(x,t) = U(\gamma) = \frac{b_1 \pm \sqrt{b_1^2 - 4(\beta - a)b_2}}{-2b_2} \pm \sqrt{\frac{4km^2}{b_2}} cn(\gamma, m), \gamma = x - \beta\frac{t^\alpha}{\alpha} \tag{4.20}$$

which is an exact periodic solution of the BBM-B equation with conformable derivative. For $m \to 1$,
$cn(\gamma, m) \to sech(\gamma)$ and the above exact periodic solution is degenerated into a new form of solution that can be written as

$$u(x,t) = U(\gamma) = \frac{b_1 \pm \sqrt{b_1^2 - 4(\beta - a)b_2}}{-2b_2} \pm \sqrt{\frac{4km^2}{b_2}} sech(\gamma), \gamma = x - \beta\frac{t^\alpha}{\alpha} \tag{4.21}$$

Fig.3 and Fig.4 show the exact solution of solitary waves taking $\beta = 1$, $a = -2$, $b_1 = b_2 = 1$, $k = 2$ and $c = 1$.

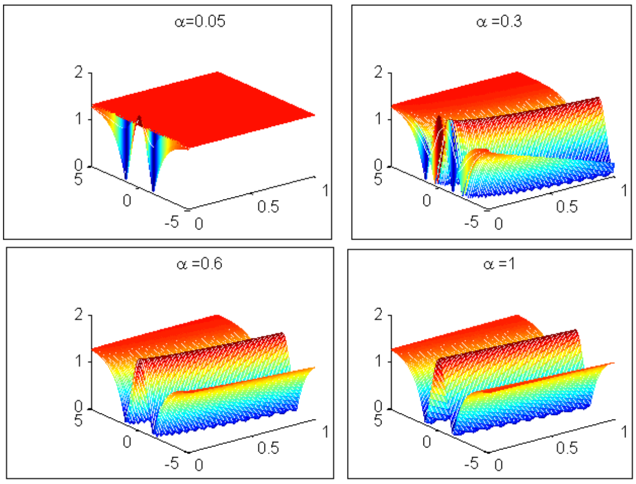

Figure 3: The solitary wave solution of BBM-B if $\delta_0 = \frac{b_1+\sqrt{b_1^2-4(\beta-a)b_2}}{-2b_2}$

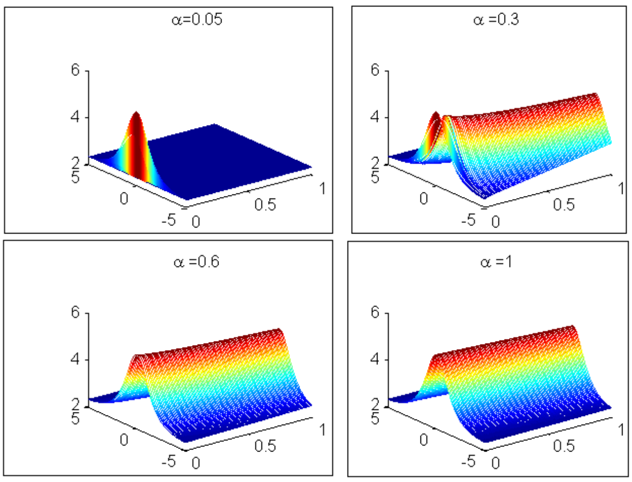

Figure 4: The solitary wave solution of BBM-B if $\delta_0 = \frac{b_1-\sqrt{b_1^2-4(\beta-a)b_2}}{-2b_2}$

5. CONCLUSION

In this paper we found new Jacobi elliptic function wave solutions of conformable fractional Benjamin-Bona-Mahoney-Burgers equation, without use Jumaries modified Riemann-Liouville derivative. The results show that there is a harmony between our solutions and the analytical solutions when $\alpha \to 1$.The method can be applied to solve many non nonlinear fractional problems. This may be done in caming works.

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