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A NEW PROOF OF CLASSICAL WATSON'S SUMMATION THEOREM FOR THE SERIES ${}_3F_2(1)$

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ABSTRACT. The aim of this short research note is to provide a new proof of classical Watson's summation theorem for the series ${}_3F_2(1)$. The theorem is obtained by evaluating an infinite integral and making use of classical Gauss's first and second summation theorems for the series ${}_2F_1$.

Key words and phrases: Generalized hypergeometric function; Gauss first and second summation theorems; Watson theorem.

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1. INTRODUCTION

In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role. Applications of the above mentioned theorems are well known now. For interesting applications, we refer a paper by Bailey [1].

Here we shall mention the following summation theorems that will be required in our present investigation.

Gauss's summation theorem [1, 2, 6, 9] :

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.1}$$

provided $\mathrm{Re}(c-a-b)>0$.

Gauss's second summation theorem [1, 2, 6, 9] :

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix};\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{1.2}$$

A special case of (1.1) [6, p.49] :

$$ {}_2F_1\left[\begin{matrix} -\frac{n}{2},\ -\frac{n}{2}+\frac{1}{2} \\ c+\frac{1}{2} \end{matrix};1\right] = \frac{2^n\,(c)_n}{(2c)_n}. \tag{1.3}$$

The aim of this short research note is to provide a new proof of the following classical Watson's summation theorem for the series ${}_3F_2$ [9] viz.

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix};1\right] \\ &\qquad = \frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\,\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})\,\Gamma(c-\frac{1}{2}a+\frac{1}{2})\,\Gamma(c-\frac{1}{2}b+\frac{1}{2})}, \end{aligned} \tag{1.4}$$

provided $\mathrm{Re}(2c-a-b)>-1$.

The theorem is obtained by evaluating an infinite integral in two ways and making use of the known summation theorems (1.1) to (1.3).

2. DERIVATION OF (1.4)

In order to establish (1.4), we proceed as follows. Consider the infinite integral

$$ I = \int_0^\infty e^{-t}\,t^{d-1}\,{}_3F_3\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+1),\ d,\ 2c \end{matrix};t\right]dt,$$

provided $\mathrm{Re}(d)>0$.

Now, expressing ${}_3F_3$ as a series, changing the order of integration and summation (which is easily seen to be justified due to uniform convergence of the series involved in the process), we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(c)_n}{(\frac{1}{2}(a+b+1))_n\,(d)_n\,(2c)_n\,n!} \int_0^{\infty} e^{-t}\,t^{d+n-1}dt.$$

Evaluating the gamma integral and using the relation

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

we have, after some algebra

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(c)_n}{(\frac{1}{2}(a+b+1))_n\,(2c)_n\,n!}. \tag{2.1}$$

Summing up the series, we have

$$I = \Gamma(d)\,{}_3F_2\left[\begin{matrix} a\,, & b\,, & c \\ \frac{1}{2}(a+b+1), & 2c & \end{matrix} ;1\right]. \tag{2.2}$$

On the other hand, writing (2.1) in the form

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(\frac{1}{2}(a+b+1))_n\,2^n\,n!} \left\{\frac{2^n\,(c)_n}{(2c)_n}\right\}.$$

Using the result (1.3), we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(\frac{1}{2}(a+b+1))_n\,2^n\,n!}\,{}_2F_1\left[\begin{matrix} -\frac{1}{2}n, \; -\frac{1}{2}n+\frac{1}{2} \\ c+\frac{1}{2} \end{matrix} ;1\right].$$

Now expressing ${}_2F_1$ as a series, we have after some simplification

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_n\,(b)_n\,(-\frac{1}{2}n)_m\,(-\frac{1}{2}n+\frac{1}{2})_m}{(\frac{1}{2}(a+b+1))_n\,2^n\,(c+\frac{1}{2})_m\,m!\,n!}.$$

Using the identity

$$(-n)_{2m} = 2^{2m}\left(-\frac{1}{2}n\right)_m\left(-\frac{1}{2}n+\frac{1}{2}\right)_m = \frac{n!}{(n-2m)!},$$

we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{(a)_n\,(b)_n}{(\frac{1}{2}(a+b+1))_n\,(c+\frac{1}{2})_m\,2^{2m+n}\,m!\,(n-2m)!}.$$

Now replacing n by $n+2m$ and using a known result [6, Equ.8, p.57]

$$\sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} A(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m,n+2m),$$

we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_{n+2m}\,(b)_{n+2m}}{(\frac{1}{2}(a+b+1))_{n+2m}\, 2^{n+4m}\,(c+\frac{1}{2})_m\, m!\, n!}.$$

Using the identity

$$(a)_{n+2m} = (a)_{2m}(a+2m)_n$$

and after some simplification, we have

$$I = \Gamma(d) \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b)_{2m}}{(\frac{1}{2}(a+b+1))_{2m}\, 2^{4m}\,(c+\frac{1}{2})_m\, m!} \sum_{n=0}^{\infty} \frac{(a+2m)_n\,(b+2m)_n}{(\frac{1}{2}(a+b+1)+2m)_n\, 2^n\, n!}.$$

Summing up the inner series, we have

$$I = \Gamma(b) \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b)_{2m}}{(\frac{1}{2}(a+b+1))_{2m}\, 2^{4m}\,(c+\frac{1}{2})_m\, m!}\, {}_2F_1\left[\begin{matrix} a+2m,\ b+2m \\ \frac{1}{2}(a+b+1)+2m \end{matrix}; \frac{1}{2}\right]. \tag{2.3}$$

We observe here that the ${}_2F_1$ on the right-hand side of (2.3) can be evaluated with the help of Gauss's second summation theorem (1.2) and making use of the identity

$$(a)_{2m} = 2^{2m}\left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a+\frac{1}{2}\right)_m,$$

we have after some simplification,

$$I = \Gamma(d) \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_m\,(\frac{1}{2}b)_m}{(c+\frac{1}{2})_m\, m!}.$$

Summing up the series, we have

$$I = \Gamma(d) \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\ \frac{1}{2}b \\ c+\frac{1}{2} \end{matrix}; 1\right].$$

Finally, evaluating ${}_2F_1$ with the help of classical Gauss's summation theorem (1.1), we have

$$I = \Gamma(d) \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\,\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})\,\Gamma(c-\frac{1}{2}a+\frac{1}{2})\,\Gamma(c-\frac{1}{2}b+\frac{1}{2})}. \tag{2.4}$$

Hence from equation (2.2) and (2.4), we at once get Watson's summation theorem (1.4). This completes the proof of Watson's summation theorem.

Remark: For other proofs of Watson's summation theorem, we refer [3, 4, 5, 7, 8, 10, 11]

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