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FIXED POINT RESULTS FOR INTEGRAL TYPE CONTRACTIONS IN R-METRIC SPACE

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ABSTRACT. The main aim of the research is to establish some invariant point (fixed point) results under the purview of R-Metric Spaces, for integral type R-contractive mappings. To serve this purpose, concepts of R-continuity, R-convergence and R-preservation has been used. Finally, the obtained results has been used to deduce some invariant point results for Banach, Kannan and Chatterjea type mappings in R-Metric Space. Also some examples and applications have been illustrated to support the findings discussed.

Key words and phrases: Invariant point (Fixed point); Metric space (M_S); R-Metric Space ($R - M_S$); R-Sequence; R-Convergence; R-Contraction; R-Preserving; Subadditivity.

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1. INTRODUCTION

The concept of M_S came to reality when a French mathematician named Maurice Fréchet [1, 2] initiated its study in the early twentieth century. Yet he did not coin the term "Metric spaces" though. This was done by another German mathematician named Felix Hausdorff [3] in 1914. Since then, the study of non-linear functional analysis took a kick-start as several more mathematicians took their signature steps into the topic.

For instance in 1922, a polish mathematician named Stefan Banach [11] established the concept of contraction, and proved a invariant point theorem which is commonly known as the Banach fixed-point theorem. With respect to functional analysis, the fixed point or invariant point x of a function g is a point such that $g(x) = x$. Many other mathematicians such as Ravindran Kanan [12, 13] in 1968 and 1969, Ljubomir Ćirić [10] in 1974, Shoutir Kishore Chatterjea [5] in 1972 and Erdal Karapinar [20] in 2012, obtained motivation from Banach's discovery and established their own invariant point theorems.

Simultaneously many other generalizations of the concepts given by Fréchet and Hausdorff were established. Such as Kramosil [14] in 1975 discovered Fuzzy metric and Statistical M_S, Mustafa [43] in 2006 discovered $G - M_S$, Kirisci [15] in 2020 discovered Neutrosophic M_S, Park [4] in 2004 discovered Intuitionistic fuzzy M_S, Beaulaa [16] in 2015 introduced Soft fuzzy M_S, Das [6] in 2013 introduced Soft M_S, Jeyaraman [8] in 2023 introduced Neutrosophic soft M_S, Tian [18] in 2020 discovered Tripled fuzzy M_S, Backhtin [26] in 1989 gave the idea of b-M_S, George [25] in 2015 established Rectangular b-M_S, Sonam [24] in 2023 introduced Soft rectangular b-M_S along with an invariant point result in it, and Ghosh [27] in 2024 established the Neutrosophic fuzzy M_S along with its properties.

Combining the concepts of such new generalizations with the concepts of invariant point theory, many other mathematicians [7, 8, 17][[20]-[23]][[28]-[31]][[37]-[40]] have shown their interests in non-linear functional analysis. Notably, Branciari [32] in 2001 and Berinde [19] in 2004 introduced the concepts of integral type contractive mapping and weak contraction respectively, and also utilized these to deduce some invariant point results. Both of these establishments act as an important milestone in the path of research related to contraction mappings, still to this day.

Now $R - M_S$ is another generalization of Fréchet's concept of conventional M_S, which has recently been introduced by Simak Khalehoghli [9] in 2020. In that research he also extended Banach's and Brouwer's invariant point theorems for the newly introduced space. Since then no other mathematician except [42] has shown interest to work on the newly established space. Particularly in [42], authors have derived fixed point results for Kannan and Chatterjea type mappings with respect to the said space. So undoubtedly it can be said that, the concept of integral type contractive mapping in $R - M_S$ is novel and unique.

2. PRELIMINARIES

Definition 2.1. ($R - M_S$) [9] If $(S, \mathscr{D})$ be a M_S, R is a relation for S. Then triplet $(S, \mathscr{D}, R)$ is regarded as an $R - M_S$.

Example 2.1. *Let $S = \mathbb{R}^+ \cup \{0\}$ be endowed with $\mathscr{D}$, the standard distance metric. Define xRy if $x + y \geq xy$. Then, $(S, \mathscr{D}, R)$ becomes an $R - M_S$.*

Definition 2.2. (R-Sequence) [9] Let $\{x_n\}$ be a sequence in $(S, \mathscr{D}, R)$. Then, it is called a R-seq. $\iff$ $x_n R x_{n+m} \; \forall \, n, m \in \mathbb{N}$.

Definition 2.3. (R-Convergence) [9] A R-sequence $\{x_n\}$ in $(S, \mathscr{D}, R)$ R-converges to some $x \in S$ if $\forall \, \epsilon > 0 \; \exists \, K \in \mathbb{Z}$ such that $d(x_n, x) < \epsilon$ where $n \geq K$.

Definition 2.4. (R-Cauchy sequence) [9] A R-sequence $\{x_n\}$ in $(S, \mathscr{D}, R)$ is known as R-Cauchy seq. if it is R-convergent to some $x \in S$. That is, $\forall\ \epsilon > 0\ \exists\ K \in \mathbb{Z}$ such that $\mathscr{D}(x_m, x_n) < \epsilon$ where $m, n \geq K$. As a result, $x_n R x_m$ or $x_m R x_n$.

Definition 2.5. (R-Continuous) [9] Let $T : S \to S$ be a mapping. T is proclaimed to be R-continuous at some point say $v \in S$, if for every R-sequence $\{v_n\}$ that R-converges to $v \in S$, it can be said that $\{T(v_n)\}$ converges to $T(v)$. In this context it is said that, T is R-continuous on S if and only if T is R-continuous $\forall\ v \in S$.

Remark 2.1. [9] Since every R-sequence is a sequence, it can be said that every continuous mapping $T : S \to S$ is R-continuous. The converse is not necessarily true.

Definition 2.6. (R-Contraction) [9] Let $(S, \mathscr{D}, R)$ be a $R - M_S$. A R-contraction is a self mapping $H : S \to S$ where $\forall\ v, w \in S$ such that vRw, it can be said that,

$$\mathscr{D}(H(v), H(w)) \leq \alpha \mathscr{D}(v, w)\ \forall\ \alpha \in (0, 1)$$

In this context, the constant α is also known as the Lipschitz contant.

Definition 2.7. (R-Preserving) [9] A self mapping $T : S \to S$ is apparently R-preserving, iff $xRy \implies (T(x))R(T(y))\ \forall\ x, y \in S$.

Example 2.2. *Let $S = \mathbb{R}^+$ with it's usual standard topology and xRy is defined when $xy \in \{x, y\}$, s.t. $y, x \in S$.*
Suppose the mapping $T : S \to S$ is defined as,

$$T(x) = \begin{cases} x^{-1} & : 1 \geq x \\ 1 & : 1 < x \end{cases}$$

.
Now, let xRy. Thus $x = 1$ or $y = 1$. Therefore, either $T(x) = 1$ or $T(y) = 1$.
Implies that, $(T(x))R(T(y))$. From here it is clear that, $xRy \implies (T(x))R(T(y))$.
Hence it can be said that, T is R-preserving.

Definition 2.8. (Picard Operator) [41] Any self mapping on a metric space, possesing a unique invariant point where every repeated application of the mapping converges the sequence to its invariant point is known as a Picard operator.

Theorem 2.1. [9] *Suppose $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete, and $\alpha \in (0, 1)$. Let $H : S \to S$ be R-continuous, R-preserving, R-contraction with Lipschitz constant α. Assume $\exists\ x_0 \in S$ such that $x_0 R y\ \forall\ y \in H(S)$. Then H has an unique invariant point. And also, H is a Picard operator.*

Definition 2.9. (Integral type contraction) [32, 33, 35, 36] A mapping $H : S \to S$ where $(S, \mathscr{D})$ is a M_S is said to be an integral type contraction if $\exists\ \alpha \in [0, 1)$ such that $\forall\ p, q \in S$,

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \alpha \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0) d\tau_0$$

Where $\eta_0(\tau_0) : [0, \infty) \to [0, \infty)$ is a Lesbegue integrable mapping.

Definition 2.10. (Subadditivity) [34] A Lesbegue integrable mapping $\eta(\tau) : [0, \infty) \to [0, \infty)$ is known to be subadditive on each $[m, n] \subset \mathbb{R}^+ \cup \{0\}$ if,

$$\int_0^{m+n} \eta_0(\tau_0) d\tau_0 \leq \int_0^{m} \eta_0(\tau_0) d\tau_0 + \int_0^{n} \eta_0(\tau_0) d\tau_0.$$

3. ESTABLISHED RESULTS

The concept for integral type R-contraction and fixed point results have been established.

On the basis of Definition 2.6 and Definition 2.7, the concept of integral type R-contraction can be defined as follows.

Definition 3.1. A mapping $H : S \to S$ where $(S, \mathscr{D}, R)$ is a $R - M_S$ is said to be an integral type R-contraction if $\exists\, \alpha \in [0, 1)$ such that $\forall\, p, q \in S$ having the property pRq,

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \alpha \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0) d\tau_0$$

Where $\eta(\tau) : [0, \infty) \to [0, \infty)$ is a Lesbegue integrable mapping.

Remark 3.1. Any integral type R-contractive condition can be reduced to a normal R-contractive condition by taking $\eta_0(\tau_0) = 1$.

Theorem 3.1. *Suppose $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete and $H : S \to S$ be a R-preserving, R-continuous self map which is an integral type R-contraction, where $p \neq q$ and the Lesbegue integrable mapping considered is subadditive. Also let $\exists\, p_0 \in S$ such that $p_0 R q\ \forall\, q \in H(S)$. Then H contains unique fixed point in $(S, \mathscr{D}, R)$.*

Proof. Let $p_0 \in S$ be arbitary. Define by induction $H^n : S \to S$ by $H^0 p_0 = p_0$, $H^1 p_0 = p_1$, $H^2 p_0 = H(Hp_0) = Hp_1 = p_2$, ... ,$H^n p_0 = p_n$ where $n \geq 0$. Thus an iterative sequence $\{p_n\}$ associated with H is obtained.

Let n, m are in $\mathbb{N}$ and n is less than m, substitute $m - n = k$.

We have $p_0 R H^k(p_0)$. Since H is R-preserving, $[H^n(p_0)]\, R\, \left[H^{n+k}(p_0)\right]$. Which implies, $p_n R p_m$. Hence $\{p_n\}$ is a R-sequence. Now since H is an integral type R-contraction, it can be said that,

$$\begin{aligned}\int_0^{\mathscr{D}(p_n,p_{n+1})} \eta_0(\tau_0)\, d\tau_0 &= \int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0)\, d\tau_0 \\ &\leq \alpha \int_0^{\mathscr{D}(H^{n-1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0 \\ &\leq \alpha^2 \int_0^{\mathscr{D}(H^{n-2} p_0, H^{n-1} p_0)} \eta_0(\tau_0) d\tau_0\end{aligned}$$

Continuing in this way,

$$\int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0) d\tau_0 \leq \alpha^n \int_0^{\mathscr{D}(p_0, Hp_0)} \eta_0(\tau_0)\, d\tau_0.$$

As a result, since $0 < \alpha < 1$,

$$\int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0)\, d\tau_0 \text{ tends to } 0 \text{ as } n \to \infty.$$

Or,

$$\mathscr{D}(H^n p_0, H^{n+1} p_0) \to 0 \text{ as } n \to \infty.$$

Now let,

$$\lim_{n\to\infty} \sup \mathscr{D}(H^n p_0, H^{n+1} p_0) = \epsilon > 0$$

Then $\exists\, v_\epsilon \in \mathbb{N}$, and a seq. $(H^{n_v} p_0)_{v \geq v_\epsilon} : \mathscr{D}(H^{n_v} p_0, H^{n_v+1} p_0) \to \epsilon > 0$ as $v \to +\infty$ and $\mathscr{D}(H^{n_v} p_0, H^{n_v+1} p_0) \geq \epsilon/2\ \forall\, v \geq v_\epsilon$. Thus we have the following contradiction:

$$0 = \lim_{v\to\infty} \int_0^{\mathscr{D}(H^{n_v} p_0, H^{n_v+1} p_0)} \eta_0(\tau_0)\, d\tau_0 \geq \int_0^{\epsilon/2} \eta_0(\tau_0)\, d\tau_0$$

Now it has to be proved that, $(H^n p_0)_{n\in\mathbb{N}}$ is R-Cauchy $\forall\, p_0 \in S$. Or,

$$\forall\, \epsilon > 0\, \exists\, v_\epsilon \in \mathbb{N} : \forall\, m, n \in \mathbb{N}, m > n > v_\epsilon;\ \mathscr{D}(H^m p_0, H^n p_0) < \epsilon.$$

Let $\exists\, \epsilon > 0 : \forall\, v \in \mathbb{N}\, \exists\, m_v, n_v \in \mathbb{N}$ such that $\mathscr{D}(H^{m_v} p_0, H^{n_v} p_0) \geq \epsilon$ where $m_v > n_v > v$. Then $(m_v)_{v\in\mathbb{N}}$, $(n_v)_{v\in\mathbb{N}}$ are chosen : m_v is infinitesimal $\forall\, v \in \mathbb{N}$ in the sense that,

$$\mathscr{D}(H^{m_v} p_0, H^{n_v} p_0) \geq \epsilon \text{ but } \mathscr{D}(H^r p_0, H^{n_v} p_0) < \epsilon\ \forall\, r \in \{n_v + 1, ..., m_v - 1\}.$$

Now it can be written that,

$$\begin{aligned}\epsilon &< \mathscr{D}(H^{m_v} p_0, H^{n_v} p_0)\\ &\leq \mathscr{D}(H^{m_v} p_0, H^{m_v-1} p_0) + \mathscr{D}(H^{m_v-1} p_0, H^{n_v} p_0)\\ &< \mathscr{D}(H^{m_v} p_0, H^{m_v-1} p_0) + \epsilon\end{aligned}$$

which tends to $\epsilon+$ as v tends to $+\infty$.
Further $\exists\, \mu \in \mathbb{N}$ such that $\forall\, v > \mu\ (v \in \mathbb{N})$ one can get $\mathscr{D}(H^{m_v+1} p_0, H^{n_v+1} p_0) < \epsilon$. Actually, if $\exists\, (v_k)_{n\in\mathbb{N}} \subseteq \mathbb{N} : \mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) \geq \epsilon$, it can be written that,

$$\begin{aligned}\epsilon &\leq \mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0)\\ &\leq \mathscr{D}(H^{m_{v_k}+1} p_0, H^{m_{v_k}} p_0) + \mathscr{D}(H^{m_{v_k}} p_0, H^{n_{v_k}} p_0) + \mathscr{D}(H^{n_{v_k}} p_0, H^{n_{v_k}+1} p_0)\end{aligned}$$

which tends to ϵ as k tends to $+\infty$.
Using the contractive condition, it can be written that,

$$\int_0^{\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0)} \eta_0(\tau_0)\, d\tau_0 \leq \alpha \int_0^{\mathscr{D}(H^{m_{v_k}} p_0, H^{n_{v_k}} p_0)} \eta_0(\tau_0)\, d\tau_0$$

Taking $k \to +\infty$ in each side of the above inequality, one can obtain the contradiction $\int_0^\epsilon \eta_0(\tau_0)\, d\tau_0 \leq \alpha \int_0^\epsilon \eta_0(\tau_0)\, d\tau_0$ [$\because \alpha \in (0, 1)$, the integral is +ve]. So for particular $\mu \in \mathbb{N}$, $\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) < \epsilon\ \forall\, v \in \mu$. Finally it is required to prove that, $\exists\, s_\epsilon \in (0, \epsilon)$ and $v_\epsilon \in \mathbb{N}$ such that for every $v > v_\epsilon$ $(v \in \mathbb{N})$, one can obtain $\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) < \epsilon - s_\epsilon$.
Now let (v_k) exist such that $\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) \to \epsilon-$ as $k \to \infty$. Then by taking $k \to \infty$ in,

$$\int_0^{\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0)} \eta_0(\tau_0)\, d\tau_0 \leq \alpha \int_0^{\mathscr{D}(H^{m_{v_k}} p_0, H^{n_{v_k}} p_0)} \eta_0(\tau_0)\, d\tau_0,$$

One can again obtain the contradiction $\int_0^\epsilon \eta_0(\tau_0)\, d\tau_0 \leq \alpha \int_0^\epsilon \eta_0(\tau_0)\, d\tau_0$ [Since, $\alpha \in (0, 1)$ and the integral is positive]. So, for every natural number $v > v_\epsilon$ (v_ϵ as above),

$$\begin{aligned}\epsilon &\leq \mathscr{D}(H^{m_v} p_0, H^{n_v} p_0)\\ &\leq \mathscr{D}(H^{m_v} p_0, H^{m_v+1} p_0) + \mathscr{D}(H^{m_v+1} p_0, H^{n_v+1} p_0) + \mathscr{D}(H^{n_v+1} p_0, H^{n_v} p_0)\\ &\leq \mathscr{D}(H^{m_v} p_0, H^{m_v+1} p_0) + (\epsilon - s_\epsilon) + \mathscr{D}(H^{n_v+1} p_0, H^{n_v} p_0)\end{aligned}$$

Which tends to $\epsilon - s_\epsilon$ as $v \to \infty$. Thus $\epsilon \leq \epsilon - s_\epsilon$, which is a contradiction.
Hence $(H^n p_0)_{n\in\mathbb{N}}$ is R-Cauchy $\forall\, p_0 \in S$.

Now by R-completeness of S it can be said that, $\{p_n\}$ R-converges to some $t \in S$. And since H is R-continuous, Hp_n R-converges to Ht. Therefore,

$$\begin{aligned}Ht &= H\left(\lim_{n\to\infty} p_n\right)\\ &= \lim_{n\to\infty} Hp_n\\ &= \lim_{n\to\infty} p_{n+1} = t\end{aligned}$$

Hence t will be fixed point on H.
Uniqueness, let $t' \in S$ be a invariant point of H. As a result, $p_0 R t'$.
Hence $(H^n p_0 = p_n) R (H^n t' = t') \; \forall \; n \in \mathbb{W}$. So by the virtue of triangular inequality it can be written that,

$$\begin{aligned}\int_0^{\mathscr{D}(t,t')} \eta_0(\tau_0) d\tau_0 &\leq \int_0^{\mathscr{D}(t,H^n p_0)} \eta_0(\tau_0) d\tau_0 + \int_0^{\mathscr{D}(t',H^n p_0)} \eta_0(\tau_0) d\tau_0 \\ &\leq \int_0^{\mathscr{D}(H^n t,H^n p_0)} \eta_0(\tau_0)\, d\tau_0 + \int_0^{\mathscr{D}(H^n t',H^n p_0)} \eta_0(\tau_0)\, d\tau_0 \\ &\leq \alpha^n \int_0^{\mathscr{D}(t,p_0)} \eta_0(\tau_0)\, d\tau_0 + \alpha^n \int_0^{\mathscr{D}(t',p_0)} \eta_0(\tau_0)\, d\tau_0\end{aligned}$$

Which tends to 0 as $n \to \infty$. [Since, $\alpha < 1$]
And thus, $t = t'$. Hence, the fixed point is unique. ∎

Lemma 3.2. *The mapping defined in Theorem 3.1 is a Picard operator.*

Proof. Let $p \in S$ be arbitrary. Since $p_0 R(Hp)$ and H is R-preservative, it can be written that $(H^n p_0) R (H^{n+1} p) \; \forall \; n \in \mathbb{W}$.
Therefore,

$$\begin{aligned}\int_0^{\mathscr{D}(t,H^n p)} \eta_0(\tau_0) d\tau_0 &= \int_0^{\mathscr{D}(H^n t,H^n p)} \eta_0(\tau_0) d\tau_0 \\ &\leq \int_0^{\mathscr{D}(H^{n-1} t,H^{n-1} p_0)} \eta_0(\tau_0) d\tau_0 + \int_0^{\mathscr{D}(H^{n-1} p_0,H^{n-1}(Hp))} \eta_0(\tau_0) d\tau_0 \\ &\leq \alpha^{n-1} \int_0^{\mathscr{D}(t,p_0)} \eta_0(\tau_0) d\tau_0 + \alpha^{n-1} \int_0^{\mathscr{D}(p_0,Hp)} \eta_0(\tau_0) d\tau_0\end{aligned}$$

Which tends to 0 as $n \to \infty$. [Since, $\alpha < 1$]

$$\text{Hence, } \lim_{n\to\infty} H^n p = t$$

∎

Remark 3.2. By the application of Remark 3.1, Theorem 3.1 reduces to Banach fixed point theorem for R-metric space.

Example 3.1. *Let $S = [0, 2]$ and $H : S \to S$ be a mapping defined as, $H(x) = \frac{x}{8} \; \forall \; x \in S$. And suppose the relation R be defined as $x + y \in \{x, y\}$. Then it can be easily confirmed that (S, d, R) is an R-complete $R - M_S$ where $d = |x^2 - y^2|$, & H is R-continuous and R-preserving. Also it is taken that $\eta(\tau) = \frac{1}{1+t^2} \; \forall \; t \in S$ and $\alpha = 0.98$. Clearly, $\eta(\tau)$ is Lesbegue integrable and subadditive in S.*
Then, by further calculations it can be verified that,

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \alpha \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0) d\tau_0, \; \forall \; p, q \in S.$$

Thus, $H(x)$ & $\eta(\tau)$ fulfills all the conitions of Theorem 3.1. And in this case, $x = 0$ is a fixed point of the mapping $H(x)$, where $x \in S$.

Theorem 3.3. *Let $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \alpha \int_0^{\mathscr{D}(Hp,p)+\mathscr{D}(Hq,q)} \eta_0(\tau_0) d\tau_0 + \beta \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0) d\tau_0$$

$\forall\, p, q \in S$ *($p \neq q$), where* $2\alpha + \beta < 1$ *and* $\eta(\tau)$ *is subadditive* $\forall\, [m, n] \subset \mathbb{R}^+ \cup \{0\}$. *Also let* $\exists\, p_0 \in S$ *such that* $p_0 R q\ \forall\, q \in H(S)$. *Then H has an unique fixed point in* $(S, \mathscr{D}, R)$.

Proof. Let $p_0 \in S$ be arbitrary. Define by induction $H^n : S \to S$ by $H^0 p_0 = p_0$, $H^1 p_0 = p_1$, $H^2 p_0 = H(Hp_0) = Hp_1 = p_2$, ... ,$H^n p_0 = p_n$ where $n \geq 0$. Thus an iterative sequence $\{p_n\}$ associated with H is obtained.
Let $m, n \in \mathbb{N}$ and $n < m$, substitute $m - n = k$.
We have $p_0 R H^k(p_0)$. Since H is R-preserving, $[H^n(p_0)]\, R \left[H^{n+k}(p_0)\right]$. Which implies, $p_n R p_m$. Hence $\{p_n\}$ is a R-sequence. Now from the R-contractive condition of H it can be said that,

$$\begin{aligned}
\int_0^{\mathscr{D}(p_n,p_{n+1})} \eta_0(\tau_0) d\tau_0 &= \int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0) d\tau_0 \\
&\leq \alpha \int_0^{\mathscr{D}(H^n p_0, H^{n-1} p_0) + \mathscr{D}(H^{n+1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0 \\
&\qquad + \beta \int_0^{\mathscr{D}(H^{n-1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0 \\
&\leq \alpha \int_0^{\mathscr{D}(H^n p_0, H^{n-1} p_0)} \eta_0(\tau_0) d\tau_0 + \alpha \int_0^{\mathscr{D}(H^{n+1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0 \\
&\qquad + \beta \int_0^{\mathscr{D}(H^{n-1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0 \\
&\leq (\alpha + \beta) \int_0^{\mathscr{D}(H^{n-1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0 + \alpha \int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0) d\tau_0
\end{aligned}$$

$$\text{(3.1)} \qquad \implies \int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0) d\tau_0 \leq \left[\frac{\alpha + \beta}{1 - \alpha}\right] \int_0^{\mathscr{D}(H^{n-1} p_0, H^n p_0)} \eta_0(\tau_0) d\tau_0$$

Continuing in this way,

$$\int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0) d\tau_0 \leq c^n \int_0^{\mathscr{D}(p_0, H p_0)} \eta_0(\tau_0) d\tau_0$$

Where,

$$c = \left[\frac{\alpha + \beta}{1 - \alpha}\right] < 1.$$

As a result, since $0 < c < 1$,

$$\int_0^{\mathscr{D}(H^n p_0, H^{n+1} p_0)} \eta_0(\tau_0) d\tau_0 \to 0 \text{ as } n \to \infty.$$

Or,

$$\mathscr{D}(H^n p_0, H^{n+1} p_0) \to 0 \text{ as } n \to \infty.$$

Now let,

$$\lim_{n\to\infty} \sup \mathscr{D}(H^n p_0, H^{n+1} p_0) = \epsilon > 0$$

Then $\exists\, v_\epsilon \in \mathbb{N}$ and a sequence $(H^{n_v} p_0)_{v \geq v_\epsilon}$ such that $\mathscr{D}(H^{n_v} p_0, H^{n_v+1} p_0) \to \epsilon > 0$ as $v \to +\infty$ and $\mathscr{D}(H^{n_v} p_0, H^{n_v+1} p_0) \geq \epsilon/2\ \forall\, v \geq v_\epsilon$. Thus we have the following contradiction:

$$0 = \lim_{v\to\infty} \int_0^{\mathscr{D}(H^{n_v} p_0, H^{n_v+1} p_0)} \eta_0(\tau_0) d\tau_0 \geq \int_0^{\epsilon/2} \eta_0(\tau_0) d\tau_0$$

Now it has to be proved that, $(H^n p_0)_{n\in\mathbb{N}}$ is R-Cauchy $\forall\, p_0 \in S$. Or,

$$\forall\, \epsilon > 0\; \exists\, v_\epsilon \in \mathbb{N} : \forall\, m, n \in \mathbb{N}, m > n > v_\epsilon;\; \mathscr{D}(H^m p_0, H^n p_0) < \epsilon.$$

Let $\exists\, \epsilon > 0$ such that for every $v \in \mathbb{N}$ there are $m_v, n_v \in \mathbb{N}$ such that $\mathscr{D}(H^{m_v} p_0, H^{n_v} p_0) \geq \epsilon$ where $m_v > n_v > v$. Then $(m_v)_{v\in\mathbb{N}}$ and $(n_v)_{v\in\mathbb{N}}$ are chosen such that m_v is infinitesimal $\forall\, v \in \mathbb{N}$ in the sense that,

$$\mathscr{D}(H^{m_v} p_0, H^{n_v} p_0) \geq \epsilon \text{ but } \mathscr{D}(H^r p_0, H^{n_v} p_0) < \epsilon\; \forall\, r \in \{n_v + 1, ..., m_v - 1\}.$$

Now it can be written that,

$$\begin{aligned}\epsilon &< \mathscr{D}(H^{m_v} p_0, H^{n_v} p_0)\\ &\leq \mathscr{D}(H^{m_v} p_0, H^{m_v-1} p_0) + \mathscr{D}(H^{m_v-1} p_0, H^{n_v} p_0)\\ &< \mathscr{D}(H^{m_v} p_0, H^{m_v-1} p_0) + \epsilon\end{aligned}$$

which tends to $\epsilon+$ as v tends to $+\infty$.
Further $\exists\, \mu \in \mathbb{N}$ such that $\forall\, v > \mu$ $(v \in \mathbb{N})$ one can get $\mathscr{D}(H^{m_v+1} p_0, H^{n_v+1} p_0) < \epsilon$. Actually, if $\exists\, (v_k)_{n\in\mathbb{N}} \subseteq \mathbb{N} : \mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) \geq \epsilon$, it can be written that,

$$\begin{aligned}\epsilon &\leq \mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0)\\ &\leq \mathscr{D}(H^{m_{v_k}+1} p_0, H^{m_{v_k}} p_0) + \mathscr{D}(H^{m_{v_k}} p_0, H^{n_{v_k}} p_0) + \mathscr{D}(H^{n_{v_k}} p_0, H^{n_{v_k}+1} p_0)\end{aligned}$$

which tends to ϵ as k tends to $+\infty$.
Using Equation 3.1, it can be written that,

$$\int_0^{\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0)} \eta_0(\tau_0) d\tau_0 \leq c \int_0^{\mathscr{D}(H^{m_{v_k}} p_0, H^{n_{v_k}} p_0)} \eta_0(\tau_0) d\tau_0$$

Taking $k \to +\infty$ in both sides of the above inequality, one can obtain the contradiction $\int_0^\epsilon \eta_0(\tau_0) d\tau_0 \leq c \int_0^\epsilon \eta_0(\tau_0) d\tau_0$ [$\because 0 < c < 1$ & the integral is +ve]. Therefore for a particular $\mu \in \mathbb{N},\; \mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) < \epsilon\; \forall\, v \in \mu$. Finally it is required to prove that, $\exists\, s_\epsilon \in (0, \epsilon)$ and $v_\epsilon \in \mathbb{N}$ such that for every $v > v_\epsilon$ $(v \in \mathbb{N})$, one can obtain $\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) < \epsilon - s_\epsilon$.
Now let (v_k) exist such that $\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0) \to \epsilon-$ as $k \to \infty$. Then by taking $k \to \infty$ in,

$$\int_0^{\mathscr{D}(H^{m_{v_k}+1} p_0, H^{n_{v_k}+1} p_0)} \eta_0(\tau_0) d\tau_0 \leq c \int_0^{\mathscr{D}(H^{m_{v_k}} p_0, H^{n_{v_k}} p_0)} \eta_0(\tau_0) d\tau_0,$$

One can again obtain the contradiction $\int_0^\epsilon \eta_0(\tau_0) d\tau_0 \leq c \int_0^\epsilon \eta_0(\tau_0) d\tau_0$ [Since, $c \in (0, 1)$ and the integral is positive]. So, for every natural number $v > v_\epsilon$ (v_ϵ as above),

$$\begin{aligned}\epsilon &\leq \mathscr{D}(H^{m_v} p_0, H^{n_v} p_0)\\ &\leq \mathscr{D}(H^{m_v} p_0, H^{m_v+1} p_0) + \mathscr{D}(H^{m_v+1} p_0, H^{n_v+1} p_0) + \mathscr{D}(H^{n_v+1} p_0, H^{n_v} p_0)\\ &\leq \mathscr{D}(H^{m_v} p_0, H^{m_v+1} p_0) + (\epsilon - s_\epsilon) + \mathscr{D}(H^{n_v+1} p_0, H^{n_v} p_0)\end{aligned}$$

Which tends to $\epsilon - s_\epsilon$ as $v \to \infty$. Thus $\epsilon \leq \epsilon - s_\epsilon$, which is a contradiction.
Hence $(H^n p_0)_{n\in\mathbb{N}}$ is R-Cauchy $\forall\, p_0 \in S$.
Now by R-completeness of S it can be said that, $\{p_n\}$ R-converges to some $t \in S$. And since H is R-continuous, Hp_n R-converges to Ht. Therefore,

$$\begin{aligned}Ht &= H\left(\lim_{n\to\infty} p_n\right)\\ &= \lim_{n\to\infty} Hp_n\\ &= \lim_{n\to\infty} p_{n+1} = t\end{aligned}$$

Hence t is a fixed point of H.
For uniqueness, let $t' \in S$ be another fixed point of H. As a result, $p_0 R t'$.
Hence $(H^n p_0 = p_n) R (H^n t' = t')\ \forall\ n \in \mathbb{W}$. So by the virtue of triangular inequality it can be written that,

$$\begin{aligned}\int_0^{\mathscr{D}(t,t')} \eta_0(\tau_0)d\tau_0 &\leq \int_0^{\mathscr{D}(t,H^n p_0)} \eta_0(\tau_0)d\tau_0 + \int_0^{\mathscr{D}(t',H^n p_0)} \eta_0(\tau_0)d\tau_0 \\ &\leq \int_0^{\mathscr{D}(H^n t,H^n p_0)} \eta_0(\tau_0)d\tau_0 + \int_0^{\mathscr{D}(H^n t',H^n p_0)} \eta_0(\tau_0)d\tau_0 \\ &\leq c^n \int_0^{\mathscr{D}(t,p_0)} \eta_0(\tau_0)d\tau_0 + c^n \int_0^{\mathscr{D}(t',p_0)} \eta_0(\tau_0)d\tau_0\end{aligned}$$

Which tends to 0 as $n \to \infty$. [Since, $c < 1$]
As a result, $t = t'$. Hence, the fixed point is unique. ∎

Lemma 3.4. *The mapping defined in Theorem 3.3 is a picard operator.*

Proof. Let $p \in S$ be arbitrary. Since $p_0 R (Hp)$ and H is R-preservative, it can be written that $(H^n p_0) R (H^{n+1} p)\ \forall\ n \in \mathbb{W}$.
Therefore,

$$\begin{aligned}\int_0^{\mathscr{D}(t,H^n p)} \eta_0(\tau_0)d\tau_0 &= \int_0^{\mathscr{D}(H^n t,H^n p)} \eta_0(\tau_0)d\tau_0 \\ &\leq \int_0^{\mathscr{D}(H^{n-1} t,H^{n-1} p_0)} \eta_0(\tau_0)d\tau_0 + \int_0^{\mathscr{D}(H^{n-1} p_0,H^{n-1}(Hp))} \eta_0(\tau_0)d\tau_0 \\ &\leq c^{n-1} \int_0^{\mathscr{D}(t,p_0)} \eta_0(\tau_0)d\tau_0 + c^{n-1} \int_0^{\mathscr{D}(p_0,Hp)} \eta_0(\tau_0)d\tau_0\end{aligned}$$

Which tends to 0 as $n \to \infty$. [Since, $c < 1$]

$$\text{Hence, } \lim_{n\to\infty} H^n p = t$$

∎

Remark 3.3. By putting $\alpha = 0$, Theorem 3.3 can be reduced to Theorem 3.1.

Corollary 3.5. *Let $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0)d\tau_0 \leq \beta \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0)d\tau_0 + \gamma \int_0^{\mathscr{D}(Hq,q)} \eta_0(\tau_0)d\tau_0$$

$\forall\ p, q \in S$ where $p \neq q$ & $\beta + \gamma < 1$, given $\eta_0(\tau_0)$ is subadditive $\forall\ [m, n] \subset \mathbb{R}^+ \cup \{0\}$. Also let $\exists\ p_0 \in S$ such that $p_0 R q\ \forall\ q \in H(S)$. Then H has an unique fixed point in $(S, \mathscr{D}, R)$.

Corollary 3.6. *Let $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0)d\tau_0 \leq \alpha \int_0^{\mathscr{D}(Hp,p)} \eta_0(\tau_0)d\tau_0 + \beta \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0)d\tau_0$$

$\forall\ p, q \in S$ where $p \neq q$ & $\alpha + \beta < 1$, given $\eta_0(\tau_0)$ is subadditive $\forall\ [m, n] \subset \mathbb{R}^+ \cup \{0\}$. Also let $\exists\ p_0 \in S$ such that $p_0 R q\ \forall\ q \in H(S)$. Then H has an unique fixed point in $(S, \mathscr{D}, R)$.

Corollary 3.7. *Let $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \alpha \int_0^{\mathscr{D}(Hp,p)+\mathscr{D}(Hq,q)} \eta_0(\tau_0) d\tau_0$$

$\forall\, p, q \in S$ where $p \neq q$ & $\alpha < 1/2$, given $\eta_0(\tau_0)$ is subadditive $\forall\, [m, n] \subset \mathbb{R}^+ \cup \{0\}$. Also let $\exists\, p_0 \in S$ such that $p_0 R q\ \forall\, q \in H(S)$. Then H has an unique fixed point in $(S, \mathscr{D}, R)$.

Remark 3.4. By the application of Remark 3.1, the contractive condition in Corollary 3.7 reduces to,

$$\mathscr{D}(Hp, Hq) \leq \alpha\, [\mathscr{D}(Hp, p) + \mathscr{D}(Hq, q)]$$

$\forall\, p, q \in S$ where $p \neq q$, and $\alpha < 1/2$. Which can also be recognised as a fixed point theorem of Kannan type in R-metric space.

Example 3.2. *Let $S = [0, 1]$ and $H : S \to S$ be a mapping defined as, $H(x) = [\frac{e^x}{4}]\ \forall\, x \in S$. Suppose the relation R be defined as $x + y \in \{x, y\}$. Then it cab be easily confirmed that (S, d, R) is an R-complete $R - M_S$ where $d = |x - y|$, & H is R-continuous and R-preserving. Also it is taken that $\eta(\tau) = \frac{1}{4+t^2}\ \forall\, t \in S$ and $\alpha = 0.49,\ \beta = 0.01$. Clearly, $2\alpha + \beta = 0.99 < 1$ & $\eta(\tau)$ is Lesbegue integrable and subadditive in S.*
Then, by further calculations it can be verified that,

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \alpha \int_0^{\mathscr{D}(Hp,p)+\mathscr{D}(Hq,q)} \eta_0(\tau_0) d\tau_0 + \beta \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0) d\tau_0,\ \forall\, p, q \in S.$$

Thus, $H(x)$ & $\eta(\tau)$ fulfills all the conitions of Theorem 3.3. And in this case, $x = 0$ is a fixed point of the mapping $H(x)$, where $x \in S$.

Theorem 3.8. *Let $(S, \mathscr{D}, R)$ be a R-metric space which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition, Let $(S, \mathscr{D}, R)$ be a $R - M_S$ which is R-complete, and $H : S \to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0) d\tau_0 \leq \gamma \int_0^{\mathscr{D}(Hp,q)+\mathscr{D}(Hq,p)} \eta_0(\tau_0) d\tau_0 + \delta \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0) d\tau_0$$

$\forall\, p, q \in S$ ($p \neq q$), where $2\gamma + \delta < 1$ and $\eta(\tau)$ is subadditive $\forall\, [m, n] \subset \mathbb{R}^+ \cup \{0\}$. Also let $\exists\, p_0 \in S$ such that $p_0 R q\ \forall\, q \in H(S)$. Then H has a unique fixed point in $(S, \mathscr{D}, R)$.

Proof. Let $p_0 \in S$ be arbitary. Define by induction $H^n : S \to S$ by $H^0 p_0 = p_0$, $H^1 p_0 = p_1$, $H^2 p_0 = H(Hp_0) = Hp_1 = p_2$, ... ,$H^n p_0 = p_n$ where $n \geq 0$. Thus an iterative sequence $\{p_n\}$ associated with H is obtained.
Let $m, n \in \mathbb{N}$ and $n < m$, substitute $m - n = k$.
We have $p_0 R H^k(p_0)$. Since H is R-preserving, $[H^n(p_0)]\, R\, \left[H^{n+k}(p_0)\right]$. Which implies, $p_n R p_m$.

Hence $\{p_n\}$ is a R-sequence. Now from the R-contractive condition of H it can be said that,

$$\begin{aligned}\int_0^{\mathscr{D}(p_n,p_{n+1})}\eta_0(\tau_0)d\tau_0 &= \int_0^{\mathscr{D}(H^n p_0,H^{n+1}p_0)}\eta_0(\tau_0)d\tau_0\\ &\leq \gamma\int_0^{\mathscr{D}(H^n p_0,H^n p_0)+\mathscr{D}(H^{n+1}p_0,H^{n-1}p_0)}\eta_0(\tau_0)d\tau_0\\ &\qquad + \delta\int_0^{\mathscr{D}(H^{n-1}p_0,H^n p_0)}\eta_0(\tau_0)d\tau_0\\ &\leq \gamma\int_0^{\mathscr{D}(H^{n+1}p_0,H^n p_0)}\eta_0(\tau_0)d\tau_0 + \gamma\int_0^{\mathscr{D}(H^{n-1}p_0,H^n p_0)}\eta_0(\tau_0)d\tau_0\\ &\quad + \delta\int_0^{\mathscr{D}(H^{n-1}p_0,H^n p_0)}\eta_0(\tau_0)d\tau_0\\ &\leq (\gamma+\delta)\int_0^{\mathscr{D}(H^{n-1}p_0,H^n p_0)}\eta_0(\tau_0)d\tau_0 + \gamma\int_0^{\mathscr{D}(H^n p_0,H^{n+1}p_0)}\eta_0(\tau_0)d\tau_0\end{aligned}$$

$$(3.2) \qquad \Longrightarrow \int_0^{\mathscr{D}(H^n p_0,H^{n+1}p_0)}\eta_0(\tau_0)d\tau_0 \leq \left[\frac{\gamma+\delta}{1-\gamma}\right]\int_0^{\mathscr{D}(H^{n-1}p_0,H^n p_0)}\eta_0(\tau_0)d\tau_0$$

Continuing in this way,

$$\int_0^{\mathscr{D}(H^n p_0,H^{n+1}p_0)}\eta_0(\tau_0)d\tau_0 \leq c^n\int_0^{\mathscr{D}(p_0,Hp_0)}\eta_0(\tau_0)d\tau_0$$

Where,

$$c=\left[\frac{\gamma+\delta}{1-\gamma}\right]<1.$$

As a result, since $0<c<1$,

$$\int_0^{\mathscr{D}(H^n p_0,H^{n+1}p_0)}\eta_0(\tau_0)d\tau_0 \to 0 \text{ as } n\to\infty.$$

Or,

$$\mathscr{D}(H^n p_0,H^{n+1}p_0)\to 0 \text{ as } n\to\infty.$$

Now let,

$$\lim_{n\to\infty}\sup\mathscr{D}(H^n p_0,H^{n+1}p_0)=\epsilon>0$$

Then $\exists\, v_\epsilon\in\mathbb{N}$ and a sequence $(H^{n_v}p_0)_{v\geq v_\epsilon}:\mathscr{D}(H^{n_v}p_0,H^{n_v+1}p_0)\to\epsilon>0$ as $v\to+\infty$ and $\mathscr{D}(H^{n_v}p_0,H^{n_v+1}p_0)\geq\epsilon/2\ \forall\, v\geq v_\epsilon$. Thus we have the following contradiction:

$$0=\lim_{v\to\infty}\int_0^{\mathscr{D}(H^{n_v}p_0,H^{n_v+1}p_0)}\eta_0(\tau_0)d\tau_0 \geq \int_0^{\epsilon/2}\eta_0(\tau_0)d\tau_0$$

Now it has to be proved that, $(H^n p_0)_{n\in\mathbb{N}}$ is R-Cauchy $\forall\, p_0\in S$. Or,

$$\forall\,\epsilon>0\ \exists\, v_\epsilon\in\mathbb{N}:\forall\, m,n\in\mathbb{N}, m>n>v_\epsilon;\ \mathscr{D}(H^m p_0,H^n p_0)<\epsilon.$$

Let $\exists\,\epsilon>0:\forall\, v\in\mathbb{N}\ \exists\, m_v,n_v\in\mathbb{N}$ such that $\mathscr{D}(H^{m_v}p_0,H^{n_v}p_0)\geq\epsilon$ where $m_v>n_v>v$. Then $(m_v)_{v\in\mathbb{N}}$ and $(n_v)_{v\in\mathbb{N}}$ are chosen so that m_v is infinitesimal $\forall\, v\in\mathbb{N}$ in the sense that,

$$\mathscr{D}(H^{m_v}p_0,H^{n_v}p_0)\geq\epsilon \text{ but } \mathscr{D}(H^r p_0,H^{n_v}p_0)<\epsilon\ \forall\, r\in\{n_v+1,...,m_v-1\}.$$

Now it can be written that,

$$\begin{aligned}\epsilon &< \mathscr{D}(H^{m_v}p_0, H^{n_v}p_0)\\ &\le \mathscr{D}(H^{m_v}p_0, H^{m_v-1}p_0) + \mathscr{D}(H^{m_v-1}p_0, H^{n_v}p_0)\\ &< \mathscr{D}(H^{m_v}p_0, H^{m_v-1}p_0) + \epsilon\end{aligned}$$

which tends to $\epsilon+$ as v tends to $+\infty$.
Further $\exists\ \mu \in \mathbb{N}$ such that $\forall\ v > \mu\ (v \in \mathbb{N})$ one can get $\mathscr{D}(H^{m_v+1}p_0, H^{n_v+1}p_0) < \epsilon$. Actually, if $\exists\ (v_k)_{n\in\mathbb{N}} \subseteq \mathbb{N} : \mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0) \ge \epsilon$, it can be written that,

$$\begin{aligned}\epsilon &\le \mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0)\\ &\le \mathscr{D}(H^{m_{v_k}+1}p_0, H^{m_{v_k}}p_0) + \mathscr{D}(H^{m_{v_k}}p_0, H^{n_{v_k}}p_0) + \mathscr{D}(H^{n_{v_k}}p_0, H^{n_{v_k}+1}p_0)\end{aligned}$$

which tends to ϵ as k tends to $+\infty$.
Using Equation 3.2, it can be written that,

$$\int_0^{\mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0)} \eta_0(\tau_0)d\tau_0 \le c\int_0^{\mathscr{D}(H^{m_{v_k}}p_0, H^{n_{v_k}}p_0)} \eta_0(\tau_0)d\tau_0$$

Taking $k \to +\infty$ in both sides of the above inequality, one can obtain the contradiction $\int_0^\epsilon \eta_0(\tau_0)d\tau_0 \le c\int_0^\epsilon \eta_0(\tau_0)d\tau_0$ [$\because 0 < c < 1$ & the integral is +ve]. Therefore for a particular $\mu \in \mathbb{N},\ \mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0) < \epsilon\ \forall\ v \in \mu$. Finally it is required to prove that, $\exists\ s_\epsilon \in (0,\epsilon)$ and $v_\epsilon \in \mathbb{N}$ such that for every $v > v_\epsilon\ (v \in \mathbb{N})$, one can obtain $\mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0) < \epsilon - s_\epsilon$.
Now let (v_k) exist such that $\mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0) \to \epsilon-$ as $k \to \infty$. Then by taking $k \to \infty$ in,

$$\int_0^{\mathscr{D}(H^{m_{v_k}+1}p_0, H^{n_{v_k}+1}p_0)} \eta_0(\tau_0)d\tau_0 \le c\int_0^{\mathscr{D}(H^{m_{v_k}}p_0, H^{n_{v_k}}p_0)} \eta_0(\tau_0)d\tau_0,$$

One can again obtain the contradiction $\int_0^\epsilon \eta_0(\tau_0)d\tau_0 \le c\int_0^\epsilon \eta_0(\tau_0)d\tau_0$ [$\because 0 < c < 1$ & the integral is +ve]. So, for every natural number $v > v_\epsilon$ (v_ϵ as above),

$$\begin{aligned}\epsilon &\le \mathscr{D}(H^{m_v}p_0, H^{n_v}p_0)\\ &\le \mathscr{D}(H^{m_v}p_0, H^{m_v+1}p_0) + \mathscr{D}(H^{m_v+1}p_0, H^{n_v+1}p_0) + \mathscr{D}(H^{n_v+1}p_0, H^{n_v}p_0)\\ &\le \mathscr{D}(H^{m_v}p_0, H^{m_v+1}p_0) + (\epsilon - s_\epsilon) + \mathscr{D}(H^{n_v+1}p_0, H^{n_v}p_0)\end{aligned}$$

Which tends to $\epsilon - s_\epsilon$ as $v \to \infty$. Thus $\epsilon \le \epsilon - s_\epsilon$, which is a contradiction.
Hence $(H^n p_0)_{n\in\mathbb{N}}$ is R-Cauchy $\forall\ p_0 \in S$.
Now by R-completeness of S it can be said that, $\{p_n\}$ R-converges to some $t \in S$. And since H is R-continuous, Hp_n R-converges to Ht. Therefore,

$$\begin{aligned}Ht &= H\left(\lim_{n\to\infty} p_n\right)\\ &= \lim_{n\to\infty} Hp_n\\ &= \lim_{n\to\infty} p_{n+1} = t\end{aligned}$$

Hence t is a fixed point of H.
For uniqueness, let $t' \in S$ be another fixed point of H. As a result, $p_0 R t'$.
Hence $(H^n p_0 = p_n) R (H^n t' = t')\ \forall\ n \in \mathbb{W}$. So by the virtue of triangular inequality it can be

written that,

$$\begin{aligned}\int_0^{\mathscr{D}(t,t')}\eta_0(\tau_0)d\tau_0 &\leq \int_0^{\mathscr{D}(t,H^np_0)}\eta_0(\tau_0)d\tau_0+\int_0^{\mathscr{D}(t',H^np_0)}\eta_0(\tau_0)d\tau_0\\ &\leq \int_0^{\mathscr{D}(H^nt,H^np_0)}\eta_0(\tau_0)d\tau_0+\int_0^{\mathscr{D}(H^nt',H^np_0)}\eta_0(\tau_0)d\tau_0\\ &\leq r^n\int_0^{\mathscr{D}(t,p_0)}\eta_0(\tau_0)d\tau_0+r^n\int_0^{\mathscr{D}(t',p_0)}\eta_0(\tau_0)d\tau_0\end{aligned}$$

Which tends to 0 as $n\to\infty$. [Since, $r<1$]
As a result, $t=t'$. Hence, the fixed point is unique. ∎

Lemma 3.9. *The mapping defined in Theorem 3.8 is a picard operator.*

Proof. Let $p\in S$ be arbitrary. Since $p_0R(Hp)$ and H is R-preservative, it can be written that $(H^np_0)R(H^{n+1}p)\;\forall\, n\in\mathbb{W}$.
Therefore,

$$\begin{aligned}\int_0^{\mathscr{D}(t,H^np)}\eta_0(\tau_0)d\tau_0 &= \int_0^{\mathscr{D}(H^nt,H^np)}\eta_0(\tau_0)d\tau_0\\ &\leq \int_0^{\mathscr{D}(H^{n-1}t,H^{n-1}p_0)}\eta_0(\tau_0)d\tau_0+\int_0^{\mathscr{D}(H^{n-1}p_0,H^{n-1}(Hp))}\eta_0(\tau_0)d\tau_0\\ &\leq r^{n-1}\int_0^{\mathscr{D}(t,p_0)}\eta_0(\tau_0)d\tau_0+r^{n-1}\int_0^{\mathscr{D}(p_0,Hp)}\eta_0(\tau_0)d\tau_0\end{aligned}$$

Which tends to 0 as $n\to\infty$. [Since, $r<1$]

$$\text{Hence, } \lim_{n\to\infty}H^np=t$$

∎

Remark 3.5. Putting $\gamma=0$, Theorem 3.8 can be reduced to Theorem 3.1.

Corollary 3.10. *Let $(S,\mathscr{D},R)$ be a $R-M_S$ which is R-complete, and $H:S\to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)}\eta_0(\tau_0)d\tau_0\leq\gamma\int_0^{\mathscr{D}(Hq,p)}\eta_0(\tau_0)d\tau_0+\delta\int_0^{\mathscr{D}(p,q)}\eta_0(\tau_0)d\tau_0$$

$\forall\, p,q\in S$ *where* $p\neq q$ *&* $2\gamma+\delta<1$, *given* $\eta(\tau)$ *is subadditive* $\forall\,[m,n]\subset\mathbb{R}^+\cup\{0\}$. *Also let* $\exists\, p_0\in S$ *such that* $p_0Rq\;\forall\, q\in H(S)$. *Then H has an unique fixed point in* $(S,\mathscr{D},R)$.

Corollary 3.11. *Let $(S,\mathscr{D},R)$ be a $R-M_S$ which is R-complete, and $H:S\to S$ be a R-preserving, R-continuous self map that satisfies the integral type R-contractive condition,*

$$\int_0^{\mathscr{D}(Hp,Hq)}\eta_0(\tau_0)d\tau_0\leq\gamma\int_0^{\mathscr{D}(Hp,q)+\mathscr{D}(Hq,p)}\eta_0(\tau_0)d\tau_0$$

$\forall\, p,q\in S$ *where* $p\neq q$ *&* $\gamma<1/2$, *given* $\eta(\tau)$ *is subadditive* $\forall\,[m,n]\subset\mathbb{R}^+\cup\{0\}$. *Also let* $\exists\, p_0\in S$ *such that* $p_0Rq\;\forall\, q\in H(S)$. *Then H has an unique fixed point in* $(S,\mathscr{D},R)$.

Remark 3.6. By the application of Remark 3.1, the contractive condition in Corollary 3.11 reduces to,

$$\mathscr{D}(Hp,Hq)\leq\gamma\left[\mathscr{D}(Hp,q)+\mathscr{D}(Hq,p)\right]$$

$\forall\, p,q\in S$ where $p\neq q$, and $\gamma<1/2$. Which can also be recognised as a fixed point theorem of Chatterjea type in R-metric space.

Example 3.3. *Let $S = [1, 3]$ and $H : S \to S$ be a mapping defined as, $H(x) = |\sqrt{x}|\ \forall\ x \in S$. Suppose the relation R be defined as $x + y \in \{x + 1, y + 1\}$. Then can be easily verified that (S, d, R) is an R-complete $R - M_S$ where $d = |x^4 - y^4|$, & H is R-continuous and R-preserving. Also it is taken that $\eta(\tau) = \frac{1}{36+t^2}\ \forall\ t \in S$ and $\gamma = 0.49,\ \delta = 0.01$. Clearly, $2\gamma + \delta = 0.99 < 1$ & $\eta(\tau)$ is Lesbegue integrable and subadditive in S.*
Then, by further calculations it can be verified that,

$$\int_0^{\mathscr{D}(Hp,Hq)} \eta_0(\tau_0)d\tau_0 \leq \gamma \int_0^{\mathscr{D}(Hp,q)+\mathscr{D}(Hq,p)} \eta_0(\tau_0)d\tau_0 + \delta \int_0^{\mathscr{D}(p,q)} \eta_0(\tau_0)d\tau_0,\ \forall\ p, q \in S.$$

Thus, $H(x)$ & $\eta(\tau)$ fulfills all the conitions of Theorem 3.8. And in this case, $x = 1$ is a fixed point of the mapping $H(x)$, where $x \in S$.

4. APPLICATION TO INTEGRAL EQUATIONS

In this section, the outcomes of Therorem 3.3 and Theorem 3.8 are used to demonstrate the existence and uniqueness of solution of the integral equation given below.

$$n(r) = z(r) + \int_\kappa^\rho \mathfrak{M}(r, \mu, n(\mu))d\mu \tag{4.1}$$

where $\rho, \kappa \in \mathbb{R} : \kappa \leq \rho$, $n \in C[\kappa, \rho]$ (the set of all continuous real functions on the interval $[\kappa, \rho]$). Also $z : [\kappa, \rho] \to [\kappa, \rho]$ and $M : [\kappa, \rho]^3 \to [\kappa, \rho]$ are functions which are continuous.

Let $S = C[\kappa, \rho]$ be endowed with the metric defined as,

$$\mathscr{D}(n, q) = \sup_{r\in[\kappa,\rho]} |n(r) - q(r)|,\ \text{for any } n, q \in S.$$

And suppose, $H : S \to S$ be defined as,

$$Hn(r) = z(r) + \int_\kappa^\rho \mathfrak{M}(r, \mu, n(\mu))d\mu,\ \text{where } r \in [\kappa, \rho]. \tag{4.2}$$

So, evidently $(S, \mathscr{D})$ is a metric space which is complete. Now let R be a relation defined in S such that the R-Metric space $(S, \mathscr{D}, R)$ is R-complete, R-continuous and R-preserving.

It is to be noted that, the solution of Equation 4.1 is similar to the fixed point of H in Equation 4.2. Accordingly, the following result has been deduced.

Theorem 4.1. *The integral equation 4.1 has a unique solution if $\exists\ \rho > 0$ such that,*

$$|\mathfrak{M}(r, \mu, n(\mu)) - \mathfrak{M}(r, \mu, q(\mu))| \leq \frac{\gamma}{(\rho - \kappa)}[|Hn - q| + |Hq - n|].$$

where γ is a constant in the interval $(0, 1/2)$.

Proof. It is known that,

$$\begin{aligned} &|Hn - Hq| \\ &= \left| \int_\kappa^\rho \mathfrak{M}(r, \mu, n(\mu))d\mu - \mathfrak{M}(r, \mu, q(\mu))d\mu \right| \\ &\leq \int_\kappa^\rho \big|\mathfrak{M}(r, \mu, n(\mu)) - \mathfrak{M}(r, \mu, q(\mu))\big| d\mu \\ &\leq \frac{\gamma}{(\rho - \kappa)} \int_\kappa^\rho [|Hn - q| + |Hq - n|]d\mu \\ &= \gamma[|Hn - q| + |Hq - n|]. \end{aligned}$$

Therefore,

$$|Hn - Hq| \leq \gamma[|Hn - q| + |Hq - n|], \text{ where } \gamma < 1/2. \tag{4.3}$$

Now taking supremum on both sides in the above equation,

$$\begin{aligned}\mathscr{D}(Hn, Hq) &= \sup_{r\in[\kappa,\rho]} |Hn - Hq| \\ &\leq \sup_{r\in[\kappa,\rho]} \left[\gamma\left\{|Hn - q| + |Hq - n|\right\}\right] \\ &= \gamma\left[\sup_{r\in[\kappa,\rho]} |Hn - q| + \sup_{r\in[\kappa,\rho]} |Hq - n|\right] \\ &= \gamma\left[\mathscr{D}(Hn, q) + \mathscr{D}(Hq, n)\right].\end{aligned}$$

It implies that,

$$\mathscr{D}(Hn, Hq) \leq \gamma\left[\mathscr{D}(Hn, q) + \mathscr{D}(Hq, n)\right],$$

which means that, the mapping H fulfills all the requirements of Remark 3.6. Therefore the existence of fixed point is guaranteed. Hence $\exists\ n(r) \in C[\kappa, \rho] : n(r) = Hn(r)$, which is a unique solution of Equation 4.1. ∎

Theorem 4.2. *The integral equation 4.1 has a unique solution if* $\exists\ \rho > 0$ *such that,*

$$|\mathfrak{M}(r, \mu, n(\mu)) - \mathfrak{M}(r, \mu, q(\mu))| \leq \frac{\alpha}{(\rho - \kappa)}[|Hn - n| + |Hq - q|].$$

where α *is a constant in the interval* $(0, 1/2)$.

Proof. It is known that,

$$\begin{aligned}&|Hn - Hq| \\ &= \left|\int_\kappa^\rho \mathfrak{M}(r, \mu, n(\mu))d\mu - \mathfrak{M}(r, \mu, q(\mu))d\mu\right| \\ &\leq \int_\kappa^\rho \big|\mathfrak{M}(r, \mu, n(\mu)) - \mathfrak{M}(r, \mu, q(\mu))\big| d\mu \\ &\leq \frac{\alpha}{(\rho - \kappa)} \int_\kappa^\rho [|Hn - n| + |Hq - q|]d\mu \\ &= \alpha[|Hn - n| + |Hq - q|].\end{aligned}$$

Therefore,

$$|Hn - Hq| \leq \alpha[|Hn - n| + |Hq - q|], \text{ where } \alpha < 1/2. \tag{4.4}$$

Now taking supremum on both sides in the above equation,

$$\begin{aligned}\mathscr{D}(Hn, Hq) &= \sup_{r\in[\kappa,\rho]} |Hn - Hq| \\ &\leq \sup_{r\in[\kappa,\rho]} \left[\alpha\left\{|Hn - n| + |Hq - q|\right\}\right] \\ &= \alpha\left[\sup_{r\in[\kappa,\rho]} |Hn - n| + \sup_{r\in[\kappa,\rho]} |Hq - q|\right] \\ &= \alpha\left[\mathscr{D}(Hn, n) + \mathscr{D}(Hq, q)\right].\end{aligned}$$

It implies that,

$$\mathscr{D}(Hn, Hq) \leq \alpha\left[\mathscr{D}(Hn, n) + \mathscr{D}(Hq, q)\right],$$

which means that, the mapping H fulfills all the requirements of Remark 3.4. Therefore the existence of fixed point is guaranteed. Hence $\exists\ n(r) \in C[\kappa, \rho] : n(r) = Hn(r)$, which is a unique solution of Equation 4.1. ∎

5. CONCLUSION

$R - M_S$ is a generalization of M_S and unlike metric Space it uses the concepts of relations combined with the set and a distance metric $\mathscr{D}$'.

The current research fulfills the objective by establishing three invariant point results in $R - M_S$, one for an integral type R-contractive condition of Banach type, and two other for integral type contractions combining Kannan and Chatterjea type with Banach type respectively. For this purpose, some other concepts such as R-preservation, R-continuity, R-convergence, R-sequence, R-Cauchy sequence, and R-contraction are also brought to use.

Also, some illustrations and applications have been established, in support of the results deduced. Notably, the obtained results can also be generalized for rough M_S, soft M_S, G-M_S and neutrosophic fuzzy M_S.

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