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THE DEGREE OF HOMOGENEITY: DEFINITION, GEOMETRIC INTERPRETATION AND NEW, DIRECT PROOFS OF EULER'S THEOREM

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ABSTRACT. We show how the degree of homogeneity of a function is a highly useful, precise measure of the sensitivity of a function to a change in its variables. This measure can be evaluated directly thanks to a simple geometric construct, entirely independently of measurement units. The usefulness of this concept is also illustrated by the fact that it leads to new, direct proofs, geometric as well as algebraic, of Euler's theorem; this is in contrast to the traditional approach that requires a limiting process.

Key words and phrases: Homogeneous functions; Degree of homogeneity; Euler's theorem.

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1. INTRODUCTION AND MOTIVATION.

While the total differential and the directional derivative are always very precisely defined and given geometric representations in 3-d space, apparently this is never done in the case of the degree of homogeneity k, also a measure of the sensitivity of a function to a change in its variables. This small neglect can be considered as unfortunate, because defining and picturing this quantity offers quite a number of advantages:

1. It provides at a glance, *independently of any measurement units*, an order of magnitude of this sensitivity.

2. The simple geometric construct of k leads to an immediate geometric proof of Euler's theorem in 3-d space.

3. From the very definition of k we can directly obtain an algebraic proof of Euler's theorem in n-dimensional space, without any recourse to a limiting process.

Our plan is as follows. In Section 2, we define k as a limit, with important properties. Section 3 describes how homogeneous functions can be graphically generated, whatever their degree. We then offer, in Section 4, a geometric interpretation of k, starting with the case $0 < k < 1$. We will observe that this construct has general value for any real k. As an application, in Section 5 we give a geometric proof of Euler's theorem, extended algebraically, in Section 6, to functions of n variables.

2. DEFINING THE DEGREE OF HOMOGENEITY AS A LIMIT.

Consider a homogenous n-variable function $y = f(x_1, ..., x_n)$, i.e. such that $\lambda^k y = f(\lambda x_1, ..., \lambda x_n)$; while λ is a positive number, the power k – the degree of homogeneity – can be any real number. We suppose that point $(x_1, ..., x_n, y)$ does not contain any zero. Let $\Delta\lambda/\lambda = \Delta x_i/x_i = (\lambda x_i - x_i)/x_i = \lambda - 1$, $i = 1, .., n$ designate a *common relative* increase given to each variable x_i, and $\Delta y/y$ the resulting relative increase of y, equal to $\left(\lambda^k y - y\right)/y = \lambda^k - 1$.We note that when all $\Delta x_i \to 0$ simultaneously, $\lambda \to 1$.

Definition. *The degree of homogeneity k is the limit of the ratio of the relative increase of the function to the common relative increase given to each variable when the latter tends to zero.*

Forming the ratio $(\Delta y/y) / (\Delta x_i/x_i)$ and taking limits indeed leads to:

$$\lim_{\Delta x_i \to 0} \frac{\Delta y/y}{\Delta x_i/x_i} = \lim_{\lambda \to 1} \frac{\Delta y/y}{\Delta\lambda/\lambda} \equiv \frac{dy/y}{d\lambda/\lambda} = \lim_{\lambda \to 1} \frac{\lambda^k - 1}{\lambda - 1} = k, \tag{2.1}$$

by L'Hospital's rule.

Conversely, if at any of its points $x_1^0, ..., x_n^0$ a function $y = f(x_1, ..., x_n)$ is such that the ratio $(dy/y) / (d\lambda/\lambda)$ is constant and equal to k, then $y = f(x_1, ..., x_n)$ is homogeneous of degree k. This follows immediately from the integration of

$$\frac{dy}{y} = k\frac{d\lambda}{\lambda}, \tag{2.2}$$

giving

$$\ln y = k \ln \lambda + \ln C,$$

or

$$y = C\lambda^k = g(\lambda). \tag{2.3}$$

The stricly positive constant of integration C is identified by setting $g(1) = f(x_1^0, ..., x_n^0) = C$; we thus have

$$y = g(\lambda) = f(x_1^0, ..., x_n^0)\lambda^k = y_0\lambda^k, \tag{2.4}$$

a power function of λ, whose power is its degree of homogeneity.

The formulation $k = \frac{dy/y}{d\lambda/\lambda}$ sets in full light the significance of k as a sensitivity measure of $y = f(x_1, ..., x_n)$ with respect to a common relative change in its variables $x_1, ..., x_n$. A natural question now is: what kind of precision do we have when we say that a one percent increase in the variables generates in linear approximation a k percent increase in the function? This approximation of the exact value turns out to be excellent, as the following example gives a first indication.

Suppose that our function is $y = 10x_1^{0.4}x_2^{0.2}$, homogeneous of degree $k = 0.6$. Consider an initial point $(x_1^0, x_2^0) = (1000, 500)$, with $y_0 = f(x_1^0, x_2^0) = 549.\,280\,3$. Let us give to these variables a one percent increase, implying $\lambda = 1.01$ and $f(\lambda x_1^1, \lambda x_2^1) = f(1010, 505) = 552.\,569\,4$, i.e. a relative increase $\Delta y/y = 0.599\%$. On the other hand the relative increase in linear approximation is given by $dy/y = kd\lambda/\lambda = 0.6\%$, corresponding to an estimated value of the function $y^* = 552.\,5760$, and a relative error as small as $1.\,19 \times 10^{-5}$.

We will now show that k remains an excellent tool for the estimate of $\Delta y/y$ even when k and λ are significantly different from 1. Let us determine the relative error, denoted ϵ, made by using the degree of homogeneity to determine the new value of the function. At point λ, the exact value of the function is $y = g(\lambda) = y_0\lambda^k$; on the other hand, its linear approximation around $\lambda = 1$, denoted y^*, is

$$y^* = y_0 + g'(1)\,(\lambda - 1) = y_0[1 + k\,(\lambda - 1)] \tag{2.5}$$

and therefore the relative error is

$$\epsilon(k, \lambda) = \frac{y^* - y}{y} = \frac{1 + k\,(\lambda - 1)}{\lambda^k} - 1. \tag{2.6}$$

We first observe that the error made is entirely independent of the *structure* of the function $y = f(x_1, ..., x_n)$, a structure that may be very complicated; nor will it depend on the point $(x_1^0, ..., x_n^0, y_0)$ we start from. It will solely depend on the degree of homogeneity k and on the common relative increase given to each variable.

The following table therefore illustrates the precision generated by k in measuring the sensitivity of *any* homogeneous function of degree k at *any* of its points.

λ \ k	0.6	0.8	1	1.2	1.4
0.8	0.006	0. 004	0	-0.006	-0.016
0.9	0.001	0.0009	0	-0.0013	-0.003
0.99	$1.21 * 10^{-5}$	$8.09 * 10^{-6}$	0	$-1.22 * 10^{-5}$	$-2.85 * 10^{-5}$
1	0	0	0	0	0
1.01	$1.19 * 10^{-5}$	$7.91 * 10^{-6}$	0	$-1.18 * 10^{-5}$	$-2.76 * 10^{-5}$
1.1	0.001	0.0007	0	-0.001	-0.002
1.2	0.004	0.003	0	-0.004	-0.008

Table 1. *The smallness of the error made by k in measuring the sensitivity of any homogeneous function at any of its points.*

Finally, let us observe that the degree of homogeneity k is a pure number, independent of any measurement units. This means that if we can represent it geometrically, we will be able to have at first sight of a homogeneous function a numerical order of magnitude of its sensitivity to changes in its variables – something we cannot do with the directional derivative or the differential, which are unit dependent. In order to do so, let us first show how we can generate a homogeneous function of any degree k.

3. CONSTRUCTING HOMOGENEOUS FUNCTIONS OF DEGREE k.

We first depict, with an example, how such functions can be generated. This will lead to a geometric interpretation of k in a natural, direct way. The benefit of such analysis is that we will be led to a direct, new proof of Euler's theorem, without any recourse to a limit process.

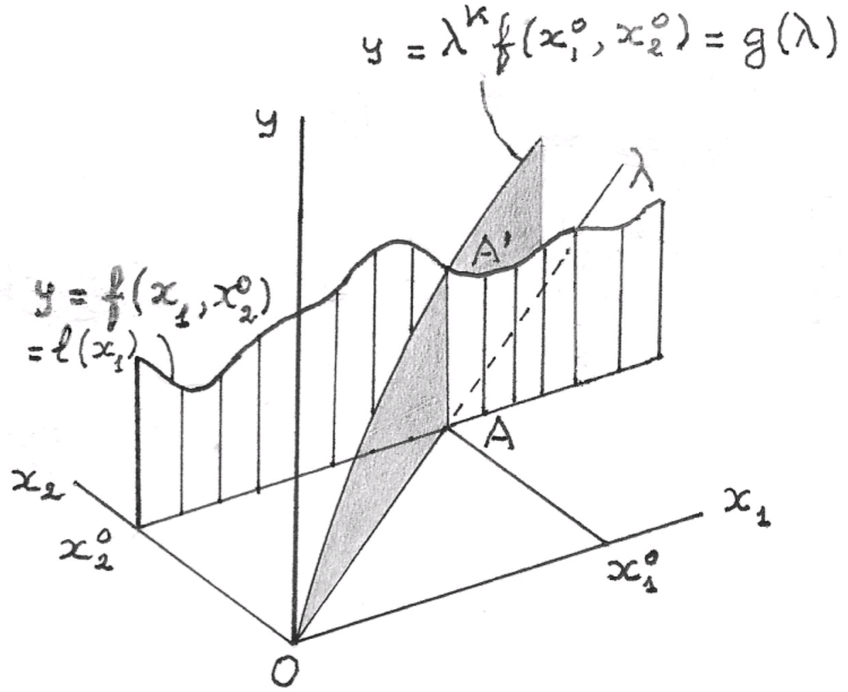

Figure 1: Generating a function homogeneous of degree k. The generating curve $y = \lambda^k f(x_1^0, x_2^0) = g(\lambda)$ sweeps all points of the leading curve $y = f(x_1, x_2^0) = l(x_1)$.

In space (y, x_1, x_2), consider a single variable function $y = f(x_1, x_2^0) \equiv l(x_1)$; we call this curve a *leading* curve. As illustrated in figure 1, $l(x_1)$ may be, on various intervals, increasing or decreasing, concave or convex. Let A $= (x_1^0, x_2^0)$ be any point on the horizontal $x_2 = x_2^0$, and A′ the point defined by $(x_1^0, x_2^0, f(x_1^0, x_2^0))$.

Let OA designate the ray from the origin to point A and beyond; this ray will serve as an axis, denoted λ. Consider now the curve corresponding to a positive, power function $y = g(\lambda) = \lambda^k f(x_1^0, x_2^0)$, where k is any real number; we call this curve a *generating* curve, and $g(1) = f(x_1^0, x_2^0)$. This curve will be concave or convex according to $0 < k < 1$ or $k > 1$. In Figure 1, $g(\lambda)$ is concave.

Choosing all possible values of x_1^0, we now make this concave curve sweep all points of the leading curve $y = f(x_1, x_2^0) \equiv l(x_1)$. In our example, this action generates a highly undulating surface corresponding to a function $y = f(x_1, x_2)$ homogeneous of degree k. Indeed, from any point (x_1, x_2^0), multiplying x_1^0 and x_2^0 by a factor λ will multiply $f(x_1^0, x_2^0)$ by λ^k.

We can verify this algebraically. Indeed, the simple knowledge of $f(x_1, x_2^0) \equiv l(x_1)$ and $y = g(\lambda) = \lambda^k f(x_1^0, x_2^0)$ is sufficient to determine the expression of the entire function $f(x_1, x_2)$; this can be done, for instance, as follows. With

$$f(\lambda x_1^0, \lambda x_2^0) = \lambda^k f(x_1^0, x_2^0) \tag{3.1}$$

we can set the common growth factor λ of the variables as

$$\lambda = \frac{x_1}{x_1^0} = \frac{x_2}{x_2^0}; \tag{3.2}$$

and write (3.1) as

$$f(\frac{x_1}{x_1^0}x_1^0, \frac{x_2}{x_2^0}x_2^0) = f(x_1, x_2) = (\frac{x_2}{x_2^0})^k f(x_1^0,\ x_2^0). \tag{3.3}$$

We can make both arguments x_1 and x_2 appear in the last term of (3.3) by using $x_1^0 = x_2^0(x_1/x_2)$, for instance; we then obtain the function

$$f(x_1, x_2) = (\frac{x_2}{x_2^0})^k f(x_2^0\frac{x_1}{x_2},\ x_2^0), \tag{3.4}$$

which can be ascertained to be homogeneous of degree k.

We should stress that the leading curve $l(x_1) = f(x_1, x_2^0)$ chosen at the outset in our example may have more complicated features than those just mentioned. It can be, on various intervals, negative, not differentiable, or discontinuous, thus leading to functions homogeneous of degree k with the same properties.

4. THE DEGREE OF HOMOGENEITY: A SIMPLE CONSTRUCTION.

4.1. The case $0 < k < 1$.

As in figure 1, $g(\lambda)$ is pictured in figure 2 as the concave section of the $f(x_1, x_2)$ surface by a vertical plane intersecting the horizontal plane along ray OA. We draw BA′ as the tangent to $g(\lambda)$ at A′.

. a first approach

From the very definition of the degree of homogeneity, given by equation 1, we can write

$$k = \frac{dy/y}{d\lambda/\lambda} = \frac{dy}{d\lambda}.\frac{\lambda}{y} \tag{4.1}$$

and therefore

$$k = \frac{\mathrm{DC}}{\mathrm{CA'}}.\frac{\mathrm{OA}}{\mathrm{OC}} = \frac{\mathrm{DC}}{\mathrm{OA}}.\frac{\mathrm{OA}}{\mathrm{OC}} = \frac{\mathrm{DC}}{\mathrm{OC}}. \tag{4.2}$$

As an example of a negative homogeneous function, consider the refreshing ice-cream cone you were called upon to hand to your daughter. Needless to say, she knows how to please her father. She told you: " Imagine I put the tip of my ice-cream at the origin on your diagram, and its height h on axis x_2; r is the radius of the circle at height h, where I just let you take a bite." She made your day when adding: "In your formula (3.3), if I set $x_2^0 = h$, $k = 1$, keeping $x_1^0 \leq r$, I can replace $f(x_1^0, x_2^0)$ by $\pm\sqrt{r^2 - (x_1^0)^2} = \pm\sqrt{r^2 - \left(h\frac{x_1}{x_2}\right)^2}$. You will certainly agree that the equation of my ice-cream is $f(x_1, x_2) = \pm\frac{x_2}{h}\sqrt{r^2 - \left(h\frac{x_1}{x_2}\right)^2}$; of course $f(x_1, x_2)$ is defined only when $\frac{h}{r}x_1 \leq x_2 \leq h$, and I guarantee that f is homogeneous of degree one".

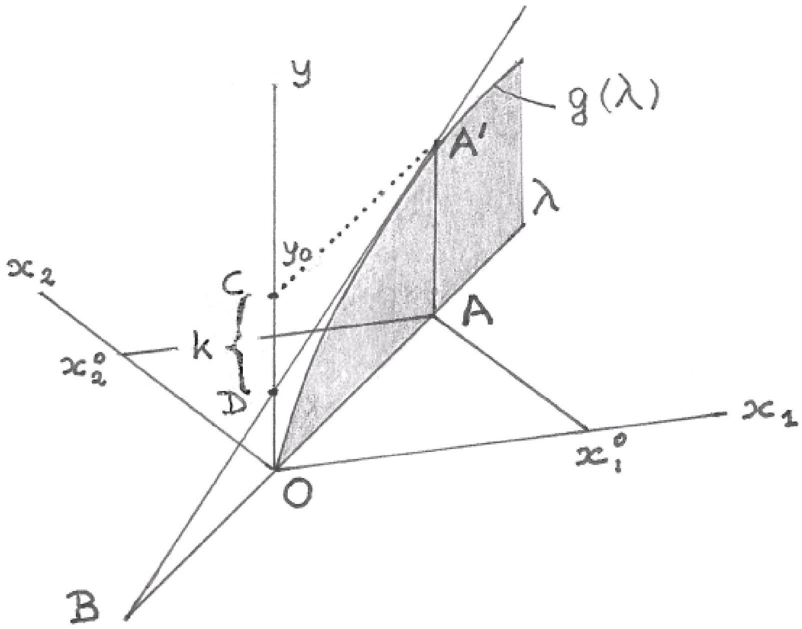

Figure 2: The degree of homogeneity k can be shown to be equal to the ratio DC/OC. Without loss of generality, we can normalize OC (equal to $y_0 = f(x_1^0, x_2^0)$) to 1; k is then just equal to DC. In this diagram, k is equal to 0.6.

We will not lose any generality if we normalize OC, equal to y_0, to one. We thus have the very simple representation of the degree of homogeneity:

$$k = \text{DC}. \tag{4.3}$$

. a second approach

I owe to Ernst Hairer an alternate, very elegant way of deriving the result $k = \text{DC/OC}$. We have, with $g'(\lambda) = k\lambda^{k-1}y_0$

$$g'(1) = ky_0, \tag{4.4}$$

and

$$g'(1) = \frac{\text{DC}}{\text{CA'}} = \frac{\text{DC}}{\text{OA}} = \text{DC}, \tag{4.5}$$

remembering that OA $= 1$; hence $ky_0 =$ DC, and the result $k =$ DC/$y_0 =$ DC/OC.

We can illustrate how the degree of homogeneity measures the sensitivity of $y = f(x_1, x_2)$ to a common, relative change in its variables. Suppose that, as in Fig. 2, $k = 0.6$. This implies that an increase of 1% in x_1 and x_2 generates an increase of y equal to 0.6% in linear approximation, and an increase exactly equal to 0.6% on tangent BA'.

We can now show that this representation remains valid for any real value of k, the only proviso being that if k is negative, DC is the absolute value of k.

4.2. The case $k > 1$.

The vertical section $y = \lambda^k f(x_1^0, x_2^0) = \lambda^k y_0 = g(\lambda)$ is now a power function of λ with power $k > 1$, pictured in Figure 2 as the convex curve going through O and A'. Its tangent at A' intersects the λ axis at B, and the ordinate y at D. We can verify that B will be on the segment

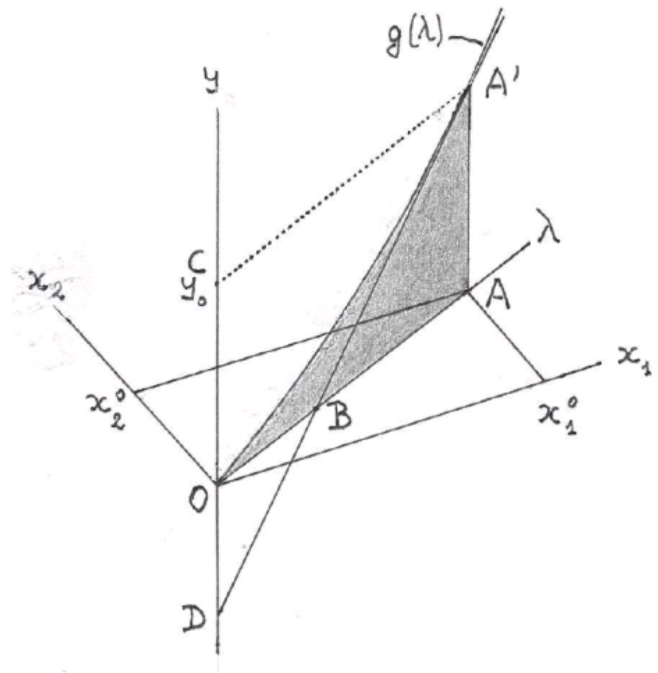

Figure 3: The case $k > 1$. Normalizing OC $= y_0$ to one, as in the case $0 < k < 1$, the degree of homogeneity k is equal to DC. In this diagram, k is approximately equal to 1.6.

OA as follows: as before, triangles AA'B and CDA' are similar; equation (4.1) applies again, and $k =$ OA/BA > 1 implies OA $>$ BA. Moreover equation (4.2) applies as well, leading to $k =$ DC/OC, and we find again $k =$ DC if OC, equal to y_0, is normalized to one.

Before considering the situation where k is negative, let us take up two particular cases that offer special interest: $k = 1$, and $k = 0$.

4.3. The case $k = 1$.

Complex as this surface may be, cutting it with a vertical plane through the origin O results in an intersection given by ray OA'; $g(\lambda)$ becomes the linear function $y = f(x_1^0, x_2^0)\lambda$; as k tends toward 1, it can be seen either from Figure 1 or Figure 2 that points B and D tend to merge with the origin, and DC/OC $= 1$.

4.4. The case $k = 0$.

As we stressed at the end of Section 3, *all* homogeneous functions, whatever their degree, may well exhibit complicated surfaces; homogeneity of degree zero does not create an exception. This may be surprising if we simply look at the equation defining homogeneity, in this case

$$f(\lambda x_1^0, \lambda x_2^0) == \lambda^0 f(x_1^0, x_2^0) = f(x_1^0, x_2^0) = y_0. \tag{4.6}$$

We should not forget that the function $f(x_1, x_2)$ turns out to be a constant y_0 *along axis O*λ; our generating curve is

$$y = g\left(\lambda\right) = y_0; \tag{4.7}$$

this is just one member of an infinite number of horizontals emanating from the vertical axis Oy, sweeping a leading curve $y = l(x_1) = f(x_1, x_2^0)$ that may well exhibit the complicated features we referred to.

To illustrate, consider the function

$$y = f(x_1, x_2) = \frac{ax_1^n - bx_2^n}{cx_1^n + dx_2^n}, \tag{4.8}$$

homogeneous of degree zero; along any ray $x_2 = \mu x_1$, the value of y is

$$y = f(x_1, \mu x_1) = \frac{a - b\mu^n}{c + d\mu^n}, \tag{4.9}$$

a constant independent of x_1 and x_2, but a highly dependent function of μ, the angular coefficient of ray $x_2 = \mu x_1$. The surface $f(x_1, x_2)$ cuts the horizontal plane along ray $x_2 = (a/b)^{1/n}\, x_1$.

Geometrically, we can see how any of these horizontal lines is generated. Let us refer to Figure 2, for instance. With $k =$ OA$/$BA, since OA is fixed, when $k \to 0$ the length BA must tend to infinity. As a consequence, DA$'$ tends to the horizontal CA$'$, and the length DC collapses to zero, as it should.

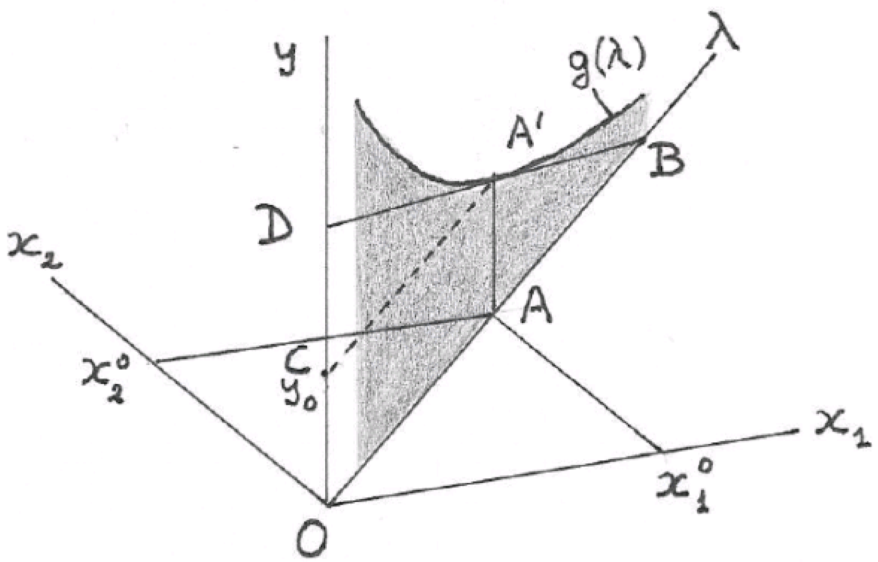

Figure 4: The case $k < 0$. Normalizing OC $=\ y_0$ to one, the length DC is equal to the degree of homogeneity in absolute value. Here k is equal to $-$ 1.2; DC $= |k| = 1.2$.

4.5. The case $k < 0$.

With $y_0 > 0$, $g(\lambda) = \lambda^k y_0$ is now decreasing if $k < 0$; the tangent DB crosses the ordinate at a point D higher than C, and meets the λ axis at B, above A. The slope of $g(\lambda)$ being negative, we have at point A' $dg/d\lambda = -$ AA'/AB; therefore, we can write

$$k = \frac{dy/y}{d\lambda/\lambda} = \frac{dy}{d\lambda}.\frac{\lambda}{y} = -\frac{\mathrm{AA}'}{\mathrm{BA}}.\frac{\mathrm{OA}}{\mathrm{AA}'} = -\frac{\mathrm{OA}}{\mathrm{BA}}. \tag{4.10}$$

Due to the similarity of triangles AA'B and CDA', we have

$$k = -\frac{\mathrm{OA}}{\mathrm{BA}} = -\frac{\mathrm{DA}'}{\mathrm{BA}'} = -\frac{\mathrm{DC}}{\mathrm{AA}'} = -\frac{\mathrm{DC}}{\mathrm{OC}}; \tag{4.11}$$

thus, normalising OC = y_0 to one, $k = -$ DC.

We can conclude that in all cases CD is equal to the degree of homogeneity k in absolute value. We thus are set up to prove Euler's theorem in 3-d space.

5. A FIRST APPLICATION: A DIRECT, GEOMETRIC PROOF OF EULER'S THEOREM.

We assume that $y = f(x_1, x_2)$ is differentiable. Let us come back to Figure 2 and to the last equality in (4.2), that can be expressed as

$$k = \frac{\mathrm{DC}}{\mathrm{OC}} = \frac{\mathrm{DC}}{f(x_1^0, x_2^0)};$$

it will prove particularly useful to obtain Euler's theorem. We first can write

$$\mathrm{DC} = kf(x_1^0, x_2^0). \tag{5.1}$$

Now doubling x_1^0 and x_2^0 results in an increase on the tangent BA$'$ equal to the differential $\frac{\partial f}{\partial x_1}(x_1^0, x_2^0)\, x_1^0 + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)\, x_2^0$, symmetrical and equal to DC. We thus also have

$$\mathrm{DC} = \frac{\partial f}{\partial x_1}(x_1^0, x_2^0)\, x_1^0 + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)\, x_2^0 \tag{5.2}$$

and therefore

$$kf(x_1^0, x_2^0) = \frac{\partial f}{\partial x_1}(x_1^0, x_2^0)\, x_1^0 + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)\, x_2^0. \tag{5.3}$$

Since point $A(x_1^0, x_2^0)$ was arbitrarily chosen in plane (x_1, x_2), equation (5.3) can be written as

$$kf(x_1, x_2) = \frac{\partial f}{\partial x_1}(x_1, x_2)\, x_1 + \frac{\partial f}{\partial x_2}(x_1, x_2)\, x_2, \tag{5.4}$$

which is Euler's theorem.

6. A DIRECT, ALGEBRAIC PROOF OF EULER'S THEOREM IN $n-$ SPACE.

Your daughter had barely touched her ice-cream when she told you: "I vividly remember that when our excellent teacher introduced us to homogeneous functions, he demonstrated Euler's theorem by differentiating their definitional property with respect to λ, and finally taking the limit when $\lambda \to 1$.

"It seems to me, she added, that you can obtain the theorem directly, without recourse to a limiting process. Consider the differential

$$dy = \sum_{i=1}^{n} (\partial f/\partial x_i) dx_i;$$

dividing by y, you can write

$$\frac{dy}{y} = \frac{1}{y}\left[\frac{\partial f}{\partial x_1} x_1 \frac{dx_1}{x_1} + ... + \frac{\partial f}{\partial x_n} x_n \frac{dx_n}{x_n}\right]. \tag{6.1}$$

Since $dx_1/x_1 = ... = dx_n/x_n$, these can be denoted $d\lambda/\lambda$ and factored out, giving

$$\frac{dy}{y} / \frac{d\lambda}{\lambda} = \frac{1}{y}\left[\frac{\partial f}{\partial x_1} x_1 + ... + \frac{\partial f}{\partial x_n} x_n\right]; \tag{6.2}$$

Using your definition of the degree of homogeneity given by (2.1), replace in (6.2) the left-hand side $(dy/y) \,/\, (d\lambda/\lambda)$ by k, and multiply both sides by y; (6.2) then becomes the Euler theorem

$$ky = \frac{\partial f}{\partial x_1} x_1 + ... + \frac{\partial f}{\partial x_n} x_n. \tag{6.3}$$

She went two steps further: “In fact, she added, I never thought that the designation of k as a "degree of homogeneity" was illuminating, or even appropriate. Would it mean that a function with $k = 2$ is "more homogeneous" than a function with $k = 1$? We might even consider, quite on the contrary, that $k = 1$ is an indication of *more* homogeneity, not less, because in this case the function behaves *exactly* as its variables, while any function with $k \neq 1$ does not. The higher k, the more pronounced becomes the *divergence* of the function’s behavior from that of its variables! The above mentioned difference just means that the first function is more sensitive than the second to a common change in the variables. Therefore it seems to me that we rather should call k the "degree of *sensitivity*" of the function.”

With the faintest of smiles, you thought that your daughter’s suggestions were not devoid of merit. You also decided that it was time to take her to the tennis court, as you had promised. Then, on your way there, you thought: “Why shouldn’t I one day write a paper with her?”