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ON THE PATHWISE UNIQUENESS OF SOLUTIONS TO STOCHASTIC INTEGRAL EQUATIONS OF ITÔ TYPE

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ABSTRACT. We give sufficient conditions for the pathwise uniqueness of solutions to nonlinear stochastic integral equations of Itô type. The result concerns a relaxation of the classical Lipschitz condition by allowing for a Nagumo-type fast-growing time dependence as the initial time is approached. We also propose a special subclass of functions $\mathcal{N} \subset \mathcal{M}$ which shows that our considerations go beyond the classical contraction case. Moreover, they assure the facilities for to prove the existence and uniqueness of solutions and for the existence of the fixed points for some class of operators associated to stochastic integral equations.

Key words and phrases: stochastic integral equations of Itô type; pathwise uniqueness of the solutions; Nagumo-type results; Lipschitz's inequality; Gronwall's inequality; Cauchy-Schwarz inequality.

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1. INTRODUCTION

We consider a stochastic integral equation of Itô type

$$X(t,\omega) = X_0(\omega) + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dW(s,\omega),\ t \in [0,a], \tag{1.1}$$

$a > 0$, $f, g : [0,a] \times \mathbb{R}^n$ are continuous functions, $X(t,\omega)$ is, the unknown, stochastic process and $W(t,\omega)$ is a n-dimensional $\mathcal{F}_t$-Brownian motion on the probability space $(\Omega, \mathcal{K}, P, \mathcal{F})$ where $\mathcal{F} = \{\mathcal{F}_t,\ t \in [0,a]\}$, $\mathcal{F}_s \subset \mathcal{F}_t$, $0 \le s \le t \le a$ is a non-anticipating family of sub-σ-algebras of $\mathcal{K}$ with respect to n-dimensional Browninan motion $W(t,\omega)$. For $t \in [0,a]$ we denote by $\mathcal{F}_t$ the smallest σ-algebra with respect to which X_0 and random vector $W(s,\omega)_{0\le s\le t}$ are measurable. To simplify the notation, we do not write out explicitly the dependence of a stochastic process on $\omega \in \Omega$. We also tacitely assume the separability of all stochastic processes that we consider in the paper.

We are concerned with the stochastic integral equation (1.1) where X_0 is $\mathcal{F}_0$-measurable, $\mathbb{R}^n$-valued function independent of $W(t)$, $0 \le t \le a$ and $E(|X_0|^2 = \int_\Omega |X_0(s)|^2 dP < \infty$ and the second integral is in the sense of Itô ([15]). Here $f(t,x)$ is a $\mathbb{R}^n$-valued measurable function defined on $[0,a] \times \mathbb{R}^n$ and $g(t,x)$ is a $n \times n$ matrix-valued continuous function on $[0,a] \times \mathbb{R}^n$, Borel measurable on $[0,a] \times \mathbb{R}^n$.

An a.s. continuous stochastic process $X : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is called a solution to (1.1) if (see [2])

i.) $X(t)$ is $\mathcal{F}_t$-measurable (i.e. $\mathcal{F}_t$-adapted) stochastic process for $t \in [0,a]$;

ii.) a.s. we have

$$\int_0^t \left[|f(s,X(s)| + |g(s,X(s)|\right]^2 ds < \infty$$

so that the Riemann integral $\int_0^t f(s,X(s))ds$ and the Itô stochastic integral $\int_0^t g(s,X(s))dW(s)$ are well defined;

iii.) the stochastic integral equation of Itô type (1.1) holds a.s. for every $t \in [0,a]$.

The existence and uniqueness of a stochastic process X solving (1.1) is known to be guaranteed under conditions that the drift coefficient $f(t,x)$ and the diffusion coefficient $g(t,x)$ are continuous and satisfy a Lipschitz condition in the x-variable (see [2, 11, 16, 18]). The uniqueness holds in the strong sense of pathwise uniqueness, in the sense that if $X(t)$ and $Y(t)$ are two solutions with $X(0,\cdot) = Y(0,\cdot) = X_0$ a.s. , then

$$P\left(\sup_{0\le t\le a} |X(t,\cdot) - Y(t,\cdot)| > 0\right) = 0 .$$

Note that the pathwise uniqueness ensures that solution are also unique in the law sense, but pathwise uniqueness and law uniqueness are not equivalent (see [27],[29]). It is well known that equations of type (1.1) are important in the formulation and analysis of the stochastic models in engineering, in the theory of automatic systems and numerous other domains of biology, physics, life sciences (see [9, 11]). The pathwise uniqueness is of particular interest for applications in the mathematical finances (see [26]) such as Black-Scholes ([4]) model or Cox-Ingersoll-Ross model (see [10]) which describes stochastic evolution of interest rate $\{r_t\}_{t\ge 0}$ by the stochastic integral equations.

The pathwise uniqueness of solutions is important in establishing the validity of a given stochastic model for a real problem, because without uniqueness the system of equations and its solutions cannot be used to make predictions about the behaviour of physical systems. The uniqueness implies continuous dependence on initial data which is useful on obtaining conditions for continuation and periodicity of solution (see [13]).

Moreover, it is known (see [27], [28], [29]) that in order to define a diffusion process through a solution of the stochastic differential equation, it is sufficient to verify the uniqueness in the sense of the probability law of solution. But, the pathwise uniqueness implies the uniqueness in the sense of probability law and thus, the solution defines a unique process. Some examples of constructing diffusion processes through solutions of stochastic differential equations by verifying the pathwise uniqueness are given in [28, 29]. Other construction of such processes seems to be much more difficult (see [13, 24]).

Thus, it is appropriate to consider the question of uniqueness apart of the existence.

2. PRELIMINARY

We will introduce a class $\mathcal{M}$ of functions which will prove to be very useful in the investigation of the existence and uniqueness of solution for the stochastic integral equations and to assure the convergence of the successive approximations to solutions.

Definition 2.1. Let $\mathcal{M}$ be the class of all continuous nondecreasing functions $\varphi : [0,\infty) \to [0,\infty)$ such that $x \mapsto x - \varphi(x)$ is nonnegative and strictly increasing on $[0,\infty)$ and satisfies $\sum_{n=1}^{\infty} \varphi^{*n}(x_0) < \infty$ for some $x_0 > 0$, where φ^{*n} is the n^{th} iterate of φ (see [5],[20], [22]).

Note that $\varphi(0) = 0$ if $\varphi \in \mathcal{M}$ while $0 \le \varphi(x) < \infty$ for all $x > 0$. Moreover, $\sum_{n=1}^{\infty} \varphi^{*n}(x) < \infty$ for all $x \in [0, x_0]$. The condition $\sum_{n=1}^{\infty} \varphi^{*n}(x) < \infty$ for some $x_0 > 0$ is not implied by other properties defining the class $\mathcal{M}$, as one from example of $\varphi(x) = \dfrac{x}{1+x}$ for $x \ge 0$, in which case formula $\varphi^{*n}(x) = \dfrac{x}{1+nx}$, for $n \ge 1$ and $x \ge 0$, easily checked by induction, ensures that $\sum_{n=1}^{\infty} \varphi^{*n}(x_0) = \infty$ for all $x_0 > 0$. Indeed, given $x_0 > 0$, if the integer $N \ge 1$ is such that $x_0 \ge \frac{1}{N}$ then

$$\sum_{n=1}^{\infty} \varphi^{*n}(x_0) = \sum_{n=1}^{\infty} \frac{x_0}{1+nx_0} \ge \sum_{n=N}^{\infty} \frac{1}{2n} = \infty \ .$$

As examples of the functions $\varphi \in \mathcal{M}$ are: $\varphi(x) = \alpha x$, $x \ge 0$, $\alpha \in (0,1)$; $\eta(x) = \dfrac{1}{\lambda + x}$ for $x > 0$, $\lambda > 1$, $\eta(0) = 0$ and for the divergence case $\theta(x) = \dfrac{x}{1+\delta x}$, $\delta > 0$, $x \ge 0$.

Further on, we identify a subclass of the class of functions $\mathcal{M}$, $\mathcal{N} \subset \mathcal{M}$, generated by aid of a set of positive strictly decreasing sequences $\{a_n\}_{n\ge1}$ with the certain properties [20].

This class $\mathcal{N}$ of functions allows to formulate new classes of stochastic integral equations with the important properties regarding different types of convergence of the sequences of successive approximations to solutions. In particular, a relaxation of the classical Lipschitz conditions on their coefficients is given by allowing a suitably controlled growth in the time variable. The class $\mathcal{N}$ can also be used to study the concept of stability of random solution of certain class of stochastic integral equations.

Let $\mathcal{S}$ be the set of strictly decreasing positive sequences $\{a_n\}_{n\geq 1}$ with the following properties:

$$a_n - a_{n+1} \geq a_{n+1} - a_{n+2} \text{ for all } n \geq 1, \quad \lim_{n\to\infty} a_n = 0, \ \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

$$\lim_{n\to\infty} \frac{a_{n+1} - a_{n+2}}{a_n - a_{n+1}} = 1, \quad \text{and} \quad \sum_{n\geq 1} a_n < \infty \ .$$

Lets to consider the sequences

$$\alpha_n = \frac{a_{n+1} - a_{n+2}}{a_n - a_{n+1}} \quad \text{and} \quad \beta_n = \frac{a_n a_{n+2} - (a_{n+1})^2}{a_n - a_{n+1}}, \quad n \geq 1 \ .$$

We define the function $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(0) = 0$ and

$$\varphi(x) = \begin{cases} a_2, & a_1 \leq x \\ \alpha_n x + \beta_n, & a_{n+1} < x \leq a_n, \quad n \geq 1 \end{cases}$$

The piecewise linear function φ is continuous on $[0,\infty)$, satisfies $\varphi(a_n) = a_{n+1}$ for all $n \geq 1$ and $x \in [a_{n+1}, a_n]$ ensures $\varphi(x) \in [a_{n+2}, a_{n+1}]$. Since the sequence $\{a_n\}_{n\geq 1}$ is nonnegative, strictly decreasing and converging to zero, this implies that $x > \varphi(x)$ for all $x > 0$.

On the other hand, $\lim\limits_{x\to 0} [x - \varphi(x)] = 0$ since $\varphi(x) \leq a_{n+1}$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} a_{n+1} = 0$, while $\lim\limits_{x\to\infty} [x - \varphi(x)] = \infty$ since $\varphi(x) = a_2$ for all $x \geq a_1$.

Furthermore, since by construction $\varphi(a_n) = a_{n+1}$ for all $n \geq 1$, for the iterate function $\varphi^{*n}(x)$ we obtain that

$$\sum_{n=1}^{\infty} \varphi^{*n}(a_1) = \sum_{n=1}^{\infty} \varphi(a_n) = \sum_{n=1}^{\infty} a_{n+1} < \infty \ .$$

Moreover, for $x > a_1$, we have that $\varphi^{*n}(x) = a_{n+1}$, $n \geq 1$, and $\sum\limits_{n=1}^{\infty} \varphi^{*n}(x) = \sum\limits_{n=1}^{\infty} a_{n+1} < \infty$.

Also, for the iterate function φ^{*n} and all $x \in (a_{n+1}, a_n]$ we have that
$\varphi(x) = \varphi^{*1}(x) \in (a_{n+2}, a_{n+1}]$, $\varphi^{*1}(x) \leq a_{n+1}$, $\varphi^{*2}(x) \in (a_{n+3}, a_{n+2}]$, $\varphi^{*2}(x) \leq a_{n+2}$, $\ldots$ $\varphi^{*n}(x) \in (a_{2n+1}, a_{2n}]$, $\varphi^{*n}(x) \leq a_{2n}$.

Hence, for each $x \in (0, a_1]$ there exists an $k \geq 1$ such that $x \in (a_{k+1}, a_k]$ with $\varphi(x) \in (a_{k+2}, a_{k+1}]$, and $\varphi^{*k}(x) \in (a_{2k+1}, a_{2k}]$.

Then we have for $m \geq k$

$$\varphi^{*m}(x) \leq a_{2m} \quad \text{and} \quad \sum_{m\geq k} \varphi^{*m}(x) \leq \sum_{m\geq k} a_{2m} < \infty, \quad \text{for all} \ \ x \in (0, a_1]$$

and $\sum\limits_{n=1}^{\infty} \varphi^{*n}(x) < \infty$, for all $x \in (0, a_1]$.

On the other hand, it is easily to deduce that the function $\varphi(x)$ is derivable on the complement of the countable subset $\{a_n\}_{n\geq 1}$ of $[0,\infty)$ and $\varphi'(x) = \alpha_n > 0$ on (a_{n+1}, a_n), $n \geq 1$, hence φ is an increasing function and therefore $\varphi \in \mathcal{N} \subset \mathcal{M}$. Note that there does not exist $\gamma \in (0,1)$ such that $\varphi(x) \leq \gamma x$, for all $x \geq 0$, since $\varphi(a_n) = a_{n+1} \leq \gamma a_n$, for all $n \geq 1$, and so $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ is impossible.

The fact that the sequence $\{\alpha_n\}_{n\geq 1}$ is strictly increasing implies that $\varphi'(x)$ is a nonincreasing function and that $\varphi(x)$ is concave function (see [14]).

To see this more clearly, one can have a look on Figure 1, which compares the function φ with the identity map for $x \in [a_{20}, a_2 + 0.2]$, Supplementary, the graph of $\varphi'(x)$ in given in Figure 2.

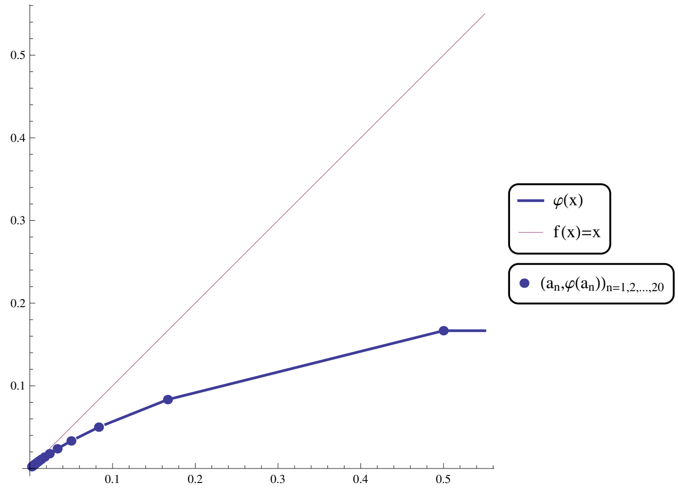

Figure 1: The graphs of $\varphi(x)$ *(thick line) and the identity map* $f(x) = x$ *(dotted line), with some points* $(a_n, \varphi(a_n))$ *located (dots)*

As an example for a sequence with the above properties, we choose the following particular sequence

$$\{a_n\}_{n\geq 1} = \left\{ \frac{1}{n(n+1)} \right\}_{n\geq 1}$$

for which we have that $a_n - a_{n+1} \geq a_{n+1} - a_{n+2}$, which is ensured by

$$\frac{6}{n(n+1)(n+2)(n+3)} \geq 0\,,$$

and $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, $\sum\limits_{n=1}^{\infty} a_n < \infty$.

Defining

$$\alpha_n = \frac{a_{n+1} - a_{n+2}}{a_n - a_{n+1}} = \frac{n}{n+3}, \quad \alpha_{n+1} = \frac{n+1}{n+4}$$

$$\beta_n = \frac{1}{(n+1)(n+2)(n+3)}$$

we see that $\lim\limits_{n\to\infty} \alpha_n = 1$, with $\lim\limits_{n\to\infty} \beta_n = 0$. Note that

$$\alpha_n - \alpha_{n+1} = -\frac{3}{(n+3)(n+4)} < 0,$$

i.e. $\alpha_n < \alpha_{n+1}$ for $n \geq 1$.

Other examples of sequences with similarly properties with the sequence $\{a_n\}_{n\geq 1}$ are: $b_n = \sum\limits_{k>n+1} \frac{1}{k^3}$, $c_n = \frac{1}{n^2}$, $d_n = \frac{1}{n^2 - n + 1}$, $e_n = \frac{1}{n\sqrt{n}}$, $n \geq 1$.

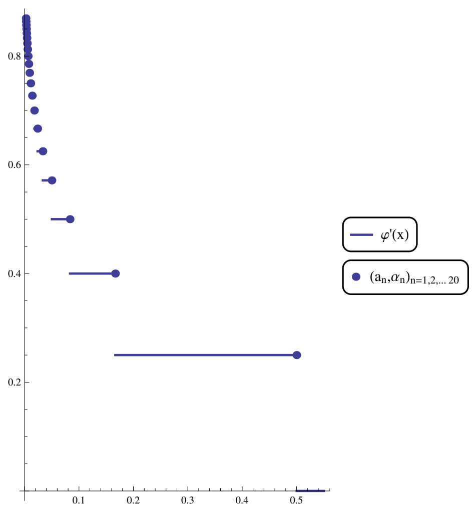

Figure 2: The graphs of $\varphi'(x)$ (thin line) with some points $(a_n, \alpha_n))$ located (dots)

3. Main results

In this section we give a result concerning a relaxation of the Lipschitz condition, proving a Nagumo-type result (see [3], [6], [7], [8], [19]) for deterministic counterpart. We improve Theorem 1 and Theorem 2 from [22] which ensures the existence and pathwise uniqueness for an appropriate case.

For example, it is known that the classical Itô theorem establishes pathwise uniqueness of solutions to equation (1.1) if the functions f and g are uniformly Lipschitz in the second variable:

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le K|x-y|,\ t \in [0,1],\ x, y, \in \mathbb{R} \quad (L)$$

for some $K > 0$ (see [16], [17]). Also, in [17] is proved that the Lipschitz condition (L) can be slightly weakened to allow for a blowup in time:

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le h(t)|x-y|,\ t \in [0,1],\ x, y \in \mathbb{R} \quad (L1)$$

where the function $h : [0,1] \to \mathbb{R}_+$ belongs to $L_2([0,1], \mathbb{R}_+)$. Indeed, if X and Y are two solutions of (1.1), then for every $t \in [0,1]$

$$E\left(|X_t - Y_t|^2\right) \le C \int\limits_0^t h(s)^2 E\left(|X_s - Y_s|\right) ds,$$

which by Gronwall's inequality implies $E\left(|X_s - Y_s|^2\right) = 0$ and thus $X_t = Y_t$ a.s., thanks to the assumption $\int\limits_0^t k(s)^2 ds < \infty$.

We shall introduce a new class of functions-factor on the right side of $(L1)$,

$$\mathcal{K} = \left\{ k : \mathbb{R}_+ \to \mathbb{R}_+,\ k(t) = \frac{\theta'(t)}{\theta(t)} \notin L_2([0,1], \mathbb{R}_+) \text{ and satisfies } \ (L1) \right\}$$

and we denote with

$$\mathcal{H} = \left\{ h : \mathbb{R}_+ \to \mathbb{R}_+,\ h(t) \in L_2([0,1], \mathbb{R}_+) \text{ and satisfies } \ (L1) \right\} \ .$$

These classes are complementary and they imply different classes of coefficients f, respective g, for equation (1.1).

Moreover, we replace the operation "+" with "$\vee$" ($\max\{f,g\} \overset{def}{=} f \vee g$) in the left side of $(L1)$, $h \longrightarrow k$, $h \in \mathcal{H}$, $k \in \mathcal{K}$, $|x-y|^2 \longrightarrow \varphi(|x-y|^2)$, $\varphi \in \mathcal{N}$ and we obtain a new relation $i)$ which assure that the solution of (1.1) has pathwise uniqueness property in $\mathcal{B}_{[0,1]}$, but the classical Lipschitz condition is not satisfied.

Following Rodkina (see [25]) and Taniguchi (see [27]) we consider the Banach space $B_{[0,a]}$ which is the set of functions $h : [0,a] \times \Omega \to \mathbb{R}^n$ measurable in the second variable for each fixed $t \in [0,a]$ and continuous in the first variable for a.s. fixed $\omega \in \Omega$ and with

$$E\left(\sup_{t\in[0,a]}\left\{|h(t,\cdot)|^2\right\}\right) < \infty$$

endowed with the norm

$$\|h(t,\cdot)\|_{B_{[0,a]}} = \left\{E\left(\sup_{t\in[0,a]}\left\{|h(t,\cdot)|^2\right\}\right)\right\}^{1/2} .$$

Throughout in this note, by $|x| = \sqrt{x_1 + \cdots + x_n}$ we denote the Euclidean norm of the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

We have following

Theorem 3.1. *Assume that there exists an absolutely continuous function $\theta : [0,a] \to [0,\infty)$ with $\theta(0) = 0$ and $\theta(t) > 0$ for $t > 0$, having an integrable derivative θ' on $[0,a]$ with $\theta'(t) > 0$ for $t > 0$ and $\lim_{t\to 0_+} \theta'(t) = \infty$. and such that*

i)

$$|f(t,x) - f(t,y)|^2 \vee |g(t,x) - g(t,y)|^2 \le \frac{1}{4(a+1)}\frac{\theta'(t)}{\theta(t)}\varphi(|x-y|^2)$$

for all $t \in [0,a]$, $x, y \in \mathbb{R}^n$ and $\varphi \in \mathcal{N}$;

ii) *there exists a constant $K > 0$ and such that*

$$|f(t,x)|^2 \vee |g(t,x)|^2 \le K(1 + |x|^2),\ x \in \mathbb{R}^n,\ t \in [0,a] .$$

If $X_0 \in L^2(\Omega, \mathcal{K}, P)$ is independent of the Brownian motion $W(t,\cdot)$, $a \le t \le a$ and f, g are continuous with $f(t,0) = g(t,0) = 0$ for all $t \in [0,a]$, then the solution to (1.1) has pathwise uniqueness property in $B_{[0,a]}$.

Proof. Let X and Y be two solutions of (1.1) in $B_{[0,a]}$.

As in the proof of Theorem 1 from [22], we define the operator $T : B_{[0,a]} \to B_{[0,a]}$ by

$$T(X(t,\cdot)) = X_0(0,\cdot) + \int_0^t f(s, X(s,\cdot))ds + \int_0^t g(s, X(s,\cdot))dW(s,\cdot) ,$$

for which it is easy to see that this operator is well defined. Also, by using the inequality $(\alpha+\beta)^2 \le 2(\alpha^2 + \beta^2)$ for $\alpha, \beta \in R$, the Cauchy-Schwartz inequality for the square-integrable functions and Itô's isometry formula (see [22]) we obtain

$$\|T(X(t,\cdot)) - T(Y(t,\cdot))\|^2_{B_{[0,a]}} = E\left[\sup_{t\in[0,a]}\left\{|T(X(t,\cdot)) - T(Y(t,\cdot))|^2\right\}\right] =$$

$$= E\left[\sup_{t\in[0,a]}\left\{\left|\int_0^t [f(s,X(s,\cdot)) - f(s,Y(s,\cdot))]ds + \right.\right.\right.$$

$$+\int_0^t [g(s,X(s,\cdot))-g(s,Y(s,\cdot))]dW(s,\cdot)\Bigg|^2\Bigg\}\Bigg] \leq$$

$$\leq 2a\int_0^a E\left[|f(s,X(s,\cdot))-f(s,Y(s,\cdot))|^2\right]ds+$$

$$+2\int_0^a E\left[|g(s,X(s,\cdot))-g(s,Y(s,\cdot))|^2\right]ds \leq$$

$$\leq 4(a+1)\int_0^a E\left[|f(s,X(s,\cdot))-f(s,Y(s,\cdot))|^2 \vee |g(s,X(s,\cdot))-g(s,Y(s,\cdot))|^2\right]ds\ .$$

We define, for $n\in\mathbb{N}$, the stopping time (see [1, 7, 12, 16])

$$\tau_n(t)=\inf\{0\leq t\leq a\ :\ |X(t,\cdot)|>n \text{ or } |Y(t,\cdot)|>n\}\wedge a$$

and set

$$Z_n(t)=E\left[\sup_{0\leq s\leq \tau_n\wedge t}|X(s,\cdot)-Y(s,\cdot))|^2\right],\ 0\leq t\leq a\ .$$

Then, from the condition $i)$, we can write for $t\in[0,a]$ that

$$Z_n(t)\leq 2\Bigg[E\Bigg(t\int_0^{\tau_n\wedge t}|f(s,X(s,\cdot))-f(s,Y(s,\cdot))|^2ds+$$

$$+\int_0^{\tau_n\wedge t}|g(s,X(s,\cdot))-g(s,Y(s,\cdot))|^2ds\Bigg)\Bigg]\leq$$

$$\leq 4(t+1)\left[E\left(\int_0^{\tau_n\wedge t}[|f(s,X(s,\cdot))-f(s,Y(s,\cdot))|^2\vee|g(s,X(s,\cdot))-g(s,Y(s,\cdot))|^2]ds\right)\right]$$

$$\leq 4(a+1)\left\{\int_0^t \frac{1}{4(a+1)}\frac{\theta'(s)}{\theta(s)}\varphi\left(E(\sup_{0\leq s\leq\tau_n\wedge t}|X(s,\cdot)-Y(s,\cdot))|^2\right)ds\right\}=$$

$$=\int_0^t\frac{\theta'(s)}{\theta(s)}\varphi(Z_n(s))ds\ .$$

Now, let $\varepsilon>0$. Choose $\delta>0$ such that

$$|f(t,x)|^2\vee|g(t,x)|^2\leq\frac{\varepsilon\theta'(t)}{16(a+1)},\ \text{for } t\in(0,\delta],\ |x|\leq n$$

so that

$$Z_n(t)\leq 2\Bigg[tE\Bigg\{\int_0^{\tau_n\wedge t}|f(s,X(s,\cdot))-f(s,Y(s,\cdot))|^2ds+$$

$$+\int_0^{\tau_n\wedge t}|g(s,X(s,\cdot))-g(s,Y(s,\cdot))|^2ds\Bigg\}\Bigg]\leq$$

$$\leq 4(t+1)\left[E\left\{\int_0^{\tau_n\wedge t}(|f(s,X(s,\cdot))|^2+|f(s,Y(s,\cdot))|^2ds+\right.\right.$$

$$\left.\left.+\int_0^{\tau_n\wedge t}(|g(s,X(s,\cdot))|^2+|g(s,Y(s,\cdot))|^2)ds\right\}\right]\leq$$

$$\leq 4(t+1)\left[E\left\{\int_0^{\tau_n\wedge t}\left(|f(s,X(s,\cdot))|^2\vee|g(s,X(s,\cdot))|^2+|f(s,Y(s,\cdot))|^2\vee|g(s,Y(s,\cdot))|^2+\right.\right.\right.$$

$$\left.\left.\left.+|g(s,X(s,\cdot))|^2\vee|f(s,X(s,\cdot))|^2+|g(s,Y(s,\cdot))|^2\vee|f(s,Y(s,\cdot))|^2\right)ds\right\}\right]\leq$$

$$\leq 8(t+1)\left[E\left\{\int_0^{\tau_n\wedge t}(|f(s,X(s,\cdot))|^2\vee|g(s,X(s,\cdot))|^2+|f(s,Y(s,\cdot))|^2\vee|g(s,Y(s,\cdot))|^2)ds\right\}\right]\leq$$

$$\leq 8(t+1)\int_0^t\frac{2\varepsilon\theta'(s)}{16(a+1)}ds\leq\varepsilon\int_0^t\theta'(s)ds=\varepsilon\theta(t),\ \ 0<t\leq\delta\ .$$

Moreover, if we define

$$\alpha_n(t)=\int_0^t\frac{\theta'(s)}{\theta(s)}\varphi(Z_n(s))ds,\ t\in[0,a]\ ,$$

we see that α_n is continuous and $\lim\limits_{t\to 0}\dfrac{\alpha_n(t)}{\theta(t)}=0$.

On the other hand, since $\varphi(x)<x$ for all $x>0$ we have that

$$\alpha_n(t)=\int_0^t\frac{\theta'(s)}{\theta(s)}\varphi(Z_n(s))ds\leq\int_0^t\frac{\theta'(s)}{\theta(s)}Z_n(s)ds\leq\int_0^t\frac{\theta'(s)}{\theta(s)}\varepsilon\theta(s)ds=\varepsilon\theta(t),\ t\in(0,\delta]$$

Moreover, for $t>0$ we have

$$\alpha_n'(t)=\frac{\theta'(t)}{\theta(t)}\varphi(Z_n(t))\leq\frac{\theta'(t)}{\theta(t)}Z_n(t)\leq\frac{\theta'(t)}{\theta(t)}\int_0^t\frac{\theta'(s)}{\theta(s)}\varphi(Z_n(s))ds=\frac{\theta'(t)}{\theta(t)}\alpha_n(t)$$

hence

$$\frac{\alpha_n'(t)}{\alpha_n(t)}\leq\frac{\theta'(t)}{\theta(t)}\ \text{ and }\ \left(\frac{\alpha_n(t)}{\theta(t)}\right)'\leq 0,\ t\in(0,a]$$

Thus, the nonnegative continuous function $\dfrac{\alpha_n(t)}{\theta(t)}$ is decreasing on $(0,a]$ and $\lim\limits_{t\to 0^+}\frac{\alpha_n(t)}{\theta(t)}=0$ and therefore $\alpha_n(t)$ must vanish on $(0,a]$ meaning $\alpha_n(t)\equiv 0$, for $t\in(0,a]$.

As $Z_n(t)\leq\alpha_n(t)$, for $t\in(0,a]$, we deduce that $Z_n(t)\equiv 0$ on $[0,a]$.

Since $n\geq 1$ is arbitrary, we have that $X(t,\cdot)\equiv Y(t,\cdot)$ a.s. for every fixed $t\in[0,a]$ and hence for a countable dense set S in $[0.a]$. By the continuity of $X(t,\cdot)$ and $Y(t,\cdot)$, coincidence on S implies coincidence throughout the entire interval $[0,a]$ and the proof is complete ([6, 22]). ∎

Corollary 3.1. *Assume that there exists a constant α, $0 < \alpha < \frac{1}{16}$ such that*

$$|f(t,x) - f(t,y)|^2 \vee |g(t,x) - g(t,y)|^2 \leq \frac{2\alpha}{t}\varphi(|x-y|^2),\ t \in (0,1],\ x, y, \in \mathbb{R}^n$$

and a constant $K > 0$ such that

$$|f(t,x)|^2 \vee |g(t,x)|^2 \leq K(1+|x|^2),\ x \in \mathbb{R}^n,\ t \in [0,1]\ .$$

If f and g are continuous functions on $[0,1] \times \mathbb{R}^n$ and $X(0,\cdot) \in L^2(\Omega, \mathcal{K}, P; \mathbb{R}^n)$ is independent of Brownian motion $W(t,\cdot)$, for $t \in [0,1]$, then equation (1.1) has the pathwise uniqueness property.

Proof. In order to prove, apply the above Theorem 3.1 for $\theta(t) = t^{16\alpha}$, $0 < \alpha < \frac{1}{2}$, $\alpha = \frac{1}{18}$, which implies that the right part of inequality $i)$ from Theorem 3.1 becomes

$$\frac{1}{4(a+1)}\frac{\theta'(t)}{\theta(t)}\varphi(|x-y|^2) = \frac{1}{4\cdot 2}\frac{16\alpha}{t}\varphi(|x-y|^2) = \frac{2\alpha}{t}\varphi(|x-y|^2)\ .$$

∎

Remark 3.1. We observe that if we elude to use the function φ and the operation $\vee \longrightarrow +$ in enunciation of Theorem 3.1, the proof becomes similar with the except of condition $i)$ from Theorem 3.1, which becomes

$$i')\ \ |f(t,x) - f(t,y)|^2 + |g(t,x) - g(t,y)|^2 \leq \frac{1}{4(a+1)}\frac{\theta'(t)}{\theta(t)}|x-y|^2$$

that offer the facilities to give interested examples (see [7], [8], [22], [23]).

Remark 3.2. For the existence of solutions for equation (1.1) with similar conditions, see [7, 8, 22, 23]. On the other hand, Theorem 3.1 is a more general with respect to the problem of pathwise uniqueness than the result given in [7]. Also, we see for example that neither Corollary 1 nor Proposition 1 from [27] can't be applied .

4. Some detailed examples

It is appropriate to insist on the pathwise uniqueness property of the solution with some examples of stochastic integral equations of (1.1) type to which Theorem 3.1 and Corollary 3.1 can be applied and for which the drift coefficients, as well as the diffusions coefficients, are not Lipschitz (see [6, 7, 9, 22, 23]).

Example 4.1. *Let $f, g : [0,1] \times \mathbb{R}_+$ be given by*

$$f(t,x) = g(t,x) = \begin{cases} 0,\ t = 0, x \in \mathbb{R}_+ \\ \frac{\varphi\left[\left(\frac{x-t}{2}\right)^2\right]}{8\sqrt{t}},\ \textit{for}\ x \leq t,\ t \in (0,1],\ x \in \mathbb{R}_+ \\ 0,\ \textit{for}\ x > t,\ t \in (0,1],\ x \in \mathbb{R}_+ \end{cases}$$

where $\varphi \in \mathcal{N}$, i.e. $\varphi(0) = 0$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the map $t \mapsto t - \varphi(t)$ is nonnegative and strictly increasing on $\mathbb{R}_+$, $\varphi(t) < t$ for $t > 0$, $\varphi(t) \leq a_2$ where a_2 is the second element of the generatrix sequence $\{a_n\}_{n\geq 1}$ of the function φ, usually $a_2 \leq 1$.

Then we have that

$$|f(t,x)-f(t,y)|^2 \vee |g(t,x)-g(t,y)|^2 = |f(t,x)-f(t,y)|^2 =$$

$$= \left| \frac{\varphi\left[\left(\frac{x-t}{2}\right)^2\right]}{8\sqrt{t}} - \frac{\varphi\left[\left(\frac{y-t}{2}\right)^2\right]}{8\sqrt{t}} \right|^2 \le 2\frac{\left\{\varphi\left[\left(\frac{x-t}{2}\right)^2\right]\right\}^2}{64t} + \frac{\left\{\varphi\left[\left(\frac{y-t}{2}\right)^2\right]\right\}^2}{64t} \le$$

$$\le 4\frac{\left\{\varphi\left[\left(\frac{x-t}{2}\right)^2\right]\right\}^2 \vee \left\{\varphi\left[\left(\frac{y-t}{2}\right)^2\right]\right\}^2}{64t} =$$

$$= \begin{cases} \frac{\left\{\varphi\left[\left(\frac{x-t}{2}\right)^2\right]\right\}^2}{16t}, & \text{for } x \ge y,\ t \in (0,1],\ x,y \in \mathbb{R}_+ \\ \frac{\left\{\varphi\left[\left(\frac{y-t}{2}\right)^2\right]\right\}^2}{16t}, & \text{for } x < y,\ t \in (0,1],\ x,y \in \mathbb{R}_+ \end{cases} \le$$

$$\le \begin{cases} \frac{\left\{\varphi\left[\left(\frac{x-y}{2}\right)^2\right]\right\}^2}{16t}, & \text{for } y < t,\ t \in (0,1],\ x,y \in \mathbb{R}_+ \\ \frac{\left\{\varphi\left[\left(\frac{y-x}{2}\right)^2\right]\right\}^2}{16t}, & \text{for } x < t,\ t \in (0,1],\ x,y \in \mathbb{R}_+ \end{cases} \le$$

$$\le \begin{cases} \frac{\varphi\left[(x-y)^2\right]}{16t}, & \text{for } y < t,\ t \in (0,1],\ x,y \in \mathbb{R}_+ \\ \frac{\varphi\left[(y-x)^2\right]}{16t}, & \text{for } x < t,\ t \in (0,1],\ x,y \in \mathbb{R}_+ \end{cases}$$

since $(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2)$ for all $\alpha, \beta \in R$ and $|\varphi(u)|^2 \le |\varphi(u)| = \varphi(u) \le a_2 \le 1$, from the definition of φ.

Hence, Theorem 3.1 or Corollary 3.1 are satisfied since we can choose $\theta(t) = t^{2\alpha}$, $0 < \alpha < \frac{1}{2}$, for example $\alpha = \frac{1}{3}$. Then the second part of the inequality $i)$ of Theorem 3.1, becomes

$$\frac{1}{4(a+1)}\frac{\theta'(t)}{\theta(t)}\varphi(|x-y|^2) = \frac{1}{4\cdot 2}\frac{2}{3}\frac{t^{\frac{2}{3}-1}}{t^{2/3}}\varphi(|x-y|^2) = \frac{1}{12t}\varphi(|x-y|^2), \text{ for } a = 1\ ,$$

and our example has for the second part of the inequality $i)$ of the form

$$\frac{1}{16t}\varphi(|x-y|^2) \le \frac{1}{12t}\varphi(|x-y|^2)$$

i.e. the hypotheses of Theorem 3.1 are satisfied and the solution of equation (1.1) with the above coefficients has the pathwise property.

Nevertheless, the Lipschitz condition

$$|f(t,x)-f(t,y)|^2 \le L|x-y|^2, \text{ for } t \in [0,1],\ x,y \in \mathbb{R}_+,\ L > 0$$

is not satisfied, since for $x = \frac{1}{2}$ and $y = t$ implies

$$|f(t,\frac{1}{2}) - f(t,t)|^2 = \frac{\left\{\varphi\left[\left(\frac{\frac{1}{2}-t}{2}\right)^2\right]\right\}^2}{64t} \le L|\frac{1}{2}-t|^2$$

i.e. $\left\{\varphi\left[\left(\frac{\frac{1}{2}-t}{2}\right)^2\right]\right\}^2 \le 64tL|\frac{1}{2}-t|^2$ and for $t \to 0$ implies that $|\varphi(\frac{1}{4})|^4 \le 0$ which it is impossible since $\varphi(t) = 0$ only for $t = 0$.

Example 4.2. *Let $f, g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ be given by*

$$f(t,x) = g(t,x) = \begin{cases} \frac{x\sqrt[4]{\sin t}}{7\sqrt{t}}, \ 0 < x < t, \ t \in (0,1], \\ \frac{\sqrt{t}\sqrt[4]{\sin t}}{7}, \ x \geq t, \ t \in (0,1], \\ 0, \ x = 0, \\ 0, \ t = 0 \end{cases}$$

The hypotheses of Corollary 3.1 are satisfied for $\theta(t) = t^{2\alpha}$, $0 < \alpha < \frac{1}{2}$, $\alpha = \frac{1}{7}$ and then $\dfrac{2\alpha}{t} = \dfrac{2}{7t}$, but the classical Lipschitz condition is not satisfied.

Indeed we have that

$$2|f(t,x) - f(t,y)|^2 = 2\left|\frac{x\sqrt[4]{\sin t}}{7\sqrt{t}} - \frac{y\sqrt[4]{\sin t}}{7\sqrt{t}}\right|^2 = 2\frac{|x-y|^2}{49t}\sqrt{\sin t} \leq 2\frac{|x-y|^2}{49t},$$

for $0 < x < y < t$ or $0 < y < x < t$;

$$2|f(t,x) - f(t,y)|^2 = \left|\frac{x\sqrt[4]{\sin t}}{7\sqrt{t}} - \frac{t\sqrt[4]{\sin t}}{7\sqrt{t}}\right|^2 \leq 2\frac{|x-t|^2\sqrt{\sin t}}{49t} \leq 2\frac{|x-y|^2|}{49t},$$

for $0 < x < t < y$ or $0 < y < t < x$;

$$2|f(t,x) - f(t,y)|^2 = 2\left|\frac{t\sqrt[4]{\sin t}}{7\sqrt{t}} - \frac{t\sqrt[4]{\sin t}}{7\sqrt{t}}\right|^2 = 0.$$

for $0 < t < x < y$ or $0 < t < y < x$.

On the other hand we have that

$$\frac{2\alpha}{t}|x-y|^2 = \frac{2}{7t}|x-y|^2 \geq \frac{2}{49t}|x-y|^2 ,$$

i.e. the solution of equation (1.1) with the above coefficients has the pathwise property.

For the second part, we observe that if it would hold, we would have

$$|f(t,x) - f(t,y)|^2 \leq L|x-y|^2, \ 1 \geq t > 0, \ x, y \in \mathbb{R}_+, \ L > 0 ,$$

and for $x = t$ and $y = 0$, it would be

$$\left|\frac{\sqrt{t}\sqrt[4]{\sin t}}{7} - 0\right|^2 \leq L|t-0|^2$$

which implies that

$$\frac{\sqrt{\sin t}}{49} \leq Lt \text{ and } 1 \leq 49L\sqrt{t}\frac{\sqrt{t}}{\sqrt{\sin t}} .$$

This is impossible since we would have

$$1 \leq 49t \lim_{t\to 0}\sqrt{t} \lim_{t\to 0}\frac{\sqrt{t}}{\sqrt{\sin t}} = 0$$

which is absurd.

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