Aust. J. Math. Anal. Appl.
Vol. **22** (2025), No. 2, Art. 13, 16 pp.
AJMAA

THE DIFFERENCE BETWEEN TWO APPROXIMATE AND ACCURATE SOLUTIONS OF THE STOCHASTIC DIFFERENTIAL DELAY EQUATION UNDER WEAK CONDITIONS

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Received 29 July, 2025; accepted 10 November, 2025; published 28 November, 2025.

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ABSTRACT. In this paper we investigates the existence of approximate solutions derived from the Caratheodory's and the Euler-Maruyama's scheme under a uniform Lipschitz condition and a weakened linear growth condition. And by analyzing the continuity and convergence of these approximate solutions, we would like to provide reliable results to approximate the unique solutions of stochastic functional differential delay equations. In particular, we investigate how quickly the approximate solution by the Carateodori and Euler-Maruyama approximation methods approaches the accurate solution of the equation.

Key words and phrases: Doob's martingale inequality, Hölder inequality, Moment inequality, Approximate solutions, Stochastic functional differential equation.

2000 *Mathematics Subject Classification.* 60H05, 60H10, 60H20.

ISSN (electronic): 1449-5910

This research was supported by Changwon National University in 2025-2026.

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1. INTRODUCTION

Stochastic Functional Differential Equations(SFDEs) are an advanced class of mathematical models that extend stochastic differential equations by incorporating dependence not only on the current state of a system but also on its past history. This historical dependence allows SFDEs to more accurately represent systems in which memory, delay, or aftereffects significantly influence their evolution. (See the references to this [2]- [8] and [10]-[16]).

SFDEs are particularly useful for modeling phenomena in which dynamics are shaped by both random fluctuations and time-lagged interactions. For instance, in biology, SFDEs are applied to describe gene regulation, neural feedback mechanisms, and population dynamics that depend on earlier states. In engineering and control theory, they are used to model systems with delayed feedback or processing lags under uncertain conditions. SFDEs also appear in economics, where the impact of past decisions or delayed market responses must be taken into account. (See the references to this [10], [11], and [12]).

Based on the results presented in the aforementioned paper, by integrating stochasticity with memory effects, SFDEs provide a powerful and flexible framework for analyzing complex dynamical systems whose future behavior is influenced by both random noise and past states.

Prior to investigating the existence, continuity, and convergence of various types of approximate solutions, we begin by presenting the form of the differential equation under consideration.

The general functional differential equation

$$\dot{x}(t) = f(x_t, t),$$

where $x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$ is the past history of the state. Taking into account the environmental noise we are led to the stochastic functional differential equation

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t). \tag{1.1}$$

The above equation (1.1) has the following initial data;

$$x_{t_0} = \xi = \{\xi(\theta) : -\tau \le \theta \le 0\} \text{ is an } \mathcal{F}_{t_0}\text{-measurable } C([-\tau, 0]; R^d)\text{-valued random variable such that } E||\xi||^2 < \infty \tag{1.2}$$

A special but important class of stochastic functional differential equations is the stochastic functional differential delay equations (SFDDEs)(See [11]). So we can have a deeper discussion of the following delay equations

$$dx(t) = F\left(x(t), x(t-\tau), t\right)dt + G\left(x(t), x(t-\tau), t\right)dB(t) \tag{1.3}$$

on $t \in [t_0, T]$ with initial data (1.2), where $F : R^d \times R^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to R^{d\times m}$. If we define

$$f(\varphi, t) = F(\varphi(0), \varphi(-\tau), t) \text{ and } g(\varphi, t) = G(\varphi(0), \varphi(-\tau), t)$$

for $(\varphi, t) \in C([-\tau, 0]; R^d) \times [t_0, T]$, then the equation (1.3) can be written as equation (1.1), so one can apply the existence-and-uniqueness theorems established to the delay equation (1.3). In other words, if F and G satisfy the local Lipschitz condition and the linear growth condition, that is, for every integer $n \ge 1$, there exists a positive constant K_n such that for all $t \in [t_0, T]$ and all $x, y, \bar{x}, \bar{y} \in R^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le n$,

$$|F(x, y, t) - F(\bar{x}, \bar{y}, t)|^2 \vee |G(x, y, t) - G(\bar{x}, \bar{y}, t)|^2 \le K_n(|x - \bar{x}|^2 + |y - \bar{y}|^2); \tag{1.4}$$

and there is a $K > 0$ such that for all $(x, y, t) \in R^d \times R^d \times [t_0, T]$,

$$|F(x, y, t)|^2 \vee |G(x, y, t)|^2 \le K(1 + |x|^2 + |y|^2), \tag{1.5}$$

then there is a unique solution to the dealy equation (1.3). However, we can take one further step to weaken these conditions slightly. Note that on $[t_0, t_0 + \tau]$, equation (1.3) becomes

$$dx(t) = F(x(t), \xi(t - t_0 - \tau), t)dt + G(x(t), \xi(t - t_0 - \tau), t)dB(t)$$

with initial value $x(t_0) = \xi(0)$. But this is a stochastic differential equation(without delay), and it will have a unique solution if the linear growth condition (1.5) holds and $F(x, y, t), G(x, y, t)$ are locally Lipschitz continuous in x only. Once the solution $x(t)$ on $[t_0, t_0 + \tau]$ is known, we can proceed this argument on interval $[t_0 - \tau, T]$. This argument shows that it is unnecessary to require that the functions $F(x, y, t)$ and $G(x, y, t)$ be locally Lipschitz continuous in y.

On the other hand, despite the establishment of a theorem that guarantees the existence and uniqueness of this equation, in many applications of differential equationsâĂŤparticularly those involving control systems, biological models, or economic processes-the underlying dynamics may not adhere to the smoothness conditions required by classical existence theorems. Traditional methods often assume that the right-hand side function $f(t, x)$ is continuous and satisfies a Lipschitz condition, but real-world systems frequently violate these assumptions due to discontinuities, delays, or state-dependent irregularities.

As a solution to this irregular situation, we use approximate solutions of SFDEs, one of which is Caratheodory's approximate solution(See [5], [11]). Specifically, Caratheodory's approach allows f to be measurable with respect to time and continuous in the state variable, while ensuring that f is integrable over the relevant intervals. An important tool in this setting is the concept of Caratheodory's approximate solution, which involves approximating the original system by a sequence of classical problems with smoother dynamics.

Another widely used approximation method, particularly in the context of stochastic differential equations is the Euler'Maruyama method. Serving as the stochastic counterpart to the classical Euler method, it is designed to approximate solutions of SDEs for which closed-form solutions are rarely available. Instead of continuous-time evolution, the method constructs a discrete-time process that approximates the system's behavior using stepwise updates.

Given a fixed time step Δt, the Euler'Maruyama scheme generates a sequence $\{X_n\}$ according to the iterative rule:

$$X_{n+1} = X_n + f(x_n)\Delta t + g(X_n)\Delta W_n$$

where ΔW_n denotes the increment of a Wiener process over the interval Δt, typically sampled from a normal distribution with mean zero and variance Δt. Under appropriate regularity conditions on the drift term f and diffusion term g, the method converges to the true solution in either the strong or weak sense as $\Delta t \to 0$.

Despite its simplicity, the Euler'Maruyama method effectively captures key aspects of stochastic dynamics and serves as a foundation for more advanced numerical schemes. It is especially valuable for simulating systems involving randomness, memory, or delayâĂŤmaking it particularly relevant in scenarios where both stochastic effects and historical dependence shape the system's evolution.

Stochastic delay differential equations provide a rich mathematical framework for modeling systems in which both randomness and time delay play a significant role. Foundational research in this area often focuses on establishing the existence and uniqueness of solutions under standard conditions such as Lipschitz continuity and linear growth. These results are essential for ensuring that the modeled systems are well-defined and behave in a predictable manner. Another key aspect of SDDEs theory is stability analysis, including mean square stability and almost sure stability. Such analyses typically employ Lyapunov functions adapted to delayed systems(See [3], [5], [14]-[16]). In recent years, numerous examples have emerged in which the Lipschitz condition does not hold; nevertheless, existence and uniqueness of solutions can still be established. To address these cases, continuous approximation methods have been employed(See [2], [6], and [13]-[16]). However, it remains unclear whether solutions in such non-Lipschitz settings can always be approximated continuously.

In this paper, we first investigate the continuity of approximate solutions derived using the approximation method of Carathodeory and Euler-Maruyama, and we derive a convergence theorem where the approximate solutions converge to a unique solution of the equation. These results on convergence and continuity may lead to further insights, such as the stability of solutions and more general existence and uniqueness theorems under weaker conditions. In addition, extending the analysis to nonlinear or discontinuous systems, exploring higher-order approximation methods like the Milstein scheme, and examining systems with random or time-varying coefficients present promising directions for future research. The framework may also be applicable to hybrid or switching systems, as well as to problems in stochastic control and optimization.

2. PRELIMINARY

In this section, we introduce the notation of the stochastic functional differential equations that appear in the next chapter and some inequalities that are necessary to prove the results.

We are working on the given complete probability space $(\Omega, \mathcal{F}, P)$ with the filteration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and $B(t)$ is the given m-dimensional Brownian motion defined on the space. Let $\tau > 0$ and denote by $C([-\tau, 0]; R^d)$ the family of continuous function φ from $[-\tau, 0]$ to R^d with the norm $||\varphi|| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. And let

$$f : C([-\tau, 0]; R^d) \times [t_0, T] \to R^d \text{ and } g : C([-\tau, 0]; R^d) \times [t_0, T] \to R^{d\times m}$$

be both Borel measurable. Furthermore, we will consider the following d-dimensional probability function differential equations.

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \ \text{ on } \ t_0 \leq t \leq T, \tag{2.1}$$

where $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is regarded as a $C([-\tau, 0]; R^d)$-valued stochastic process.

The next definition introduced is the definition of the solution of the above equation (2.1).

Definition 2.1. ([11]) An R^d-valued stochastic process $x(t)$ on $t_0 - \tau \leq t \leq T$ is called a solution of equation (2.1) if it has the following properties:
(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;
(ii) $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; R^{d\times m})$;
(iii) $x_{t_0} = \xi$ and, for every $t_0 \leq t \leq T$,

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x_s, s)ds + \int_{t_0}^{t} g(x_s, s)dB(s) \tag{2.2}$$

A solution $x(t)$ is said to be unique if any other solution $\bar{x}(t)$ is indistinguishable from it, that is

$$P\{x(t) = \bar{x} \ \ for \ \ all \ \ t_0 - \tau \leq t \leq T\} = 1.$$

Some important known inequality theorems are introduced below. These results will be used to identify the main characteristics of the solutions of the equations, which are introduced in the next section.

Lemma 2.1. ([2])*(elementary inequality) if* $p \geq 2$*, then*

$$|a + b|^p \leq 2^{p-1} \left(|a|^p + |b|^p\right)$$

Lemma 2.2. ([11]) *(Gronwall's inequality) Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be Borel measurable bounded nonnegative function on $[0, T]$, and $v(\cdot)$ be nonnegative integrable function on $[0, T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds,\ t \in [0, T],$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right),\ t \in [0, T].$$

Lemma 2.3. ([1, 11]) *(Hölder's inequality) if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p$, and $Y \in L^q$, then*

$$(E|X^T Y|) \leq (E|X|^p)^{\frac{1}{p}} (E|X|^q)^{\frac{1}{q}} \tag{2.3}$$

Lemma 2.4. ([11]) *(Doob's martingale inequalities) Let $\{M_t\}_{t \geq 0}$ be an R^d-valued martingale. Let $[a, b]$ be a bounded interval in R_+.*
(i) If $p \geq 1$ and $M_t \in L^p(\Omega; R^d)$, then

$$P\left\{\omega : \underset{a \leq t \leq b}{sup} |M_t(\omega)| \geq c\right\} \leq \frac{E\,|M_b|^p}{c^p} \tag{2.4}$$

holds for all $c > 0$.
(ii) If $p > 1$ and $M_t \in L^p(\Omega; R^d)$, then

$$E\left(\underset{a \leq t \leq b}{sup} |M_t|^p\right)\left(\frac{p}{p-1}\right)^p E\,|M_b|^p\,.$$

Lemma 2.5. ([11])*(moment inequality) Let $p \geq 2$. Let $g \in \mathcal{M}^2([0, T]; R^{d \times m})$ such that*

$$E \int_0^T |g(s)|^p ds < \infty.$$

Then

$$E\left|\int_0^T g(s)dB(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E \int_0^T |g(s)|^p ds. \tag{2.5}$$

In particular, for $p = 2$, there is equality.

Lemma 2.6. ([11])*(moment inequality) Under the same assumptions as Lemma 2.5, we have*

$$E\left(\sup_{0 \leq t \leq T}\left|\int_0^t g(s)dB(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E \int_0^T |g(s)|^p ds. \tag{2.6}$$

3. MAIN RESULTS

Now, in this section, we want to investigate the properties of two approximate solutions by asymptotic methods based on the existence of solutions and the theory of uniqueness. First, we will show that the uniform Lipschitz condition and the weakened linear growth condition guarantee the boundedness and continuity of the approximate solution.

From here we will consider the following stochastic differential delay equations

$$dx(t) = F(x(t), x(t - \delta(t)), t)dt + G(x(t), x(t - \delta(t)), t)dB(t) \tag{3.1}$$

on $t \in [t_0, T]$ with initial data (1.2), where $\delta : [t_0, T] \to [0, \tau],\ F : R^d \times R^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to R^{d\times m}$ are all Borel measurable.

To discuss the properties of approximate solutions of this equation (3.1), we would like to impose the following two conditions, the uniform Lipschitz condition and the weakened linear growth condition. That is, there exists a $\overline{K} > 0$ such that for all $t \in [t_0, T]$ and all $x, y, \overline{x}, \overline{y} \in R^d$

$$|F(x, y, t) - F(\overline{x}, \overline{y}, t)| \vee |G(x, y, t) - G(\overline{x}, \overline{y}, t)| \leq \overline{K}(|x - \overline{x}|^2 + |y - \overline{y}|^2); \tag{3.2}$$

and there is a $K > 0$ such that for all $(x, y, t) \in R^d \times R^d \times [t_0, T]$,

$$|F(0, 0, t)|^2 \vee |G(0, 0, t)|^2 \leq K. \tag{3.3}$$

Now let's discuss the properties of equation (3.1) by the Caratheodory-method approximation process, which is one of the approximate solutions of the equation by the asymptotic method. When we discussed the Caratheodory approximation for the stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t),$$

the main idea was to replace the present state $x(t)$ with the past $x(t - 1/n)$ to obtain the delay equation

$$dx_n(t) = f(x_n(t - 1/n), t)dt + g(x_n(t - 1/n), t)dB(t)$$

and then showed that the solution $x_n(t)$ of this delay equation approximates the soultion $x(t)$ of the original equation (See [11]).

We define one approximate solution of equation (3.1), the Caratheodory approximation as follows: For each integer $n \geq 2/\tau$, define $x_n(t)$ on $[t_0 - \tau, T]$ by

$$x_n(t_0 + \theta) = \xi(\theta) \text{ for } -\tau \leq \theta \leq 0$$

and

$$\begin{aligned} x_n(t) =&\xi(0) + \int_{t_0}^t I_{D_n^c}(s)F(x_n(s - 1/n), x_n(s - \delta(s)), s)ds \\ &+ \int_{t_0}^t I_{D_n}(s)F(x_n(s - 1/n), x_n(s - \delta(s) - 1/n), s)ds \\ &+ \int_{t_0}^t I_{D_n^c}(s)G(x_n(s - 1/n), x_n(s - \delta(s)), s)dB(s) \\ &+ \int_{t_0}^t I_{D_n}(s)G(x_n(s - 1/n), x_n(s - \delta(s) - 1/n), s)dB(s) \end{aligned} \tag{3.4}$$

for $t_0 \leq t \leq T$, where

$$D_n = \{t \in [t_0, T] : \delta(t) < 1/n\} \text{ and } D_n^c = [t_0, T] - D_n.$$

It is important to note that each $x_n(\cdot)$ can be determined explicitly by the stepwise iterated It$\hat{o}$ integrals over the intervals $[t_0, t_0 + 1], (t_0 + 1/n, t_0 + 2/n]$ etc. Now let's prepare a few lemmas to show one of the main results.

Lemma 3.1. *Let (3.2) and (3.3) hold. Then, for all* $n \geq 2/\tau$

$$E\left(\sup_{t_0-\tau\leq t\leq T} |x_n(t)|^2\right) \leq \left(6E||\xi||^2 + 20C_1K(T - t_0)\right) e^{20C_1\overline{K}(T-t_0)}. \tag{3.5}$$

where $C_1 = T - t_0 + 4$.

Proof. Using the extended elementary inequality, the next inequality can be obtained from the defined Caratheodory approximation (3.1).

$$\begin{aligned}|x_n(t)|^2 \leq & 5\,|\xi(0)|^2 + 5\left|\int_{t_0}^{t} I_{D_n^c}(s)F(x_n(s-1/n),x_n(s-\delta(s)),s)ds\right|^2 \\ & + 5\left|\int_{t_0}^{t} I_{D_n}(s)F(x_n(s-1/n),x_n(s-\delta(s)-1/n),s)ds\right|^2 \\ & + 5\left|\int_{t_0}^{t} I_{D_n^c}(s)G(x_n(s-1/n),x_n(s-\delta(s)),s)dB(s)\right|^2 \\ & + 5\left|\int_{t_0}^{t} I_{D_n}(s)G(x_n(s-1/n),x_n(s-\delta(s)-1/n),s)dB(s)\right|^2.\end{aligned}$$

We can use the Hölder's inequality (2.3), then take the expected value on both sides, and then lead the next inequality through the moment inequality (2.6),

$$\begin{aligned}& E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right) \leq 5E|\xi(0)|^2 \\ & + [5(t-t_0)]E\int_{t_0}^{t} I_{D_n^c}(s)\left|F(x_n(s-1/n),x_n(s-\delta(s)),s)\right|^2 ds \\ & + [5(t-t_0)]E\int_{t_0}^{t} I_{D_n}(s)\left|F(x_n(s-1/n),x_n(s-\delta(s)-1/n),s)\right|^2 ds \\ & + 20E\int_{t_0}^{t} I_{D_n^c}(s)\left|G(x_n(s-1/n),x_n(s-\delta(s)),s)\right|^2 ds \\ & + 20E\int_{t_0}^{t} I_{D_n}(s)\left|G(x_n(s-1/n),x_n(s-\delta(s)-1/n),s)\right|^2 ds.\end{aligned}$$

Estimate the following inequality using the elementary inequality from the previous inequality.

$$\begin{aligned}& E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right) \leq 5E|\xi(0)|^2 \\ & + [10(t-t_0)]E\int_{t_0}^{t} I_{D_n^c}(s)\left|F(x_n(s-1/n),x_n(s-\delta(s)),s)-F(0,0,s)\right|^2 ds \\ & + [10(t-t_0)]E\int_{t_0}^{t} I_{D_n}(s)\left|F(x_n(s-1/n),x_n(s-\delta(s)-1/n),s)-F(0,0,s)\right|^2 ds \\ & + [10(t-t_0)]E\int_{t_0}^{t} I_{D_n^c}(s)\left|F(0,0,s)\right|^2 ds + [10(t-t_0)]E\int_{t_0}^{t} I_{D_n}(s)\left|F(0,0,s)\right|^2 ds \\ & + 40E\int_{t_0}^{t} I_{D_n^c}(s)\left|G(x_n(s-1/n),x_n(s-\delta(s)),s)-G(0,0,s)\right|^2 ds \\ & + 40E\int_{t_0}^{t} I_{D_n}(s)\left|G(x_n(s-1/n),x_n(s-\delta(s)-1/n),s)-G(0,0,s)\right|^2 ds \\ & + 40E\int_{t_0}^{t} I_{D_n^c}(s)\left|G(0,0,s)\right|^2 ds + 40E\int_{t_0}^{t} I_{D_n}(s)\left|G(0,0,s)\right|^2 ds.\end{aligned}$$

Here, using the uniform Lipschitz condition (3.2) and the weakened linear growth condition (3.3), we obtain the following.

$$E\left(\underset{t_0\le s\le t}{sup}|x_n(s)|^2\right)\le \quad 5E|\xi(0)|^2+20C_1K(t-t_0)+20C_1\overline{K}\int_{t_0}^t E\left(\underset{t_0-\tau\le r\le s}{sup}|x_n(r)|^2\right)ds,$$

where $C_1=T-t_0+4$. Therefore, considering the enlarged region of the previous inequality, we can drive it to the next inequality.

$$\begin{aligned}E\left(\underset{t_0-\tau\le s\le t}{sup}|x_n(s)|^2\right)&\le E||\xi||^2+E\left(\underset{t_0\le s\le t}{sup}|x_n(s)|^2\right)\\&\le 6E||\xi||^2+20C_1K(t-t_0)+20C_1\overline{K}\int_{t_0}^t E\left(\underset{t_0-\tau\le r\le s}{sup}|x_n(r)|^2\right)ds.\end{aligned}$$

Applying the Gronwall inequality to the above inequality results in the following inequality

$$E\left(\sup_{t_0-\tau\le s\le t}|x_n(s)|^2\right)\le\left(6E||\xi||^2+20C_1K(t-t_0)\right)e^{20C_1\overline{K}(t-t_0)}$$

and we can immediately see that it is an inequality such as the inequality (3.5) that required proof. ∎

Lemma 3.2. *Let (3.2) and (3.3) hold. Then the solution of equation (3.1) has the property*

$$E\left(\underset{t_0-\tau\le t\le T}{sup}|x(t)|^2\right)\le C_2, \tag{3.6}$$

where $C_2=\left(4E||\xi||^2+6C_1K(T-t_0)\right)e^{12C_1\overline{K}(T-t_0)}$.

Proof. The following inequality can be obtained from the solution of the defined equation (3.1) using the extended elementary inequality.

$$|x(t)|^2\le 3|x(0)|^2+3\left|\int_{t_0}^t F\left(x(s),x(s-\delta(s)),s\right)ds\right|^2+3\left|\int_{t_0}^t G\left(x(s),x(s-\delta(s)),s\right)dB(s)\right|^2.$$

We can use the Hölder's inequality (2.3), then take the expected value on both sides, and then lead the next inequality through the moment inequality (2.6).

$$\begin{aligned}&E\left(\underset{t_0\le s\le t}{sup}|x(s)|^2\right)\\&\le 3E|x(0)|^2+[3(t-t_0)]E\int_{t_0}^t|F(x(s),x(s-\delta(s)),s)-F(0,0,s)+F(0,0,s)|^2ds\\&\quad+12E\int_{t_0}^t|G(x(s),x(s-\delta(s)),s)-G(0,0,s)+G(0,0,s)|^2ds.\end{aligned}$$

Estimate the following inequality using the elementary inequality from the previous inequality

$$\begin{aligned}&E\left(\underset{t_0\le s\le t}{sup}|x(s)|^2\right)\\&\le 3E|x(0)|^2+[6(t-t_0)]E\int_{t_0}^t\left(|F(x(s),x(s-\delta(s)),s)-F(0,0,s)|^2+|F(0,0,s)|^2\right)ds\\&\quad+24E\int_{t_0}^t\left(|G(x(s),x(s-\delta(s)),s)-G(0,0,s)|^2+|G(0,0,s)|^2\right)ds.\end{aligned}$$

Here, using the uniform Lipschitz condition (3.2) and the weakened linear growth condition (3.3), we obtain the following.

$$\begin{aligned}&E\left(\sup_{t_0\le s\le t}|x(s)|^2\right)\\ \le&3E||\xi||^2+6C_1K(t-t_0)+6C_1\overline{K}\int_{t_0}^t\left(E|x(s)|^2+E|x(s-\delta(s))|^2\right)ds\\ \le&3E||\xi||^2+6C_1K(t-t_0)+12C_1\overline{K}\int_{t_0}^t E\left(\sup_{t_0-\tau\le r\le s}|x_n(r)|^2\right)ds.\end{aligned}$$

Therefore, considering the enlarged region of the previous inequality, we can drive it to the next inequality.

$$\begin{aligned}E\left(\sup_{t_0-\tau\le s\le t}|x_n(s)|^2\right)&\le E||\xi||^2+E\left(\sup_{t_0\le s\le t}|x_n(s)|^2\right)\\ &\le 4E||\xi||^2+6C_1K(t-t_0)+12C_1\overline{K}\int_{t_0}^t E\left(\sup_{t_0-\tau\le r\le s}|x_n(r)|^2\right)ds.\end{aligned}$$

Applying the Gronwall inequality to the above inequality results in the following inequality

$$E\left(\sup_{t_0-\tau\le s\le t}|x_n(s)|^2\right)\le\left(4E||\xi||^2+6C_1K(t-t_0)\right)e^{12C_1\overline{K}(t-t_0)},$$

and we can immediately see that it is an inequality such as the inequality (3.6) that required proof. ∎

Lemma 3.3. *Suppose that conditions (3.2) and (3.3) are valid for any* $t_0\le s<t\le T$ *with* $t-s\le 1$*. Then the solution of equation (3.1) has the property*

$$E|x(t)-x(s)|^2\le 4C_3(2C_2\overline{K}+K)(t-s), \tag{3.7}$$

where $C_3=t-s+4$*.*

Proof. Using the elementary inequality, the following inequality can be obtained from the solution of the defined equation (3.1).

$$|x(t)-x(s)|^2\le 2\left|\int_s^t F\left(x(r),x(r-\delta(r)),r\right)dr\right|^2+2\left|\int_s^t G\left(x(r),x(r-\delta(r)),r\right)dB(r)\right|^2.$$

Based on Lemma 3.2, the following inequality can be obtained by taking the expected value of both sides of the previous equation and then using Hölder's inequality (2.3) and the moment inequality (2.6).

$$\begin{aligned}E|x(t)-x(s)|^2\le&[2(t-s)]E\int_s^t\left|F\left(x(r),x(r-\delta(r)),r\right)-F(0,0,r)+F(0,0,r)\right|^2dr\\ &+8E\int_s^t\left|G\left(x(r),x(r-\delta(r)),r\right)-G(0,0,r)+G(0,0,r)\right|^2dr.\end{aligned}$$

Here, using the uniform Lipschitz condition (3.2) and the weakened linear growth condition (3.3), we obtain the following.

$$\begin{aligned}E|x(t)-x(s)|^2\le&4C_3\overline{K}\int_s^t\left[E|x(r)|^2+E|x(r-\delta(r))|^2\right]dr+4C_3K(t-s)\\ \le&4C_3(2C_2\overline{K}+K)(t-s).\end{aligned}$$

We can immediately see that it is an inequality such as the inequality (3.7) that required proof. ∎

The following theorem is one of the main theorems of this section, where the conditions (3.3) and (3.4) are satisfied, the Carteodori approximation allows us to estimate how fast approximate solutions of the stochastic functional differential equation (3.1) approach to the unique solution.

Theorem 3.4. *Let (3.2) and (3.3) hold. Then, between the exact solution and asymptotic solutions, the following holds.*

$$E\left(\sup_{t_0\le t\le T}|x(t)-x_n(t)|^2\right)\le (J_1+J_2)e^{5C_4\overline{K}(T-t_0)}, \tag{3.8}$$

where $C_4=4(t-t_0+4)$ *and* $C_5=4C_3(2C_2\overline{K}+K)$.

Proof. Based on Lemma 3.1 and Lemma 3.2, the difference between the exact solution and asymptotic solutions is calculated using the elementary inequality, and then the expected values are taken on both sides, and the Hölder's inequality (2.3), moment inequality(2.6) are applied sequentially to obtain the following.

$$\begin{aligned}E|x(t)-x_n(t)|^2 \le & C_4\overline{K}\int_{t_0}^t E\left|x(s)-x_n(s-1/n)\right|^2 ds\\ &+C_4\overline{K}\int_{t_0}^t I_{D_n^c}(s)E\left|x(s-\delta(s))-x_n(s-\delta(s))\right|^2 ds\\ &+C_4\overline{K}\int_{t_0}^t I_{D_n}(s)E\left|x(s-\delta(s))-x_n(s-\delta(s)-1/n)\right|^2 ds.\end{aligned}$$

where $C_4=4(t-t_0+4)$, and the Uniform Lipschitz condition (3.2) and weakened linear growth condition (3.3) were used. Here, by adding and subtracting a new term, the elementary inequality is used to obtain the following equation.

$$\begin{aligned}&E|x(t)-x_n(t)|^2\\ \le& 2C_4\overline{K}\int_{t_0}^t E\left|x(s)-x(s-1/n)\right|^2 ds+2C_4\overline{K}\int_{t_0}^t E\left|x(s-1/n)-x_n(s-1/n)\right|^2 ds\\ &+C_4\overline{K}\int_{t_0}^t I_{D_n^c}(s)E\left|x(s-\delta(s))-x_n(s-\delta(s))\right|^2 ds\\ &+2C_4\overline{K}\int_{t_0}^t I_{D_n}(s)E\left|x(s-\delta(s))-x(s-\delta(s)-1/n)\right|^2 ds\\ &+2C_4\overline{K}\int_{t_0}^t I_{D_n}(s)E\left|x(s-\delta(s)-1/n)-x_n(s-\delta(s)-1/n)\right|^2 ds\\ \le& 5C_4\overline{K}\int_{t_0}^t E\left(\sup_{t_0\le r\le s}|x(r)-x_n(r)|^2\right)ds+J_1+J_2,\end{aligned}$$

where

$$J_1=2C_4\overline{K}\int_{t_0}^t E\left|(x(s)-x(s-1/n)\right|^2 ds$$

and

$$J_2=2C_4\overline{K}\int_{t_0}^t I_{D_n}(s)E\left|(x(s-\delta(s))-x(s-\delta(s)-\frac{1}{n})\right|^2 ds.$$

Applying the Gronwall inequality to the previous inequality yields the following inequality

$$E\left(\sup_{t_0\le s\le T}|x(s)-x_n(s)|^2\right)\le (J_1+J_2)e^{5C_4\overline{K}(T-t_0)}. \tag{3.9}$$

On the other hand, using Lemma 3.3, integral J_1 is calculated as follows.

$$\begin{aligned}J_1 \le& 4C_4\overline{K}\int_{t_0}^{t_0+\frac{1}{n}} E\left[|x(s)|^2+|x(s-1/n)|^2\right]ds\\ &+2C_4\overline{K}\int_{t_0+\frac{1}{n}}^{t} E\left|x(s)-x(s-1/n)\right|^2 ds\\ \le&\frac{8C_2C_4}{n}\overline{K}+\frac{2C_4C_5}{n}t\overline{K}=2C_4\overline{K}\left(4C_2+C_5t\right)\frac{1}{n}.\end{aligned}$$

Also, setting $D_0=\{t\in[t_0,T]:\delta(t)=0\}$, the integral J_2 can be written as follows.

$$\begin{aligned}J_2 =&2C_4\overline{K}\int_{t_0}^{t_0+\frac{1}{n}} I_{D_0}(s)E\left|x(s)-x(s-1/n)\right|^2 ds\\ &+2C_4\overline{K}\int_{t_0+\frac{1}{n}}^{t} I_{D_0}(s)E\left|x(s)-x(s-1/n)\right|^2 ds\\ &+2C_4\overline{K}\int_{t_0}^{t_0+\tau+\frac{1}{n}} I_{D_n-D_0}(s)E\left|x(s-\delta(s))-x(s-\delta(s)-\frac{1}{n})\right|^2 ds\\ &+2C_4\overline{K}\int_{t_0+\tau+\frac{1}{n}}^{t} I_{D_n-D_0}(s)E\left|x(s-\delta(s))-x(s-\delta(s)-\frac{1}{n})\right|^2 ds.\end{aligned}$$

Using an elementary inequality and the Lemma 3.3 once again, we can estimate

$$\begin{aligned}J_2 \le& 8C_2C_4\overline{K}\int_{t_0}^{t_0+\frac{1}{n}} I_{D_0}(s)ds+\frac{2C_4C_5}{n}\overline{K}\int_{t_0+\frac{1}{n}}^{t} I_{D_0}(s)ds\\ &+8C_2C_4\overline{K}\int_{t_0}^{t_0+\tau+\frac{1}{n}} I_{D_n-D_0}(s)ds+\frac{2C_4C_5}{n}\overline{K}\int_{t_0+\tau+\frac{1}{n}}^{t} I_{D_n-D_0}(s)ds\\ \le&\frac{16C_2C_4\overline{K}}{n}+\frac{2C_4C_5}{n}\overline{K}\int_{t_0+\tau+\frac{1}{n}}^{t}1ds+\frac{2C_4C_5}{n}\overline{K}\int_{t_0+\frac{1}{n}}^{t_0+\tau+\frac{1}{n}} I_{D_0}(s)ds\\ &+8C_2C_4\overline{K}\mu\left([t_0,t_0+\tau]\left(D_n-D_0\right)\right)\\ \le&\frac{2C_4\left(8C_2+TC_5\right)}{n}\overline{K}+8C_2C_4\overline{K}\mu\left([t_0,t_0+\tau]\left(D_n-D_0\right)\right).\end{aligned}$$

Substituting J_1 and J_2 into (3.9) yields the required result (3.8). The proof is complete. ∎

Let us now turn to the Euler-Maruyama approximate procedure. We first give the definition of the Euler-Maruyama approximation sequence(See [11]). For each integer $n\ge 1$, define $x_n(t)$ on $[t_0-\tau,T]$ as follows:

$$x_n(t_0+\theta)=\xi(\theta)\text{ for }-\tau\le\theta\le 0$$

and

$$x_n(t) = x_n(t_0 + k/n) + \int_{t_0+k/n}^{t} F(x_n(t_0 + k/n), x_n(t_0 + k/n - \delta(s)), s)ds + \int_{t_0+k/n}^{t} G(x_n(t_0 + k/n), x_n(t_0 + k/n - \delta(s)), s)dB(s) \tag{3.10}$$

for $t_0 + k/n < t \leq [t_0 + (k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$. Clearly, $x_n(\cdot)$ can be determined explicitly by the stepwise iterated It$\hat{o}$ intergrals over the intervals $(t_0, t_0 + 1/n]$, $(t_0 + 1/n, t_0 + 2/n]$ etc. Moreover, if we define $\hat{x}_n(t_0) = x_n(t_0)$, $\widetilde{x}_n(t_0) = x_n(t_0 - \delta(t_0))$,

$$\hat{x}_n(t) = x_n(t_0 + k/n) \text{ and } \widetilde{x}_n(t) = x_n(t_0 + k/n - \delta(t))$$

for $t_0 + k/n < t \leq [t_0 + (k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$, it then follows (3.10) that

$$x_n(t) = \xi(0) + \int_{t_0}^{t} F(\hat{x}(s), \widetilde{x}(s), s)ds + \int_{t_0}^{t} G(\hat{x}(s), \widetilde{x}(s), s)dB(s) \tag{3.11}$$

for $t_0 \leq t \leq T$.

The following lemma shows that under conditions (3.2) and (3.3), the Euler-Maruyama approximation sequence is bounded in L^2.

Lemma 3.5. *Let (3.2) and (3.3) hold. Then for all* $n \geq 1$,

$$E\left(\sup_{t_0-\tau \leq t \leq T} |x_n(t)|^2\right) \leq C_6, \tag{3.12}$$

where $C_6 = [4E||\xi||^2 + 6C_1K(t - t_0)]e^{12C_1\overline{K}(T-t_0)}$

Proof. Applying the elementary inequality in the Euler-Maruyama application sequence (3.11), then taking the expected values on both sides and the H$\ddot{o}$lder's inequality (2.3), moment inequality(2.6) in turn, we get the following.

$$\begin{aligned} &E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\ \leq &3E\left|\xi(0)\right|^2 + 3(t - t_0)E\int_{t_0}^{t} \left|F\left(\hat{x}_n(s), \widetilde{x}_n(s), s\right) - F(0,0,s) + F(0,0,s)\right|^2 ds \\ &+ 12E\int_{t_0}^{t} \left|G\left(\hat{x}_n(s), \widetilde{x}_n(s), s\right) - G(0,0,s) + G(0,0,s)\right|^2 ds. \end{aligned}$$

The elementary inequality, uniform Lipschitz condition (3.2), and the weakened linear growth condition (3.3) drive the following inequality.

$$E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \leq \quad 3E\left|\xi(0)\right|^2 + 6C_1K(t - t_0) + 12C_1\overline{K}\int_{t_0}^{t} E\left(\sup_{t_0-\tau \leq r \leq s} |x_n(r)|^2\right),$$

where C_1 was defined in Lemma(3.1). Furthermore, if we extend the supremum interval, we get the following.

$$E\left(\sup_{t_0-\tau \leq t \leq T} |x_n(t)|^2\right) \leq 4E||\xi||^2 + 6C_1K(t - t_0) + 12C_1\overline{K}\int_{t_0}^{t} E\left(\sup_{t_0-\tau \leq r \leq s} |x_n(r)|^2\right).$$

The Gronwall inequality implies

$$E\left(\sup_{t_0-\tau\leq t\leq T}|x_n(t)|^2\right)\leq[4E||\xi||^2+6C_1K(t-t_0)]e^{12C_1\overline{K}(T-t_0)}.$$

We can immediately see that it is an inequality such as the inequality (3.12) that required proof. ∎

The next theorem shows that the Euler-Maruyama approximate sequence has continuity under conditions (3.2) and (3.3).

Theorem 3.6. *Suppose that conditions (3.2) and (3.3) are valid for any $t_0\leq s<t\leq T$ with $t-s\leq 1$. Then we have*

$$E\left|x_n(t)-x_n(s)\right|^2\leq 20(2\overline{K}C_6+K)(t-s), \tag{3.13}$$

where C_6 is defined in Lemma 3.5.

Proof. Applying the elementary inequality in the Euler-Maruyama approximate sequence (3.11), we get the following.

$$\left|x_n(t)-x_n(s)\right|^2\leq 2\left|\int_s^t F\left(\hat{x}_n(r),\widetilde{x}_n(r),r\right)dr\right|^2+2\left|\int_s^t G\left(\hat{x}_n(r),\widetilde{x}_n(r),r\right)dB(r)\right|^2.$$

Based on Lemma 3.5, by taking the expectation of both sides, using the H$\ddot{o}$lder's inequality (2.3) and moment inequality(2.6), we have

$$E\left|x_n(t)-x_n(s)\right|^2\leq 2(t-s)E\int_s^t\left|F\left(\hat{x}_n(r),\widetilde{x}_n(r),r\right)\right|^2dr+8E\int_s^t\left|G\left(\hat{x}_n(r),\widetilde{x}_n(r),r\right)\right|^2dr$$

By adding and subtracting a new term, the elementary inequality is used to obtain the following equation.

$$\begin{aligned}E\left|x_n(t)-x_n(s)\right|^2\leq& 4(t-s)E\int_s^t\left[\left|F\left(\hat{x}_n(r),\widetilde{x}_n(r),r\right)-F(0,0,r)\right|^2+\left|F(0,0,r)\right|^2\right]dr\\&+16E\int_s^t\left[\left|G\left(\hat{x}_n(r),\widetilde{x}_n(r),r\right)-G(0,0,r)\right|^2+\left|G(0,0,r)\right|^2\right]dr.\end{aligned}$$

Here, using the uniform Lipschitz condition (3.2) and the weakened linear growth condition (3.3), we obtain

$$E\left|x_n(t)-x_n(s)\right|^2\leq 4(t-s+4)\overline{K}\int_s^t 2C_6dr+4(t-s+4)K\int_s^t 1dr.$$

Since $t-s\leq 1$, we reach

$$E\left|x_n(t)-x_n(s)\right|^2\leq 40\overline{K}C_6(t-s)+20K(t-s)=20(2\overline{K}C_6+K)(t-s).$$

It is an inequality such as the inequality that required proof. ∎

The next theorem is another of the main theorems of this section, where conditions (3.2) and (3.3) are satisfied, It is possible to estimate how quickly the approximate solution of the stochastic functional differential equation (3.1) with the Euler-Maruyama approximate sequence converges in the unique solution.

Theorem 3.7. *Suppose that conditions (3.2) and (3.3) are valid and the initial data* $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ *is uniformly Lipschitz* L^2*-continuous, that is there is a positive constant* β *such that*

$$E\left|\xi(\theta_1) - \xi(\theta_2)\right|^2 \leq \beta(\theta_2 - \theta_1) \quad if \quad -\tau \leq \theta_1 < \theta_2 \leq 0 \tag{3.14}$$

Then the between the accurate solution $x(t)$ *and Euler-Maruyama sequence (3.11) can be estimated as*

$$E\left(\sup_{t_0 \leq t \leq T} |x(t) - x_n(t)|^2\right) \leq (J_3 + J_4)\, e^{4\overline{K}C_1(T-t_0)} \tag{3.15}$$

where C_1 *is defined in Lemma 3.1*

Proof. Based on Lemma (3.1) and Lemma (3.5), the difference between the exact solution and asymptotic solutions is calculated using the elementary inequality, and then the expected values are taken on both sides are applied sequentially to obtain the following.

$$\begin{aligned} E\left|x(t) - x_n(t)\right|^2 \leq & 2E\left|\int_{t_0}^{t} \left[F\left(\hat{x}(s), \widetilde{x}(s), s\right) ds - F\left(\hat{x}_n(s), \widetilde{x}_n(s), s\right)\right] ds\right|^2 \\ & + 2E\left|\int_{t_0}^{t} \left[G\left(\hat{x}(s), \widetilde{x}(s), s\right) ds - G\left(\hat{x}_n(s), \widetilde{x}_n(s), s\right)\right] dB(s)\right|^2. \end{aligned}$$

Using the Hölder's inequality (2.3), moment inequality (2.6), the uniform Lipschitz condition (3.2) and weakened linear growth condition (3.3), we can drive it to the next one

$$E\left|x(t) - x_n(t)\right|^2 \leq 2\overline{K}C_1 \int_{t_0}^{t} \left[E\left|\hat{x}(s) - \hat{x}_n(s)\right|^2 + E\left|\widetilde{x}(s) - \widetilde{x}_n(s)\right|^2\right] ds.$$

Since $\hat{x}(t_0) = x(t_0)$, $\widetilde{x}(t_0) = x\left(t_0 - \delta(t_0)\right)$, $\hat{x}(t) = x(t_0 + k/n)$ and $\widetilde{x}(t) = x\left(t_0 + k/n - \delta(t_0)\right)$, we can write as follows

$$\begin{aligned} E\left(\sup_{t_0 \leq s \leq t} |x(s) - x_n(s)|^2\right) \leq & J_3 + J_4 + 2\overline{K}C_1 \int_{t_0}^{t} \left[E\left|\hat{x}(s) - \hat{x}_n(s)\right|^2 + E\left|\widetilde{x}(s) - \widetilde{x}_n(s)\right|^2\right] ds \\ \leq & J_3 + J_4 + 4\overline{K}C_1 \int_{t_0}^{t} E\left(\sup_{t_0 \leq r \leq s} |x(r) - x_n(r)|^2\right) ds, \end{aligned}$$

where

$$J_3 = 2\overline{K}C_1 \int_{t_0}^{t} E\left|x(s) - \hat{x}(s)\right|^2 ds$$

and

$$J_4 = 2\overline{K}C_1 \int_{t_0}^{t} E\left|x(s - \delta(s)) - \widetilde{x}(s)\right|^2 ds.$$

By Gronwall inequality, The following can be obtained

$$E\left(\sup_{t_0 \leq s \leq t} |x(s) - x_n(s)|^2\right) \leq (J_3 + J_4)\, e^{4\overline{K}C_1(t-t_0)}. \tag{3.16}$$

On the other hand, using Lemma (3.3), integral J_3 is calculated as follows.

$$\begin{aligned} J_3 = & 2\overline{K}C_1 \sum_{k \geq 0} \int_{t_0 + \frac{k}{n}}^{\left[t_0 + \frac{k}{n}\right] \wedge T} E\left|x(s) - x(t_0 + k/n)\right|^2 ds \\ \leq & \frac{2}{n} C_1 C_5 (T - t_0), \end{aligned} \tag{3.17}$$

where C_5 is defined in theorem 3.4. Also, using an elementary inequality and the Lemma (3.3) once again, we can estimate

$$\begin{aligned}
J_4 \leq & 2\overline{K}C_1 \sum_{k\geq 0} \int_{t_0+\frac{k}{n}}^{\left(t_0+\frac{k+1}{n}\right)\wedge T} E\left|x(s-\delta(s)) - x(t_0+k/n-\delta(s))\right|^2 ds \\
\leq & \frac{2}{n}\overline{K}C_1 \sum_{k\geq 0} \int_{\left(t_0+\frac{k}{n}\right)\vee\tau}^{\left(t_0+\frac{k+1}{n}\right)\wedge T} C_5 ds \\
& + 2\overline{K}C_1 \sum_{k\geq 0} \int_{t_0+\frac{k}{n}}^{\left(t_0+\frac{k}{n}\right)\wedge\tau} E\left|x(s-\delta(s)) - x(t_0+k/n-\delta(s))\right|^2 ds,
\end{aligned}$$

where C_5 is defined in theorem 3.4. It is easy to show, by condition (3.14) and Lemma (3.3), that

$$\begin{aligned}
J_4 \leq & \frac{2}{n}\overline{K}C_1C_5(T-t_0) \\
& + 2\overline{K}C_1 \sum_{k\geq 0} \int_{t_0+\frac{k}{n}}^{\left(t_0+\frac{k}{n}\right)\wedge\tau} \beta(t_0+(k+1)/n-(t_0+k/n))ds \\
& + 2\overline{K}C_1 \sum_{k\geq 0} \int_{t_0+\frac{k}{n}}^{\left(t_0+\frac{k}{n}\right)\wedge\tau} C_5(t_0+(k+1)/n-(t_0+k/n))ds \\
\leq & \frac{2}{n}\overline{K}C_1C_5(T-t_0) + 2\overline{K}C_1 \sum_{k\geq 0} \int_{t_0+\frac{k}{n}}^{\left(t_0+\frac{k}{n}\right)\wedge\tau} (\beta/n + C_5/n)\, ds.
\end{aligned}$$

Therefore, we have

$$J_4 \leq \frac{2}{n}\overline{K}C_1\left(C_5(T-t_0) + 2\tau\left(\beta \vee C_5\right)\right). \tag{3.18}$$

Substituting (3.17)and (3.18) into (3.16) yields the required assertion (3.15). The proof is complete. ∎

4. CONCLUSION

In this study, we investigated the application of Caratheodory's and Euler'Maruyama's approximation methods to stochastic functional differential delay equations. Using the uniform Lipschitz condition (3.3) and the weakened linear growth condition (3.4), in Lemma 3.3, we estimated that the singular solution $x(t)$ of the stochastic function differential equation (refe.3.1.a) has continuity and in Theorem 3.4, we estimated how fast approximation $x_n(t)$ approaches the unique solution $x(t)$ of the stochastic functional differential equation (3.1) by the Caratheodory's approximation method.

We also rigorously analyzed the convergence of the Euler'Maruyama approximation in the presence of functional delays and stochastic perturbations. Using the same condition (3.3) and (3.4),in Lemma 3.6, we estimated that the Euler-Maruyama approximate sequence (3.11) has continuity and in Theorem 3.7, we estimated how quickly the approximate solution of the stochastic functional differential equation (3.1) with the Euler-Maruyama approximate sequence converges in the unique solution.

The combined use of these two approaches offers a flexible and robust strategy for analyzing complex stochastic systems with delay and historical dependence. Our findings suggest that even in settings where classical analytical techniques fail, the proposed framework provides both theoretical guarantees and practical applicability.

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