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ON SATURATION OF NORM CONVERGENCE OF WALSH-FOURIER MATRIX TRANSFORM MEANS

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ABSTRACT. In this paper we investigate the saturation of norm convergence issues for regular matrix transform means in case of Walsh-Paley system.

The main result is the observation of equality

$$\left\|\sigma_n^T(f) - f\right\|_p = o(a_n),$$

where a_n sequence of positive numbers tends to zero and there exists constant c, for which $t_{1,n} \geq ca_n$ for every $n \in \mathbb{P}$.

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1. DEFINITIONS AND NOTATIONS

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote the discrete cyclic group of order 2 by $\mathbb{Z}_2$. The group operation is the modulo 2 addition. Let every subset be open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given by $\mu(\{0\}) = \mu(\{1\}) = 1/2$. That is, the measure of a singleton is $1/2$. $G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_2$ and G is called the dyadic group. The elements of the dyadic group G are the sequences $0, 1$. That is, $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition (modulo 2, denoted by $+$), the normalized Haar measure μ is the product measure, and the topology is the product topology. For an other topology on the dyadic group see e.g. [9]. The dyadic intervals are defined in the usual way

$$I_0(x) := G,\ \ I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

for $x \in G, n \in \mathbb{P}$. They form a base for the neighbourhoods of G.

Let $L_p(G)$ with $1 \leq p < \infty$ denote the usual Lebesgue spaces on G (with the corresponding norm $|.|_p$).

Next, we define the modulus of continuity in $L_p(G), 1 \leq p < \infty$, of a function $f \in L_p(G)$ by

$$\omega_p(f, \delta) := \sup_{|t|<\delta} \|f(. + t) - f(.)\|_p, \quad \delta > 0,$$

with the notation

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \quad \text{for all } x \in G.$$

The Lipschitz classes in $L_p(G)$ (for all $\alpha > 0$) are defined as

$$\text{Lip}(\alpha, p, G) := \{f \in L_p(G) : \omega_p(f, \delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

We now introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_n(x) := (-1)^{x_n} \quad (x \in G, n \in \mathbb{N}).$$

The sequence of the Walsh-Paley functions is the product system of the Rademacher functions. Namely, every natural number n can be uniquely expressed in the base 2 number system in the form

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad n_k \in \{0, 1\}\ (k \in \mathbb{N}),$$

where only a finite number of n_k is nonzero. Denote the order of $n \in \mathbb{P}$ by $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$. It means $2^{|n|} \leq n < 2^{|n|+1}$. The Walsh-Paley functions are $w_0 := 1$ and for $n \in \mathbb{P}$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

It is known [18] that the Walsh-Paley system $\{w_n, n \in \mathbb{N}\}$ is the character system of $(G, +)$.

The jth Fourier-coefficient, the kth rectangular partial sum of the Fourier series, the Fejér mean, and the nth Dirichlet kernel are defined by

$$\hat{f}(j) := \int_G f w_j d\mu,\ S_k(f) := \sum_{j=0}^{k-1} \hat{f}(j) w_j,\ \sigma_n(f) := \frac{1}{n} \sum_{k=1}^{n} S_k(f),$$

$$D_n := \sum_{k=0}^{n-1} w_k,\ D_0 := 0.$$

Let $\{q_k : \ k \in \mathbb{N}\}$ be a sequence of nonnegative numbers. The nth Nörlund mean is defined by

$$t_n(f) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f;x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k \ (n \in \mathbb{P})$. It is always assumed that $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty.$$

In this case, the summability method generated by $\{q_n, n \in \mathbb{N}\}$ is regular (see [35]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

Let $T := (t_{k,n})_{k,n=0}^{\infty}$ be a doubly infinite matrix of real numbers. It is always supposed that matrix T is upper triangular. It means, that let $t_{k,n} := 0$, if $n < k$. Let us define the nth matrix transform mean (or linear mean) determined by the matrix T

$$\sigma_n^T(f) := \sum_{k=1}^{n} t_{k,n} S_k(f),$$

where $\{t_{k,n} : 1 \le k \le n, \ k \in \mathbb{P}\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$.

The nth matrix transform kernel is defined by

$$K_n^T := \sum_{k=1}^{n} t_{k,n} D_k.$$

It is easily seen that

$$\sigma_n^T(f;x) = \int_G f(u) K_n^T(u+x) d\mu(u),$$

where $x, u \in G$. This equality (and its analogous versions for special means) shows us the need to observe kernel functions.

2. HISTORICAL OVERVIEW

Matrix transform means are common generalizations of several well-known summation methods. It follows from simple consideration that Fejér (or the $(C,1)$), Cesàro (or the (C,α)), Riesz, Nörlund and weighted means are special cases of the matrix transform summation method introduced above.

For matrix transform means for the trigonometric system see e.g. results of Chandra [10] and Leindler [22], for the Walsh system, see paper of Blyumin [8].

In the classical book by Schipp, Wade, Simon and Pál [28], on p. 191. we read inequality

$$\|\sigma_{2^s}(f) - f\|_X \le \omega_X \left(f; 2^{-s}\right) + \sum_{k=0}^{s-1} 2^{k-s} \omega_X \left(f; 2^{-k}\right), \tag{2.1}$$

where σ is the Fejér mean operator, X is a homogeneous Banach space (for example, an arbitrary $L_p(G)$ space with $1 \le p < \infty$ and the space of continuous functions C) and ω_X is the modulus of continuity for functions in X, using norm of X.

Móricz and Siddiqi observed this result [25] using the Walsh-Nörlund summation method, and Móricz and Rhoades [24] using the Walsh weighted mean method. Móricz and Siddiqi [25], and later Móricz and Rhoades [24] in these papers proved their generalized results in an analogous form to inequality (2.1).

As special cases, Móricz and Siddiqi obtained the earlier results of Yano [34], Jastrebova [20] and Skvortsov [29] on the rate of the approximation by Walsh-Cesàro means. A common generalization of the two results of Móricz and Siddiqi [25] and Móricz and Rhoades [24] was given by Nagy and the author in the paper [4].

In 2008, Fridli, Manchanda and Siddiqi generalized Móricz and Siddiqi's result to homogeneous Banach spaces and dyadic Hardy spaces [12]. These results were generalized by Nagy, Salim and the author in [5].

Recently, Baramidze, Gát, Goginava, Nagy, Memić, Persson, Salim, Tephnadze, Wall and the author presented some results on the Nörlund and matrix transform means [1, 2, 5, 7, 17, 23]. See also [31, 33].

We mention that Iofina and Volosivets obtained similar results on Vilenkin systems (which are generalizations of the Walsh-Paley system) with similar assumptions and different methods (independently form technics of Móricz, Rhoades, Siddiqi, Fridli and others) for the matrix transform means in [19].

For Marcinkiewicz means and other two-dimensional results on Walsh-Paley system, see e.g. [3, 6, 26, 27], for d-dimensional ones see [13, 14, 15, 16, 30, 32].

In their paper [25] Móricz and Siddiqi proved the following, among other things. Let $\{q_k : k \in \mathbb{P}\}$ be a nondecreasing sequence non-negative numbers such that condition $nq_n/Q_n = O(1)$ is satisfied. If $f \in \text{Lip}(\alpha, p, G)$ and $\alpha > 1$, then

$$\|t_n(f) - f\|_p = O\left(\frac{1}{n}\right).$$

After that they formulated the problem: "How can one characterize those functions $f \in L_p$ such that

$$\|\sigma_n(f) - f\|_p = O\left(\frac{1}{n}\right)$$

for some $1 \le p \le \infty$?" The answer was given by Fridli [11]. Fridli [11] and Joó [21] also discussed inequality in their works

$$\|\sigma_n(f) - f\|_p = o\left(\frac{1}{n}\right). \tag{2.2}$$

In this paper we deal with the generalization of inequality (2.2) to matrix transform means.

3. RESULTS

Lemma 3.1. *Let $f \in L_p(G)$, where $1 < p < \infty$. For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \le k \le n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{3.1}$$

is satisfied. Then for any $j, n \in \mathbb{P}$

$$\left|\hat{f}_j\right| \sum_{k=1}^{j} t_{k,n} \le \left\|\sigma_n^T(f) - f\right\|_p.$$

Proof. Since

$$\sigma_n^T(f) = \sum_{k=1}^{n} t_{k,n} S_k(f) = \sum_{k=1}^{n} t_{k,n} \sum_{i=0}^{k-1} \hat{f}_i w_i$$

and $f = \sum_{s=0}^{\infty} \hat{f}_s w_s$ a.e. for $\forall f \in L_p(G)$, where $1 < p < \infty$, we get for any $j \in \mathbb{P}$, that

$$\left|\int_G \left(\sigma_n^T(f) - f\right) w_j d\mu\right| = \left|\int_G \left(\sum_{k=1}^{n} t_{k,n} \sum_{i=0}^{k-1} \hat{f}_i w_i - \sum_{s=0}^{\infty} \hat{f}_s w_s\right) w_j d\mu\right| =: (*).$$

If $j < n$, then

$$(*) = \left| \sum_{k=j+1}^{n} t_{k,n} \hat{f}_j - \hat{f}_j \right| = \left| \hat{f}_j \right| \sum_{k=1}^{j} t_{k,n},$$

if $j \geq n$, then

$$(*) = \left| -\hat{f}_j \right| = \left| \hat{f}_j \right|,$$

because of orthonormality of Walsh-Paley system and of Condition (3.1).

If $1/p + 1/q = 1$, then Hölder's inequality yields

$$\begin{aligned} \left| \int_G \left(\sigma_n^T(f) - f \right) w_j d\mu \right| &\leq \left\| \sigma_n^T(f) - f \right\|_p \|w_j\|_q \\ &= \left\| \sigma_n^T(f) - f \right\|_p, \end{aligned}$$

so Lemma 3.1 is proved. ∎

Remark 3.1. We mention that assumption of (3.1) is natural, because many well-known (see e.g. Fejér, Cesàro, Nörlund, weighted, Riesz) means satisfy it, and this equality is part of the regularity conditions [35, page 74.]

Corollary 3.2. *Let $f \in L_p(G)$, where $1 < p < \infty$. For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{3.2}$$

is satisfied. If $n \leq j$, then

$$\left| \hat{f}_j \right| \leq \left\| \sigma_n^T(f) - f \right\|_p,$$

specially,

$$\left| \hat{f}_n \right| \leq \left\| \sigma_n^T(f) - f \right\|_p.$$

Proof. Since $t_{k,n} = 0$ if $n < k$, using Lemma 3.1 and Condition (3.2) we obtain the statement of Corollary 3.2 immediately. ∎

Remark 3.2. The Riemann-Lebesgue Lemma says, that if $f \in L_1(G)$, then $\hat{f}_n \to 0$ ([28], page 24), so Corollary 3.2 does not contradict our expectations with respect to norm convergence.

Remark 3.3. It is known, that Walsh-Fejér mean (as a special case of matrix transform means) tends to every $f \in L_1(G)$ functions in norm, so Corollary 3.2 implies the Riemann-Lebesgue Lemma.

Lemma 3.3. *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{3.3}$$

is satisfied. If f is a constant function, then

$$\sigma_n^T(f) = f$$

for every $n \in \mathbb{P}$.

Proof. Using the simple fact, that

$$\int_G w_i(t)d\mu(t) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise}, \end{cases}$$

assuming $f(x) := f$ $(\forall x \in G)$ is a constant function, using Condition (3.3) we get, that

$$\begin{aligned} \sigma_n^T(f;x) =& \sum_{k=1}^{n} t_{k,n} \sum_{i=0}^{k-1} \left(\int_G f w_i(t)d\mu(t) \right) w_i(x) \\ =& f \sum_{k=1}^{n} t_{k,n} = f, \end{aligned}$$

so we are ready. ∎

Theorem 3.4. *Let $f \in L_p(G)$, where $1 < p < \infty$. For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \le k \le n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied. Let a_n any sequence of positive numbers tends to zero. If there exists $0 < c$ absolute constant, that

$$t_{1,n} \ge c a_n, \tag{3.4}$$

for $\forall n \in \mathbb{P}$, then

$$\left\| \sigma_n^T(f) - f \right\|_p = o(a_n)$$

holds if and only if f is a constant function.

Proof. Let f is a constant function. This part is implied by Lemma 3.3.

On the other hand, if

$$\left\| \sigma_n^T(f) - f \right\|_p = o(a_n),$$

then Lemma 3.1 yields

$$0 \le \frac{1}{a_n} \left| \hat{f}_j \right| \sum_{k=1}^{j} t_{k,n} \le \frac{1}{a_n} \left\| \sigma_n^T(f) - f \right\|_p \to 0,$$

so

$$\frac{1}{a_n} \left| \hat{f}_j \right| \sum_{k=1}^{j} t_{k,n} \to 0,$$

as $n \to \infty$. But based on Condition (3.4) we get for $\forall j \in \mathbb{P}$, that

$$\begin{aligned} \frac{1}{a_n} \left| \hat{f}_j \right| \sum_{k=1}^{j} t_{k,n} \ge& \frac{1}{a_n} \left| \hat{f}_j \right| t_{1,n} \\ \ge& \frac{1}{a_n} \left| \hat{f}_j \right| c a_n = c \left| \hat{f}_j \right| \ge 0. \end{aligned}$$

Summarising our results,

$$\lim_{n\to\infty} \hat{f}_j = 0$$

$\forall j \in \mathbb{P}$. It means, that only $\hat{f}_0$ can be differ from 0, but $\hat{f}_j = 0$ for $\forall j \in \mathbb{P}$. In this case $f(x) = \hat{f}_0 w_0(x) = \hat{f}_0$, hence f is a constant function. ∎

Corollary 3.5. *Let $f \in L_p(G)$, where $1 < p < \infty$. If sequence of non-negative numbers $\{t_{k,n} : 1 \leq k \leq n\}$ is non-increasing for every fixed $n \in \mathbb{P}$ such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{3.5}$$

is satisfied. Then

$$\left\| \sigma_n^T(f) - f \right\|_p = o\left(\frac{1}{n}\right)$$

holds if and only if f is a constant function.

Proof. The "if" case is trivial based on Lemma 3.3.

If sequence $\{t_{k,n} : 1 \leq k \leq n\}$ is non-increasing for every fixed $n \in \mathbb{P}$, using condition (3.5) we get

$$t_{1,n} \geq \frac{1}{n},$$

so choosing $a_n = 1/n$ and $c = 1$ in Theorem 3.5 we are ready. ∎

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