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# FUZZY IDEAL CONGRUENCES OF ADL'S

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ABSTRACT. The concept of fuzzy congruence of an ADL is introduced. Established a correspondence between fuzzy ideals and fuzzy congruences of an ADL and obtained an equivalent condition for an ADL with a maximal element is a Boolean algebra.

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## 1. INTRODUCTION

Ever since Zadeh [18] introduced the concept of a fuzzy set as a function from a set $X$ into $[0,1]$. Rosenfeld [6] formulated the concept of a fuzzy subgroup of a group. Since then a host of researchers (see for example [1, 2, 3, 4]) are engaged in fuzzifying various subalgebras of algebras. Swamy and Swamy [16] introduced the notion of a fuzzy prime ideal of a ring. Further, Swamy and Raju [13, 14] introduced the concept of irreducibility in algebraic fuzzy systems and applied a general theory of algebraic fuzzy systems to fuzzy ideals of distributive lattices.

In 1980, Swamy and Rao [15] introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of most of the existing lattice (ring) theoretic generalization of a Boolean algebra (ring). An ADL is an algebra $(A, \vee, \wedge, 0)$ satisfying the conditions : for all $a, b$ and $c \in A$,

(1) $0 \wedge a = 0$
(2) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $a \vee (a \wedge b) = a$
(5) $(a \vee b) \wedge b = b$
(6) $(a \vee b) \wedge a = a$
(7) $a \vee (b \vee c) = (a \vee b) \vee c$.

An ADL $(A, \vee, \wedge, 0)$ satisfies all the axioms of a distributive lattice, except the commutativity of the operations $\vee$ and $\wedge$ and the right distributivity of $\vee$ over $\wedge$. In fact, these three axioms are equivalent in any ADL. If any one of these axioms hold, then the ADL becomes a distributive lattice. A non-empty subset $X$ of an ADL $A$ is called an ideal if $a \vee b \in X$ and $a \wedge x \in X$ for all $a, b \in X$ and $x \in A$. $(X]$ denotes the smallest ideal of $A$ containing $X$. An equivalence relation $\theta$ on an ADL $A$ is called a congruence relation on $A$ if it is compatible with the binary operations $\vee$ and $\wedge$ on $A$. Maximal element with respect to the partial order $\leq$ on an ADL $A$ defined by $a \leq b$ iff $a \wedge b = a$ (equivalently, $a \vee b = b$) is called a maximal element of $A$.

In recent time, Swamy, Sundar Raj and Teshale [12] have introduced the notion of a fuzzy ideal of an ADL $A$ as a function $\lambda$ from $A$ into $L$ satisfying the conditions that $\lambda(a_0) = 1$ for some $a_0 \in A$ and $\lambda(a \vee b) = \lambda(a) \wedge \lambda(b)$ for any $a, b \in A$, where $L$ is a complete lattice satisfying the infinite $\wedge$-distributivity ; $x \wedge \big(\bigvee_{y \in S} y\big) = \bigvee_{y \in S} (x \wedge y)$ for any $x \in L$ and $S \subseteq L$. It is proved that the set of all fuzzy ideals of an ADL forms a complete distributive lattice under point-wise ordering. Also, Swamy, Sundar Raj and Teshale [7, 8, 9] have extended the notion of fuzzy ideals to filters of ADL's and introduced the concepts of fuzzy prime ideals (filters) and fuzzy maximal ideals (filters). Further, in [5, 10] the authors of this paper have introduced the notions of fuzzy prime spectrums and fuzzy initial and final segments of ADL's. In this paper, we extend the notion of ideal congruences of ADL's to the fuzzy ideals. Here, we introduce the concept of a fuzzy congruence of an ADL $A$, and obtain a fuzzy congruence $\theta_\lambda$ corresponding to a fuzzy ideal $\lambda$ of an ADL $A$, which we call the fuzzy ideal congruence of $A$ and establish a correspondence $\lambda \mapsto \theta_\lambda$ (not necessarily one-to-one) between the fuzzy ideals and fuzzy congruences of $A$. Further, we prove that an ADL $A$ with a maximal element is a Boolean algebra if and only if every fuzzy congruence of $A$ is a fuzzy ideal congruence.

Throughout this paper, $A$ denote an ADL $(A, \vee, \wedge, 0)$ with a maximal element $m$ and $L$ stands for a complete lattice satisfying the infinite $\wedge$-distributivity.

## 2. FUZZY CONGRUENCE

A fuzzy subset $\theta : A \times A \longrightarrow L$ is called a fuzzy relation on $A$. Following [14], a fuzzy relation $\theta$ on $A$ is said to be a fuzzy equivalence of $A$ if, for any $x, y, z \in A$, $\theta$ satisfies the following:

(i) $\theta(x, x) = 1$
(ii) $\theta(x, y) = \theta(y, x)$
(iii) $\theta(x, y) \wedge \theta(y, z) \leq \theta(x, z)$.

A fuzzy equivalence $\theta$ of $A$ is said to be a fuzzy congruence of $A$ if, for any $x_1, x_2, y_1, y_2 \in A$, the following hold:

(iv) $\theta(x_1 \vee x_2, y_1 \vee y_2) \geq \theta(x_1, y_1) \wedge \theta(x_2, y_2)$
(v) $\theta(x_1 \wedge x_2, y_1 \wedge y_2) \geq \theta(x_1, y_1) \wedge \theta(x_2, y_2)$.

It is easy to verify that a fuzzy equivalence $\theta$ of $A$ is a fuzzy congruence of $A$ if and only if $\theta$ satisfies the conditions:

(iv) $\theta(x, y) \leq \theta(x \vee z, y \vee z) \wedge \theta(x \wedge z, y \wedge z)$ and
(vii) $\theta(x, y) \leq \theta(z \vee x, z \vee y) \wedge \theta(z \wedge x, z \wedge y)$.

For any fuzzy relation $\theta$ on $A$ and $\alpha \in L$, define the $\alpha$-level set of $\theta$ by
$\theta_\alpha = \{(x, y) \in A \times A \ : \ \theta(x, y) \geq \alpha\}$.
Then one can easily observe that $\theta$ is a fuzzy congruence of $A$ if and only if $\theta_\alpha$ is a congruence relation on $A$ for each $\alpha \in L$.

For any congruence relation $\theta$ on $A$, it can be easily verified that the fuzzy subset $\chi_\theta$ of $A \times A$ defined by

$$\chi_\theta(x, y) = \begin{cases} 1 & \text{if } (x, y) \ \in \ \theta \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy congruence of $A$. It follows that $\chi_\Delta$ is a fuzzy congruence of $A$, where $(x, y) \in \Delta \Leftrightarrow x = y$. Also, $\chi_\nabla$ is a fuzzy congruence of $A$, where $\chi_\nabla(x, y) = 1$.

Recall (from [17]) that, the associativity relation $\sim$ on an ADL $A$ defined by
$\sim = \{(a, b) \in A \times A \ : \ a \wedge b = b \text{ and } b \wedge a = a\}$
is a congruence relation on $A$. From [15], for any $a \in A$, the relation $\theta^a$ defined by
$\theta^a = \{(x, y) \in A \times A \ : \ x \wedge a = y \wedge a\}$
is a congruence relation on $A$. Corresponding to the relations $\sim$ and $\theta^a$ we shall obtain the fuzzy congruences as given below.

**Lemma 2.1.** *Define $\tilde{\phi} : A \times A \to L$ by*

$$\tilde{\phi}(a, b) = \begin{cases} 1 & \textit{if } (a, b) \in \sim \\ 0 & \textit{otherwise}. \end{cases}$$

*Then $\tilde{\phi}$ is a fuzzy congruence of $A$.*

*Proof.* It follows by the fact that $\tilde{\phi} = \chi_\sim$. ∎

**Lemma 2.2.** *For any $a \in A$, define $\psi^a : A \times A \longrightarrow L$ by*

$$\psi^a(x, y) = \begin{cases} 1 & \textit{if } (x, y) \in \theta^a \\ 0 & \textit{otherwise}. \end{cases}$$

.

*Proof.* It is clear by $\psi^a = \chi_{\theta^a}$. ∎

**Definition 2.3.** *Let $\theta$ and $\phi$ be any fuzzy congruences of $A$. Define*

$$\theta \le \phi \Leftrightarrow \theta(x,y) \le \phi(x,y)$$

$$(\theta \wedge \phi)(x,y) = \theta(x,y) \wedge \phi(x,y)$$

*for all $x, y \in A$.*

It is easy to verify that the set $\mathcal{F}_L\mathcal{C}(A)$ of all fuzzy congruences of $A$ is a partially ordered set under the point-wise ordering defined above. In which, $\theta \wedge \phi$ is the $g.l.b\ \{\theta, \phi\}$. The operation $\wedge$ is called the point-wise infimum.

**Theorem 2.4.** *$(\mathcal{F}_L\mathcal{C}(A), \le)$ is a complete lattice ; in which for any $\{\theta_i \,:\, i \in \Delta\} \subseteq \mathcal{F}_L\mathcal{C}(A)$.*

$$g.l.b\ \{\theta_i \,:\, i \in \Delta\} = \bigwedge_{i\in\Delta} \theta_i, \text{ the point-wise infimum of } \theta_i\text{'s}$$

$$\text{and } l.u.b\ \{\theta_i \,:\, i \in \Delta\} \ = g.l.b\ \{\theta \in \mathcal{F}_L\mathcal{C}(A) \,:\, \theta_i \le \theta \text{ for all } i \in \Delta\}.$$

*Proof.* Proof is simple. ∎

Note that in the above lattice the fuzzy congruences $\chi_{\scriptscriptstyle\Delta}$ and $\chi_{\scriptscriptstyle\nabla}$ are the smallest and greatest elements respectively.

## 3. FUZZY IDEAL CONGRUENCE

If $\theta$ is a congruence relation on an ADL $(A, \vee, \wedge, 0)$, then the congruence class of $0$ ; that is $\theta(0) = \{a \in A \,:\, (a,0) \in \theta\}$ is an ideal of $A$ and it is the unique congruence class corresponding to $\theta$, which is an ideal of $A$. Now, given a fuzzy congruence $\theta$ of $A$, we define a fuzzy subset $\lambda_\theta$ of $A$ by $\lambda_\theta(x) = \theta(x,0)$ for all $x \in A$. Then one can easily seen that $\lambda_\theta$ is a fuzzy ideal of $A$. Also, $\theta \le \phi$ implies $\lambda_\theta \le \lambda_\phi$ for any fuzzy congruences $\theta$ and $\phi$ of $A$.

Following U.M. Swamy and G,C. Rao [15], it is known that, for a given ideal $I$ of $A$.

$$\theta_I \ := \ \{(x,y) \in A \times A \,:\, a \vee x = a \vee y \text{ for some } a \in I\}$$

is a congruence relation on $A$ and it is the smallest congruence relation on $A$ containing $I \times I$. Analogous to the congruence $\theta_I$, we introduce a fuzzy ideal congruence of $A$ as follows.

**Definition 3.1.** *For any fuzzy subset $\lambda$ of $A$, define a fuzzy relation $\theta_\lambda$ on $A$ by*

$$\theta_\lambda(x,y) = \bigvee \big\{\alpha \in L \,:\, a \vee x = a \vee y \text{ for some } a \in A \text{ such that } \lambda(a) \ge \alpha\big\}.$$

**Theorem 3.2.** *If $\lambda$ is a fuzzy ideal of $A$, then $\theta_\lambda$ is the smallest fuzzy congruence of $A$ such that $\lambda_{\theta_\lambda} = \lambda$.*

*Proof.* It is clear that $\theta_\lambda(x,x) = 1$ and $\theta_\lambda(x,y) = \theta_\lambda(y,x)$.
Consider $\theta_\lambda(x,y) \wedge \theta_\lambda(y,z)$
$= \bigvee \big\{\alpha \in L \,:\, a \vee x = a \vee y$ for some $a \in A$ with $\lambda(a) \ge \alpha\big\} \wedge$
$\bigvee \big\{\beta \in L \,:\, b \vee y = b \vee z$ for some $b \in A$ with $\lambda(b) \ge \beta\big\}$
$= \bigvee \big\{\alpha \wedge \beta \,:\, a \vee x = a \vee y,\ b \vee y = b \vee z$ for some $a, b \in A$ with $\lambda(a) \ge \alpha$ and $\lambda(b) \ge \beta\big\}$
(by the infinite $\wedge$-distributive in $L$).
Let $a, b \in A$ such that $a \vee x = a \vee y$ with $\lambda(a) \ge \alpha$ and $b \vee y = b \vee z$ with $\lambda(b) \ge \beta$. Then $\alpha \wedge \beta \le \lambda(a) \wedge \lambda(b) = \lambda(a \vee b)$. Since $a \vee b = a \vee b \vee a$, $a \vee b \vee x = a \vee b \vee z$ and it follows that $\alpha \wedge \beta \in \{\nu \in L \,:\, c \vee x = c \vee z$ foe some $c \in A$ such that $\lambda(c) \ge \nu\}$. This implies that $\theta_\lambda(x,y) \wedge \theta_\lambda(y,z) \le \theta_\lambda(x,z)$. Therefore $\theta_\lambda$ is a fuzzy equivalence of $A$. Again, let $\alpha \in L$ such that $a \vee x = a \vee y$ for some $a \in A$ with $\lambda(a) \ge \alpha$. Then $a \vee (x \wedge z) = (a \vee x) \wedge (a \vee z)$ and

$a \vee z \vee x = a \vee z \vee a \vee x = a \vee z \vee a \vee y = a \vee z \vee y$ which implies $\theta_\lambda(x, y) \le \theta_\lambda(x \vee z, y \vee z)$ and $\theta_\lambda(z \vee x, z \vee y)$. Hence

$$\theta_\lambda(x, y) \le \theta_\lambda(x \wedge z, y \wedge z) \wedge \theta_\lambda(z \wedge x, z \wedge y) \text{ and}$$

$$\theta_\lambda(x, y) \le \theta_\lambda(x \vee z, y \vee z) \wedge \theta_\lambda(z \vee x, z \vee y)$$

.
Thus $\theta_\lambda$ is a fuzzy congruence of $A$. Further, let $\alpha \in L$ such that $a \vee x = a$ and $\lambda(a) \ge \alpha$ for some $a \in A$. Then $\lambda(a) = \lambda(a \vee x) = \lambda(a) \wedge \lambda(x)$ and hence $\lambda(a) \le \lambda(x)$, so that $\alpha \le \lambda(x)$. It follows that $\lambda_{\theta_\lambda} \le \lambda$. Also, it is clear that $\lambda \le \lambda_{\theta_\lambda}$. Hence $\lambda_{\theta_\lambda} = \lambda$. Let $\theta$ be a fuzzy congruence of $A$ such that $\lambda_\theta = \lambda$. By condition (vii), $\theta_\lambda \le \theta$. ∎

We call $\theta_\lambda$, the fuzzy ideal congruence of $A$ corresponding to $\lambda$. If $\theta$ is a fuzzy congruence of $A$, then we note that $\theta_{\lambda_\theta} \le \theta$ and $\theta_\lambda \le \theta_{\lambda_\theta}$ for any fuzzy ideal $\lambda$ of $A$. By this fact, we have the following.

**Lemma 3.3.** *Let $\theta$ be any fuzzy congruence of $A$. Then $\theta = \theta_{\lambda_\theta}$ iff $\theta$ is a fuzzy ideal congruence of $A$.*

**Theorem 3.4.** *$A$ is a Boolean algebra iff every fuzzy congruence of $A$ is a fuzzy ideal congruence.*

*Proof.* Suppose that every fuzzy congruence of $A$ is a fuzzy ideal congruence. Then $\tilde{\phi}$ is a fuzzy ideal congruence of $A$. By Lemma 3.3, $\tilde{\phi} = \theta_{\lambda_{\tilde{\phi}}}$. If $x = y$, then it is clear that $\theta_{\lambda_{\tilde{\phi}}} = \chi_{_\Delta}(x, y)$. If $x \ne y$ and let $\alpha \in L$ such that $a \vee x = a \vee y$ for some $a \in A$ with $\lambda_{\tilde{\phi}} \ge \alpha$. Then $\alpha \le \tilde{\phi}(a, 0) = 0$ since $(a, 0) \notin \sim$. Hence $\alpha = 0$ which implies $\theta_{\lambda_{\tilde{\phi}}}(x, y) = 0 = \chi_{_\Delta}(x, y)$. Therefore $\tilde{\phi} = \chi_{_\Delta}$. As $(x \wedge y, y \wedge x) \in \sim$, $\chi_{_\Delta}(x \wedge y, y \wedge x) = 1$ and it follows that $x \wedge y = y \wedge x$. Therefore $A$ is a bounded distributive lattice. Further, let $a \in A$ and consider the fuzzy relation $\psi^a$. By Lemma 2.2, $\psi^a$ is a fuzzy congruence of $A$ and hence $\psi^a = \theta_\lambda$ for some fuzzy ideal $\lambda$ of $A$. Since $m \wedge a = a$, $(m, a) \in \theta^a$ so that $\psi^a(m, a) = 1$ and hence $\theta_\lambda(m, a) = 1$. So there exists $b \in A$ such that $b \vee m = b \vee a$ and $\lambda(b) = 1$. Now $a \vee b = b \vee a = m$. Since $\lambda(b) = 1$ and $\lambda_{\theta_\lambda} = \lambda$, we have $\theta_\lambda(b, 0) = 1$ so that $(b, 0) \in \theta^a$ which implies $a \wedge b = b \wedge a = 0$. Therefore $b$ is the complement of $a$. Thus $A$ is a Boolean algebra.

Conversely, if $\theta$ is a fuzzy congruence of $A$, then $\lambda_\theta$ is a fuzzy ideal of $A$ and clearly $\theta_{\lambda_\theta} \le \theta$. On the other hand, put $a = (x \wedge y') \vee (y \wedge x')$. Then $a \vee x = a \vee y$. Now,
$\theta(x, y) \le \theta(x \wedge y', 0) \wedge \theta(y \wedge x', 0)$
$= \lambda_\theta(x \wedge y') \wedge \lambda_\theta(y \wedge x')$
$= \lambda_\theta\big((x \wedge y') \vee (y \wedge x')\big)$
$= \lambda_\theta(a)$
which implies that $\theta(x, y) \le \theta_{\lambda_\theta}(x, y)$ for all $x, y \in A$. Therefore $\theta \le \theta_{\lambda_\theta}$. Hence $\theta = \theta_{\lambda_\theta}$ and thus $\theta$ is a fuzzy ideal congruence of $A$. ∎

**Theorem 3.5.** *Let $\lambda$ and $\mu$ be fuzzy ideals of $A$. Then*

(1) $\lambda \le \mu \Leftrightarrow \theta_\lambda \le \theta_\mu$
(2) $\theta_{\lambda \wedge \mu} \le \theta_\lambda \wedge \theta_\mu$
(3) $\lambda = \chi_{_{\{0\}}} \Leftrightarrow \theta_\lambda = \chi_{_\Delta}$
(4) $\lambda = \chi_{_A} \Leftrightarrow \theta_\lambda = \chi_{_\nabla}$
(5) $\theta_{\chi_{_I}} = \chi_{_{\theta_I}}$, *for any ideal $I$ of $A$.*

*Proof.* Proof is simple and straight forward verification. ∎

Let $\overline{\theta}$ denotes the smallest fuzzy congruence of $A$ generated by a fuzzy relation $\theta$ on $A$. Following [12], $\overline{\lambda}$ is the smallest fuzzy ideal of $A$ generated by $\lambda$ and is described by $\overline{\lambda}(0) = 1$ and

$$\overline{\lambda}(x) = \bigvee \Big\{ \bigwedge_{a \in X} \lambda(a) \;:\; x \in (X],\ X \text{ is a finite subset of } A \Big\}$$

**Proposition 3.6.** *$\overline{\theta_\lambda} = \theta_{\overline{\lambda}}$ for any fuzzy subset $\lambda$ of $A$.*

*Proof.* By Theorem 3.5(1), $\theta_\lambda \leq \theta_{\overline{\lambda}}$. Let $\phi$ be a fuzzy congruence of $A$ such that $\theta_\lambda \leq \phi$. Let $\alpha \in L$ such that $a \vee x = a \vee y$ for some $a \in A$ with $\overline{\lambda}(a) \geq \alpha$. Then,

$$\alpha = \bigvee \Big\{ \bigwedge_{b \in X} \lambda(b) \wedge \alpha \;:\; a \in (X],\ X \text{ is a finite subset of } A \Big\}.$$

Put $\beta = \bigwedge_{b \in X} \lambda(b) \wedge \alpha$ and $a \in (X]$, where $X$ is a finite subset of $A$. Then $a = \big( \bigvee_{i=1}^{n} b_i \big) \wedge c$ for some $b_1, b_2, ..., b_n \in X$ and $c \in A$. Now $a = \big( \bigvee_{i=1}^{n} b_i \big) \wedge a$ and hence $\big( \bigvee_{i=1}^{n} b_i \big) \vee a = \bigvee_{i=1}^{n} b_i$. As $\beta \leq \lambda(b_i)$ for all $1 \leq i \leq n$ and by the definition of $\theta_\lambda$, we have

$$\beta \leq \theta_\lambda(x,\ b_i \vee x) \text{ and } \beta \leq \theta_\lambda(y,\ b_i \vee y)$$

so that

$$\beta \leq \theta_\lambda(x,\ b_i \vee x) \wedge \theta_\lambda(y,\ b_i \vee y) \leq \phi(x,\ b_i \vee x) \wedge \phi(y,\ b_i \vee y)$$

which implies that

$$\beta \leq \bigwedge_{i=1}^{n} \phi(x,\ b_i \vee x) \wedge \bigwedge_{i=1}^{n} \phi(b_i \vee y,\ y)$$
$$\leq \phi\big(x,\ \bigvee_{i=1}^{n} (b_i \vee x)\big) \wedge \phi\big( \bigvee_{i=1}^{n} (b_i \vee y),\ y\big)$$
$$\leq \phi\big(x,\ \bigvee_{i=1}^{n} b_i \vee a \vee x\big) \wedge \phi\big( \bigvee_{i=1}^{n} b_i \vee a \vee y,\ y\big)$$
$$\leq \phi(x, y)$$

and it follows that $\theta_{\overline{\lambda}}(x, y) \leq \phi(x, y)$ for all $x, y \in A$. Therefore $\theta_{\overline{\lambda}} \leq \phi$. Thus $\overline{\theta_\lambda} = \theta_{\overline{\lambda}}$. ∎

The fuzzy congruence $\theta_\lambda$ of $A$ exhibits the properties analogous to almost all the properties of ideal congruences on ADLs ; in particular the mapping $\lambda \mapsto \theta_\lambda$ establishes a correspondence (not necessarily one-to-one) between the lattice of fuzzy ideals and the lattice of fuzzy congruences of an ADL $A$.
It can be easily verified that, for any fuzzy equivalences $\theta$ and $\phi$ of $A$,
$\theta \circ \phi = \phi \circ \theta \Leftrightarrow \theta \circ \phi = \theta \vee \phi$, the $l.u.b\ \{\theta, \phi\} \Leftrightarrow \theta \circ \phi$ is a fuzzy equivalence of $A$,
where $(\theta \circ \phi)(a, b) = \bigvee_{c \in A} \big(\phi(a, c) \wedge \theta(c, b)\big)$ for all $a, b \in A$.

**Proposition 3.7.** *Let $\lambda$ and $\mu$ be fuzzy ideals of $A$. Then*

$$\theta_\lambda \circ \theta_\mu \circ \theta_\lambda = \theta_\lambda \vee \theta_\mu = \theta_\mu \circ \theta_\lambda \circ \theta_\mu.$$

*Proof.* We observe that

$$(\theta_\lambda \circ \theta_\mu \circ \theta_\lambda)(x, y) = \bigvee_{z,\ a \in A} \big(\theta_\lambda(x, z) \wedge \theta_\mu(z, a) \wedge \theta_\lambda(a, y)\big).$$

It follows that $\theta_\lambda \circ \theta_\mu \circ \theta_\lambda$ is an $u.b\ \{\theta_\lambda, \theta_\mu\}$. Let $\theta$ be a fuzzy congruence of $A$ and $\theta = u.b\ \{\theta_\lambda, \theta_\mu\}$. Then $\theta_\lambda(x, z) \wedge \theta_\mu(z, a) \wedge \theta_\lambda(a, y) \leq \theta(x, y)$, which implies that $(\theta_\lambda \circ \theta_\mu \circ \theta_\lambda)(x, y) \leq \theta(x, y)$. Therefore $\theta_\lambda \circ \theta_\mu \circ \theta_\lambda \leq \theta$. ∎

**Lemma 3.8.** *Let $\lambda$ and $\mu$ be fuzzy ideals of $A$. Then*

$$\theta_\lambda \vee \theta_\mu = \theta_{\lambda\vee\mu}, \text{ where } \lambda \vee \mu = l.u.b\,\{\lambda, \mu\}.$$

*Proof.* Clearly $\theta_\lambda \leq \theta_{\lambda\vee\mu}$ and $\theta_\mu \leq \theta_{\lambda\vee\mu}$. Let $\phi$ be another fuzzy congruence of $A$ such that $\theta_\lambda \leq \phi$ and $\theta_\mu \leq \phi$. Then $\lambda_{\theta_\lambda} \leq \lambda_\phi$ and $\lambda_{\theta_\mu} \leq \lambda_\phi$. By Theorem 3.2, we get that $\lambda \leq \lambda_\phi$ and $\mu \leq \lambda_\phi$ so that $\lambda \vee \mu \leq \lambda_\phi$ which implies that $\theta_{\lambda\vee\mu} \leq \theta_{\lambda_\phi} \leq \phi$. Thus $\theta_\lambda \vee \theta_\mu = \theta_{\lambda\vee\mu}$. ∎

Theorem 3.5(1) and Lemma 3.8 yields the following result.

**Theorem 3.9.** *$\lambda \mapsto \theta_\lambda$ is an order isomorphism of the lattice $\mathcal{F}_L\mathcal{I}(A)$ of fuzzy ideals of $A$ onto a $\vee$-subsemilattice of the lattice $\mathcal{F}_L\mathcal{C}(A)$ of fuzzy congruences of $A$.*

## 4. CONCLUSION

It is well known that, for any lattice $(L, \wedge, \vee)$, interchanging the operations $\wedge$ and $\vee$ again yields a lattice $(L, \vee, \wedge)$, called as the dual of $L$. An ideal of the dual lattice $(L, \vee, \wedge)$ is called as the filter of the lattice $(L, \wedge, \vee)$. However, an ADL do not have the duality priciple; in the same that, by interchanging $\wedge$ and $\vee$ in an ADL $(A, \wedge, \vee, 0)$ we do not get an ADL again, the main reason is that the right distributive of $\vee$ over $\wedge$ does not hold in $A$. This necessitates a separate study of fuzzy filter congruence of an ADL in future work.

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