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REFINEMENT OF JENSEN'S INEQUALITY FOR ANALYTICAL CONVEX (CONCAVE) FUNCTIONS

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ABSTRACT. The well-known Jensen inequality and Hermite–Hadamard inequality were extended using iterated integrals by Z. Retkes in 2008 and then by P. Kórus in 2019. In this paper, we consider analytical convex (concave) functions in order to obtain new refinements of Jensen's inequality. We apply the main result to the classical HM–GM–AM, AM–RMS, triangle inequalities and present an application to the geometric series. We also give Mercer type variants of Jensen's inequality.

Key words and phrases: Analytical functions; Convex functions; Jensen's inequality, Hermite–Hadamard inequality; HM–GM–AM–RMS inequality; Triangle inequality; Geometric series; Mercer's inequality.

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1. INTRODUCTION

For an $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$ convex function, Jensen's celebrated inequality can be stated as

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

for any $x_i \in (a, b)$, $i = 1, \ldots, n$, see e.g. [6]; while the Hermite–Hadamard inequality

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx \leq \frac{f(x_1) + f(x_2)}{2}$$

holds for any $a < x_1 < x_2 < b$, see e.g. [5]. These inequalities were extended by several authors, see e.g. [1, 2, 9] and the references therein. In [7] and later in [3], the authors gave the following generalization using the notion of iterated integrals of the function f, introduced in [7].

Theorem 1.1. [3, 7] *Let $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, $x_i \in (a, b)$, $i = 1, \ldots, n$, such that $x_i \neq x_j$ if $1 \leq i < j \leq n$. Then the following refinement of Jensen's inequality holds:*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq (n-1)!\sum_{i=1}^{n} \frac{F^{[n-1]}(x_i)}{\Pi_i(x_1, \ldots, x_n)} \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i),$$

where $F^{[j]}$ is the j-th iterated integral of f and

$$\Pi_i(x_1, \ldots, x_n) = \prod_{\substack{j=1 \\ j\neq i}}^{n} (x_i - x_j).$$

In the concave case "$\leq$" is changed to "$\geq$".

An interesting corollary of the above theorem (case $f(x) = x$) is given in [7].

Formula 1.1. *Let $x_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $x_i \neq x_j$ if $1 \leq i < j \leq n$. Then*

$$\sum_{i=1}^{n} \frac{x_i^n}{\Pi_i(x_1, \ldots, x_n)} = \sum_{i=1}^{n} x_i.$$

2. MAIN RESULT

Let assume for the sake of simplicity that $f : [0, a\} \to \mathbb{R}$ is convex analytical function and $\}$ can be $)$ or $]$, in other words the domain of definition can be open or closed on its right hand side, $x_i \in [0, a\}$ for $i = 1, \ldots n$, $x_i \neq x_j$ for $i \neq j$. Then f has a power series expansion around 0 of the form

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Under these conditions, considering Theorem 1.1, we have the following refinement of Jensen's inequality.

Formula 2.1.

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq (n-1)!\sum_{k=0}^{\infty} \frac{c_k H_k(x_1, \ldots, x_n)}{(k+1)(k+2)\cdots(k+n-1)} \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i),$$

where $H_k(x_1,\ldots,x_n)$ is the complete homogeneous symmetric polynomial of order k, that is

$$H_k(x_1,\ldots,x_n) = \sum_{1\le i_1\le\ldots\le i_k\le n} x_{i_1}\cdots x_{i_k}.$$

In the concave case "$\le$" is changed to "$\ge$".

In order to prove Formula 2.1, we need the extension of the following application of Theorem 1.1. Under the conditions of Theorem 1.1,

$$\sum_{i=1}^{n} \frac{x_i^j}{\Pi_i(x_1,\ldots,x_n)} = \begin{cases} 0 & \text{if} \quad j=0,\ldots,n-2, \\ 1 = H_0(x_1,\ldots,x_n) & \text{if} \quad j=n-1, \\ \sum_{i=1}^n x_i = H_1(x_1,\ldots,x_n) & \text{if} \quad j=n, \end{cases} \tag{2.1}$$

see [7, Proposition 1]. The following lemma generalizes (2.1) to arbitrary natural exponents above n.

Lemma 2.1. *Under the conditions of Theorem 1.1 we have the following identities:*

$$\sum_{i=1}^{n} \frac{x_i^{n-1+k}}{\Pi_i(x_1,\ldots,x_n)} = H_k(x_1,\ldots,x_n)$$

for $k=0,1,2,\ldots$.

Proof. The generating function $g(t)$ of the sequence $\{H_k\}_{k=0}^{\infty}$ is given by

$$\sum_{k=0}^{\infty} H_k(x_1,\ldots,x_n)t^k = \prod_{i=1}^{n} \frac{1}{1-x_i t} = g(t). \tag{2.2}$$

Then, by the results of [8], for $g(t)$ we have

$$\begin{aligned} \prod_{i=1}^{n} \frac{1}{1-x_i t} &= (-1)^{n-1} \sum_{i=1}^{n} \frac{1}{(1-x_i t)\Pi_i(1-x_1 t,\ldots,1-x_n t)} \\ &= \sum_{i=1}^{n} \frac{1}{(1-x_i t)\Pi_i(x_1 t,\ldots,x_n t)} = \sum_{j=0}^{\infty}\sum_{i=1}^{n} \frac{x_i^j t^j}{\Pi_i(x_1 t,\ldots,x_n t)} \\ &= \sum_{j=0}^{\infty}\sum_{i=1}^{n} \frac{x_i^j t^{j-n+1}}{\Pi_i(x_1,\ldots,x_n)} = \sum_{j=n-1}^{\infty}\sum_{i=1}^{n} \frac{x_i^j t^{j-n+1}}{\Pi_i(x_1,\ldots,x_n)} \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{n} \frac{x_i^{n-1+k} t^k}{\Pi_i(x_1,\ldots,x_n)} \end{aligned} \tag{2.3}$$

by keeping (2.1) in mind for $j=0,\ldots,n-2$. Therefore, the coefficients of corresponding powers of t in (2.2) and (2.3) are equal. ∎

Proof of Formula 2.1. By virtue of Theorem 1.1, we need to evaluate the sum

$$\sum_{i=1}^{n} \frac{F^{[n-1]}(x_i)}{\Pi_i(x_1,\ldots,x_n)}. \tag{2.4}$$

For the iterated integrals we have

$$F^{[0]}(x) = f(x),\ F^{[1]}(x) = \int_0^x f(s)\,ds = \int_0^x \sum_{k=0}^{\infty} c_k s^k\,ds = \sum_{k=0}^{\infty} \frac{c_k}{k+1} x^{k+1}, \dots,$$

$$F^{[n-1]}(x) = \sum_{k=0}^{\infty} \frac{c_k}{(k+1)(k+2)\cdots(k+n-1)} x^{k+n-1}.$$

Substituting this formula into (2.4) and applying Lemma 2.1 yields

$$\begin{aligned}\sum_{i=1}^{n} \frac{F^{[n-1]}(x_i)}{\Pi_i(x_1,\dots,x_n)} &= \sum_{i=1}^{n} \frac{1}{\Pi_i(x_1,\dots,x_n)} \sum_{k=0}^{\infty} \frac{c_k}{(k+1)(k+2)\cdots(k+n-1)} x_i^{k+n-1} \\ &= \sum_{k=0}^{\infty} \frac{c_k}{(k+1)(k+2)\cdots(k+n-1)} \sum_{i=1}^{n} \frac{x_i^{k+n-1}}{\Pi_i(x_1,\dots,x_n)} \\ &= \sum_{k=0}^{\infty} \frac{c_k H_k(x_1,\dots,x_n)}{(k+1)(k+2)\cdots(k+n-1)},\end{aligned}$$

which proves Formula 2.1. ∎

In aesthetic point of view it is useful to note the following compact form of Formula 2.1:

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \sum_{k=0}^{\infty} \frac{c_k H_k(x_1,\dots,x_n)}{\binom{n-1+k}{k}} \le \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

3. APPLICATIONS

We demonstrate the usefulness of Formula 2.1 through the following examples.

Example 3.1. *Harmonic mean–geometric mean–arithmetic mean (HM–GM–AM) inequality. Let* $f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$*. Then we have*

$$\begin{aligned}e^{\frac{1}{n}\sum_{i=0}^{n} x_i} &\le (n-1)! \sum_{k=0}^{\infty} \frac{H_k(x_1,\dots,x_n)}{k!(k+1)(k+2)\cdots(k+n-1)} \\ &= 1 + \sum_{k=1}^{\infty} \frac{H_k(x_1,\dots,x_n)}{n(n+1)\cdots(n+k-1)} \le \frac{1}{n}\sum_{i=1}^{n} e^{x_i}.\end{aligned}$$

Since $e^{x_i} > 0$ *then there exists* $y_i > 0$ *such that* $e^{x_i} = y_i$ *that is* $x_i = \ln y_i$ *for* $i = 1,\dots n$*. Substituting these values leads us to the refinement of the GM–AM inequality:*

$$\sqrt[n]{y_1\cdots y_n} \le 1 + \sum_{k=1}^{\infty} \frac{H_k(\ln y_1,\dots,\ln y_n)}{n(n+1)\cdots(n+k-1)} \le \frac{y_1+\ldots+y_n}{n}.$$

If we plug $y_i = \frac{1}{x_i}$*, we have the refinement of the HM–GM inequality:*

$$\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \le \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k H_k(\ln x_1,\dots,\ln x_n)}{n(n+1)\cdots(n+k-1)}\right]^{-1} \le \sqrt[n]{x_1\cdots x_n}.$$

Example 3.2. *Arithmetic mean–root mean square (AM–RMS) inequality. Let $f(x) = x^2$ that is analytical on the whole real line with $c_k = 0$, $k = 0, 1, 3, 4, \ldots$ and $c_2 = 1$. Applying Formula 2.1 produces*

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \leq \frac{H_2(x_1, \ldots, x_n)}{\binom{n+1}{2}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

and taking the square root yields the refinement of the AM–RMS inequality:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leq \sqrt{\frac{2H_2(x_1, \ldots, x_n)}{n(n+1)}} \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}.$$

Example 3.3. *Triangle inequality. Let $f(x) = |x|$ and assume the conditions of Theorem 1.1. Since f is not analytical – in fact non differentiable at $x = 0$ – we apply Theorem 1.1 by working out the sequence of iterated integrals:*

$$F^{[0]}(x) = |x|,\ F^{[1]}(x) = \int_0^x |s|\, ds = \begin{cases} \frac{x^2}{2} & \text{if } x \geq 0 \\ -\frac{x^2}{2} & \text{if } x < 0 \end{cases} \quad = \mathit{sgn}(x) \cdot \frac{x^2}{2}.$$

Using induction simply gives the general form of

$$F^{[n-1]}(x) = \mathit{sgn}^{n-1}(x) \cdot \frac{|x|^n}{n!}$$

and applying Theorem 1.1 produces the following refinement

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} \frac{\mathit{sgn}^{n-1}(x_i)|x_i|^n}{\Pi_i(x_1, \ldots, x_n)} \leq \sum_{i=1}^{n} |x_i|.$$

Example 3.4. *Geometric series. Let $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots$. Then f is analytical in $[0, 1)$ so we might apply Formula 2.1 with $c_k = 1$, $k = 0, 1, 2, \ldots,$ and $0 \leq x_i < 1$ for $i = 1, \ldots, n$. This setting directly yields*

$$\frac{n}{n - \sum_{i=1}^{n} x_i} \leq \sum_{k=0}^{\infty} \frac{H_k(x_1, \ldots, x_n)}{\binom{n-1+k}{k}} \leq \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1 - x_i}.$$

4. MERCER TYPE RESULTS

In [4], the following variant of Jensen's inequality was proved.

Theorem 4.1. *Let f be a convex function on an interval containing the numbers $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and w_i $(1 \leq i \leq n)$ be positive weights associated with these x_i with $\sum_{i=1}^{n} w_i = 1$. Then we have*

$$f\left(x_1 + x_n - \sum_{i=1}^{n} w_i x_i\right) \leq f(x_1) + f(x_n) - \sum_{i=1}^{n} w_i f(x_i).$$

In the concave case "$\leq$" is changed to "$\geq$".

We will prove the following variant of Theorem 1.1 by extending the above theorem.

Theorem 4.2. *Let f be a convex function on an interval containing the numbers $0 < x_1 \leq x_2 \leq \ldots \leq x_n$. Then*

$$f\left(x_1 + x_n - \frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq (-1)^{n-1}(n-1)! \sum_{i=1}^{n} \frac{F^{[n-1]}(x_1 + x_n - x_i)}{\Pi_i(x_1, \ldots, x_n)} \leq f(x_1) + f(x_n) - \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{4.1}$$

In the concave case "$\leq$" is changed to "$\geq$".

Proof. If in Theorem 1.1 we substitute $x_1 + x_n - x_i$ in place of x_i, then we immediately have (4.1) by noting

$$\begin{aligned}\Pi_i(x_1, \ldots, x_n) = \prod_{\substack{j=1\\ j\neq i}}^{n} (x_i - x_j) &= (-1)^{n-1} \prod_{\substack{j=1\\ j\neq i}}^{n} ((x_1 + x_n - x_i) - (x_1 + x_n - x_j)) \\ &= (-1)^{n-1}\Pi_i(x_1 + x_n - x_1, x_1 + x_n - x_2, \ldots, x_1 + x_n - x_n)\end{aligned}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} f(x_1 + x_n - x_i) \leq f(x_1) + f(x_n) - \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

that is a consequence of the inequality

$$f(x_1 + x_n - x_i) \leq f(x_1) + f(x_n) - f(x_i)$$

proved in [4]. ∎

Assuming the conditions of Formula 2.1, we have the following variant of Jensen's inequality.

Formula 4.1.

$$\begin{aligned}&f\left(x_1 + x_n - \frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &\leq (n-1)! \sum_{k=0}^{\infty} \frac{c_k H_k(x_1 + x_n - x_1, x_1 + x_n - x_2, \ldots, x_1 + x_n - x_n)}{(k+1)(k+2)\cdots(k+n-1)} \\ &\leq f(x_1) + f(x_n) - \frac{1}{n}\sum_{i=1}^{n} f(x_i).\end{aligned}$$

In the concave case "$\leq$" is changed to "$\geq$".

Proof. The proof is analogous to that of Formula 2.1. ∎

Remark 4.1. Applications of Formula 4.1 analogous to Examples 3.1–3.4 can be given similarly.

Finally, we give an equation analogous to Formula 1.1 as a corollary of Theorem 4.2 (case $f(x) = x$).

Formula 4.2. *Let $x_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $0 < x_1 < x_2 < \ldots < x_n$. Then*

$$\begin{aligned}&(-1)^{n-1}\left[\frac{x_1^n}{\Pi_i(x_1, \ldots, x_n)} + \frac{x_n^n}{\Pi_i(x_1, \ldots, x_n)} + \sum_{i=2}^{n-1} \frac{(x_1 + x_n - x_i)^n}{\Pi_i(x_1, \ldots, x_n)}\right] \\ &= (n-1)(x_1 + x_n) - \sum_{i=2}^{n-1} x_i.\end{aligned}$$

REFERENCES

[1] M. BESSENYEI and Z. PÁLES, On generalized higher-order convexity and Hermite–Hadamard-type inequalities, *Acta Sci. Math. (Szeged)*, **70** (2004), No. 1, pp. 13–24.

[2] S. S. DRAGOMIR and C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000, [`Online: https://rgmia.org/papers/monographs/Master.pdf`].

[3] P. KÓRUS, An extension of the Hermite–Hadamard inequality for convex and s-convex functions, *Aequat. Math.*, **93** (2019), pp. 527–534.

[4] A. MCD. MERCER, A variant of Jensen's inequality, *J. Inequal. Pure Appl. Math. (JIPAM)*, **4** (2003), No. 4, Article 73.

[5] D. S. MITRINOVIĆ and I. B. LACZKOVIĆ, Hermite and convexity, *Aequat. Math.*, **28** (1985), pp. 229–232.

[6] D. S. MITRINOVIĆ, J. E. PEČARIĆ and A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[7] Z. RETKES, An extension of the Hermite–Hadamard inequality, *Acta Sci. Math. (Szeged)*, **74** (2008), pp. 95–106.

[8] Z. RETKES, Applications of the extended Hermite-Hadamard inequality, *J. Inequal. Pure Appl. Math. (JIPAM)*, **7** (2006), No. 1, Article 24.

[9] M. TARIQ, S. K. NTOUYAS and A. A. SHAIKH, A Comprehensive Review of the Hermite–Hadamard Inequality Pertaining to Fractional Integral Operators, *Mathematics*, **11** (2023), Article 1953.