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DISCRETE-TIME EVOLUTION AND STABLE EQUILIBRIA OF MULTI-COMPARTMENT DENGUE TRACKER: NONLINEAR DYNAMICS MODULATED BY CONTROLLED STOCHASTICITY

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ABSTRACT. We discuss the dynamics of solutions of a nonlinear discrete time model that will be useful in Dengue control. The proposed model may be utilized to analyze the dynamics of three variables (mosquito population, habitats and consciousness) across different parameters. Stochasticity has been introduced in realistic ways to highlight combinations of random parameters (on education and recollection) which limits the oscillatory recurrence of habitats and awareness. We propose optimal methods for implementing potential intervention strategies and offer interactive dashboards for vizualizing varied scenarios.

Key words and phrases: Local and global stability; Boundedness; Stochasticity; System of difference equations.

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1. INTRODUCTION

Dengue fever is an urban disease transmitted to humans through the bite of infected Aedes mosquitoes. The Aedes mosquito breeds in stagnant water (containers, old tires) which abound around homes and residential premises. Public policies and educational campaigns to erradicate and control mosquito breeding sites are of great importance. In addition to educating the public, other interventions have been used to reduce the mosquito breeding sites, and these interventions usually consists of indoor and outdoor spraying.

In this paper, we consider a nonlinear difference equation model that incorporates factors such as spraying and educational level. We extend previous work by Awerbuch et al. ([3]). The system consists of nonlinear equations, where the control by spraying, reduction of breeding sites and habitat control are modeled by exponential terms. A larger moquito population, prompts more awareness and this awareness will have an impact in the actions taken. A constant education parameter is present in the third equation.

$$(1.1) \qquad \begin{cases} M_{n+1} = aM_n e^{-\gamma A_n} + b(1 - e^{-sM_n})H_n \\ H_{n+1} = cH_n e^{-pA_n} + de^{-qA_n} \\ A_{n+1} = rA_n + fM_n + g \end{cases} \qquad n = 0, 1, \dots$$

The first term in the mosquito equation represents survival and elimination due to spraying, and the second term represents the creation of a new generation. The expression $(1 - e^{-sM_n})$ represents the number of mosquitoes that give birth to new offspring derived in ([1]).

Some parameter and variable definitions follow:(M_n) is the number of mosquitoes at week n; (H_n) is the number of habitats at week n; (A_n) is the level of awareness at week n; Parameter a is the survival rate of mosquitoes from week to week; b is the net reproductive rate of mosquitoes; c represents the fraction of habitats that survive from one week to the next; d is the number of new breeding sites; f is the coefficient of awareness prompted by mosquito population; r is the survival rate of awareness from previous week; s and $s = -\ln(1-\mathrm{Pr})$ where Pr is the probability that a female mosquito lays eggs in a breeding site. We also use p as a measures how sensitive the change in breeding sites that survive is to community consciousness; q as a measures how sensitive the change in breeding sites that are newly created is to individual consciousness, γ as the coefficient of mosquito decay through awareness, and g as an education parameter. The survival rates a, c and r are such that $0 < a < 1$, $0 < c < 1$, $0 < r < 1$. The parameter b, (the number of mosquitoes per female mosquito given that the female mosquito finds a breeding site) is positive. Also, s, d and f are positive. The initial conditions are considered non-negative. Parameters p and q and g are non-negative ($\gamma \geq 0$, $p \geq 0$, $q \geq 0$, $g \geq 0$).

We will rescale some of the parameters by transforming the variables. First, $M_n = \frac{1}{s}m_n$. The first equation becomes $\frac{1}{s}m_{n+1} = a\frac{1}{s}m_n e^{-\gamma A_n} + b(1-e^{-m_n})H_n$. Multiplying by s to both sides:

$$m_{n+1} = am_n e^{-\gamma A_n} + bs(1 - e^{-m_n})H_n$$

The second change, $H_n = dh_n$ ($d > 0$) gives:

$$dh_{n+1} = cdh_n e^{-pA_n} + de^{-qA_n}$$

Dividing by d to both sides, gives $h_{n+1} = h_n e^{-pA_n} + e^{-qA_n}$.

Next, call $A_n = \frac{f}{s}a_n$. Third equation becomes $\frac{f}{s}a_{n+1} = \frac{rf}{s}a_n + fM_n + g$. Multiplying by $\frac{s}{f}$ to both sides ($f \neq 0$), gives $a_{n+1} = ra_n + m_n + \tilde{g}$, where $\tilde{g} = \frac{gs}{f}$.

With this transformation the second original equation of the system changes into

$$h_{n+1} = ch_n e^{-pfa_n/s} + e^{-qfa_n/s}$$

Denote $\tilde{p} = \dfrac{pf}{s}$ and $\tilde{q} = \dfrac{qf}{s}$. The previous equation becomes $h_{n+1} = ch_n e^{-\tilde{p}a_n} + e^{-\tilde{q}a_n}$. The change of variables $H_n = dh_n$ and $A_n = \dfrac{f}{s}a_n$ in the first equation yields $m_{n+1} = am_n e^{-\gamma f a_n/s} + bsd(1-e^{-m_n})h_n$. Renaming the parameters $\tilde{\gamma} = \dfrac{\gamma f}{s}$ and $\tilde{b} = bsd$, first equation reads $m_{n+1} = am_n e^{-\tilde{\gamma}A_n} + \tilde{b}(1-e^{-m_n})h_n$. Note that with these transformations, the parameters a, c and r stayed between 0 and 1, while $\tilde{\gamma}$, $\tilde{p}$, $\tilde{q}$ and $\tilde{g}$ are non-negative. $\tilde{b}$ is positive. Summarizing, the change in variables allows us to work with a reduced parameter structure through the remaining of the paper. The system that we now work with is:

$$\text{(1.2)} \qquad \begin{cases} M_{n+1} = aM_n e^{-\gamma A_n} + b(1-e^{-M_n})H_n \\ H_{n+1} = cH_n e^{-pA_n} + e^{-qA_n} \\ A_{n+1} = rA_n + M_n + g \end{cases} \qquad n = 0, 1, \ldots$$

with $0 < a < 1$, $0 < c < 1$, $0 < r < 1$, $b > 0$ and $p, q, g, \gamma \geq 0$.

2. EXISTENCE OF EQUILIBRIUM POINTS AND BOUNDEDNESS OF SOLUTIONS

The equilibrium verifies the following nonlinear system:

$$\text{(2.1)} \qquad \begin{cases} \bar{M} = a\bar{M}e^{-\gamma\bar{A}} + b(1-e^{\bar{M}})\bar{H} \\ \bar{H} = c\bar{H}e^{-p\bar{A}} + e^{-q\bar{A}} \\ \bar{A} = r\bar{A} + \bar{M} + g \end{cases} \qquad n = 0, 1, \ldots$$

The third equation gives $(1-r)\bar{A} = \bar{M} + g$. If $\bar{M} = 0$, the first equation holds true for any positive value of $\bar{H}$. Also, $\bar{M} = 0$ in the third equation gives $\bar{A} = g/(1-r)$, which replaced in the second yields $\bar{H}(1 - ce^{-p\bar{A}}) = e^{-q\bar{A}}$. Solving for $\bar{H}$, one gets: $\bar{H} = \dfrac{e^{-q\bar{A}}}{1 - ce^{-p\bar{A}}}$. Thus, there is a degenerate equilibrium (non-negative), with coordinates $E_1\left(0, \dfrac{e^{-qg/(1-r)}}{1 - ce^{-pg/(1-r)}}, \dfrac{g}{1-r}\right)$. Our system also possesses a positive equilibrium, $E_2(\bar{M}, \bar{H}, \bar{A})$. Next we will show the existence of this equilibrium. Third equation in system (2.1) gives $\bar{H} = \dfrac{e^{-q(g+\bar{M})/(1-r)}}{1 - ce^{-p(g+\bar{M})/(1-r)}}$ (denominator is never zero since $0 < c < 1$). The first equation in (2.1) gives $(1 - ae^{-\gamma\bar{A}})\bar{M} = b(1 - e^{-\bar{M}})\bar{H}$ or

$$(1 - ae^{-\gamma(g+\bar{M})/(1-r)})\bar{M} = b(1 - e^{-\bar{M}})\bar{H}$$

Solving for $\bar{H}$, we obtain: $\bar{H} = \dfrac{(1 - ae^{-\gamma(g+\bar{M})/(1-r)})\bar{M}}{b(1 - e^{-\bar{M}})}$.

Define functions f, g: $(0, \infty) \to (0, \infty)$ such that $f(M) = \dfrac{(1 - ae^{-\gamma(g+M)/(1-r)})}{b(1 - e^{-M})}$and $g(M) = \dfrac{e^{-q(g+M)/(1-r)}}{1 - ce^{-p(g+M)/(1-r)}}$. Some properties of function f: (i) $\lim\limits_{M\to\infty} f(M) = \infty$, (ii) $\lim\limits_{M\to 0} f(M) = \lim\limits_{M\to 0} \dfrac{1 - ae^{-\gamma(g+M)/(1-r)}}{b} \lim\limits_{M\to 0} \dfrac{M}{1 - e^{-M}} = \dfrac{1 - ae^{-\gamma g/(1-r)}}{b}$ (also $f(0^+) > 0$ since $1 - ae^{-\gamma g/(1-r)} > 1 - a > 0$ and $b > 0$) and (iii) $f(x)$ is increasing.

In addition, some useful properties of function g are: (i) $\lim_{M\to\infty} g(M) = 0$, (ii) $g(0) = \dfrac{e^{-qg/(1-r)}}{1-ce^{-pg/(1-r)}}$ and (iii) $g(x)$ is decreasing. According to the properties above, we have:

(1) If $f(0^+) < g(0^+)$ then the graphs intersect only once, and the system

$$\begin{cases} \bar{H} = f(\bar{M}) \\ \bar{H} = g(\bar{M}) \end{cases} \tag{2.2}$$

has a unique solution $E_2(\bar{M}, \bar{H}, \bar{A})$.

(2) If $f(0^+) \geq g(0^+)$, then E_1 is the only equilibrium point.

In summary, expressing the above in terms of parameters, gives the lemma:

Lemma 2.1. (1) *Assume* $b \leq \dfrac{(1-ae^{-\gamma g/(1-r)})(1-ce^{-gp/(1-r)})}{e^{-gq/(1-r)}}$, *then* E_1 *is the only equilibrium.*

(2) *Assume* $b > \dfrac{(1-ae^{-\gamma g/(1-r)})(1-ce^{-gp/(1-r)})}{e^{-gq/(1-r)}}$, *then there are two equilibria,* E_1 *(degenerate) and* E_2 *(positive).*

The following lemma on boundedness and invariant interval for the non-negative solutions, it is given here for the convenience of the reader (the proof follows the same idea as in ([2]) and it will be omitted.

Lemma 2.2. i) *Let* $\{M_n, H_n, A_n\}_{n\geq 0}$ *be a positive solution of system (1.2). Parameters are such that* $0 < a < 1, 0 < c < 1$ *and* $0 < r < 1$. *Then* $\limsup_{n\to\infty} M_n \leq b/(1-a)(1-c)$, $\limsup_{n\to\infty} H_n \leq 1/(1-c)$ *and* $\limsup_{n\to\infty} A_n \leq (b/(1-a)(1-c)(1-r)) + (g/(1-r))$.

ii) *The closed set,* $[0, b/(1-a)(1-c)] \times [0, 1/(1-c)] \times [0, b/(1-a)(1-c)(1-r) + g/(1-r)]$ *is invariant in* R^3.

2.1. **Stability Analysis.** In this section we discuss the local stability of equilibrium points of system (1.2). At the degenerate equilibrium E_1= $\left(0, \dfrac{e^{-q\bar{A}}}{1-ce^{-p\bar{A}}}, \dfrac{g}{1-r}\right)$ characteristic equation is: $(ae^{-\gamma g/(1-r)} + b\bar{H} - \lambda)(ce^{-pg/(1-r)} - \lambda)(r - \lambda) = 0$. The first root of this equation is $\lambda_1 = ae^{-\gamma g/(1-r)} + \dfrac{be^{-gq/(1-r)}}{1-ce^{-pg/(1-r)}} < a + \dfrac{b}{1-ce^{-pg/(1-r)}} < a + \dfrac{b}{1-c}$. If $b \leq \dfrac{(1-ae^{-\gamma g/(1-r)})(1-ce^{-gp/(1-r)})}{e^{-gq/(1-r)}}$, then $0 < \lambda_1 < 1$. For the other two roots: $0 < \lambda_2 = ce^{-pg/(1-r)} < 1$ and $0 < \lambda_3 = r < 1$ always. So E_1 is locally asymptotically stable.

When $b > \dfrac{(1-ae^{-\gamma g/(1-r)})(1-ce^{-gp/(1-r)})}{e^{-gq/(1-r)}}$ the positive equilibrium E_2 also appears. The local asymptotic stability of the positive equilibrium $E_2(\bar{M}, \bar{H}, \bar{A})$ will follow from the conditions imposed on the eigenvalues of the Jacobian evaluated at $(\bar{M}, \bar{H}, \bar{A})$ to have modulus less than 1. Due to the nonlinearity in this equation, numerical simulations will prove handy. We created a Shinny dashboard for exploration of various combination of parameters and to vizualize the behavior of solutions under parameter changes. The dashboard can be found at `https://moinak.shinyapps.io/NonLinearDynamics/`. As a side observation, in some special cases, when $\gamma = g = 0$ (no spraying or education level entered in the system), the global attractiveness of solutions can be proved analytically. The global asymptotic stability under the general case for which $g, \gamma > 0$ is an open question.

Theorem 2.3. *Assume that $b < (1-a)(1-c)$ and $\gamma = g = 0$ then E_1 is the only equilibrium point and it is a globally asymptotically stable.*

Proof. Proof uses the same idea as in Theorem 3 in [2]. From the boundedness lemma, the sequence $\{M_n, H_n, A_n\}_{n\geq 0}$ is bounded thus all the limits below are finite. Denote $I_M = \liminf_{n\to\infty} M_n$ and $S_M = \limsup_{n\to\infty} M_n$; $I_H = \liminf_{n\to\infty} H_n$ and $S_H = \limsup_{n\to\infty} H_n$; $I_A = \liminf_{n\to\infty} A_n$ and $S_A = \limsup_{n\to\infty} A_n$. The proof goes by contradiction.

Let's suppose $S_M > 0$. From the second equation of our system, we get $S_H < cS_H + 1$ and therefore $S_H < \frac{1}{1-c}$. The first equation gives $M_{n+1} = aM_n + b(1-e^{-M_n})H_n < aM_n + bH_n$ for n=0,1,... Passing to the limit, we obtain: $S_M < aS_M + bS_H < aS_M + \dfrac{b}{1-c}$. Thus, $S_M < \dfrac{b}{(1-a)(1-c)} < 1$. Since $1-e^{-M_n} \leq M_n$ for $n = 0, 1\ldots$ we have $M_{n+1} \leq aM_n + bM_nH_n$, which gives $S_M \leq aS_M + \dfrac{b}{1-c}S_M$ or $(1-a)S_M \leq \dfrac{b}{1-c}S_M$. Dividing above by $S_M > 0$, yields $(1-a) \leq \dfrac{b}{1-c}$ or $(1-a)(1-c) \leq b$ (which is a contradiction). Thus, $S_M = 0$. Using $S_M = 0$ in third equation, we obtain $S_A = 0$.

From second equation of system (1.2), $I_H \geq cI_He^{-pS_A} + e^{-qS_A} \geq cI_H + 1$.

Solving for I_H, we obtain $I_H \geq \dfrac{1}{1-c}$. Of course, $S_H \leq \dfrac{1}{1-c} \leq I_H$, and it follows that $S_H = I_H = \dfrac{1}{1-c}$.

Based on the discussion in the Local Stability Section, when $b < (1-a)(1-c)$, the degenerate equilibrium E_1 is locally asymptotically stable. This is because when $b < (1-a)(1-c)$ then $\lambda_1 < a + \dfrac{b}{1-c} < a + \dfrac{(1-a)(1-c)}{1-c} = 1$. The other two roots $\lambda_2 = ce^{-pg/(1-r)} < 1$ and $\lambda_3 = r < 1$ always. So when $b < (1-a)(1-c)$ all the roots of characteristic equation for equilibrium E_1 are between $(0,1)$, and thus, E_1 is a locally asymptotically stable. This combined with global attractivity result, implies that E_1 is globally asymptotically stable. ∎

3. STOCHASTIC EXTENSIONS

There have been instances in the literature where controlled stochasticity have been explored to model dengue epidemics. The type of control imposed implied certain restrictions that enabled the investigation of certain parts of the system under specific ways the transition. For instance, D'Souza et al (2013) ([9]) assumed a Markovian dynamics channeled through SIS or SIR agreements which (a) makes the evolution aware of only the latest state and no more, (b) through assumptions like an individual can only either be susceptible or infected or recovered and the transitions proceed in that specific way. Attention has not been paid, for instance, to the randomness inherent in the habitat size or the level of awareness in the society. Champagne and Cazalles (2019)([10]) launched another brand of work that compared stochastic and deterministic frameworks in dengue modeling with data from Cambodia, with general recommendations that advocate for stochastic models to reflect parameters and process uncertainties. They found in deterministic setups, the uncertainty of the parameter estimates typically get underestimated. We are showing the process volatility, under our stochastic scenarios, for instance, in Fig 2. Staying still within the general Markovian framework and the SIR paradigm, Din et al. (2021)([11]) tracked expected evolutions by bringing in additional parameters such as the birth and death rates of the human population. They discovered - under these assumptions - a stochastic reproduction number for the mosquitoes - a threshold that will ensure the extinction or the persistence of the disease. Even if one is willing to ignore the assumptions temporarily, endogenous factors such as education and awareness were not considered, either on their own, or even indirectly through how they could, in principle, influence the active components in the dynamics. We remedy the above through incorporating randomness in

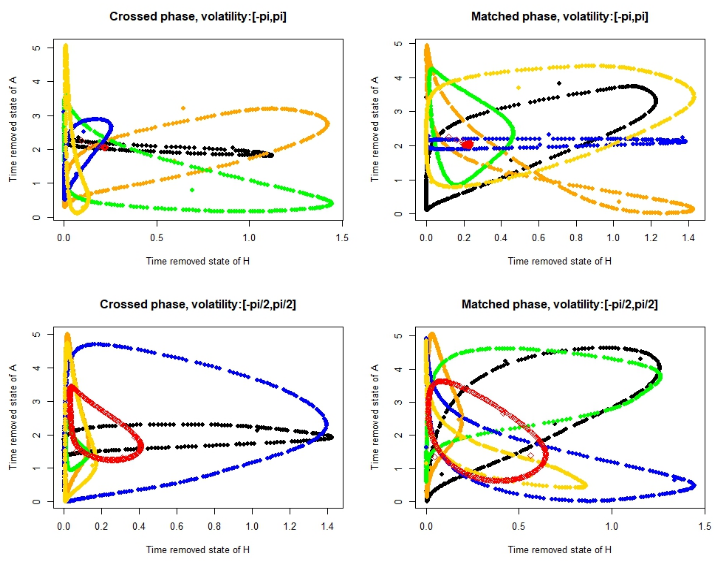

Figure 1: Phase portraits of $H(.)$ and $A(.)$ under an interacted $g(.) - r(.)$ system

the education parameter g and the recall parameter r in a more realistic way. These operate in cycles (Keating (2001), ([12])). As effort is expended in educating the public, the recollecting factor inflates. Beyond a point, the need for promoting awareness declines due to factors such as an already elevated recollection level, or the drying up of public funds. This letup eventually takes a toll on how much awareness people retain. And the cycle starts over. In addition to this general cyclical tendency, there are unpredictable factors such as scarcity of field workers which makes the imposing of a deterministic sinusoidal pattern unrealistic. To examine how the deterministic flow compares with such a periodic random trigger, we offer an interactive dashboard `https://moinak.shinyapps.io/NonLinearDynamics/` We consider the functional forms:

$$r(t) = cos(\frac{\pi}{2} - tk - U(-\frac{\pi}{2}, \frac{\pi}{2})) \tag{3.1}$$

$$g(t) = cos(t + kU(-\frac{\pi}{2}, \frac{\pi}{2})) \tag{3.2}$$

to reflect the random oscillations where $U(.)$ is a random variable with uniform probability of the specified compact support, k is a stretch factor. We note under these conditions, there opens up a possibility of limiting the rise of the number of mosquito habitats - the rise that would have been predicted by the deterministic system. Beyond the five-week mark, the effect of the deterministic pieces wither away and the random parameters take control. We recognize this is illustrative, and there may be many ways to perturb several parameters, either in isolation, or in conjunction (see the dashboard). Figure 1 showcases the phase portraits, that is the joint dynamics of the awareness level and the number of habitats under the stochastic framework. The non-red colours represent different simulations and the reds show the **average** interaction calculated through 5000 simulations. To reflect a possible unsureness about how long after

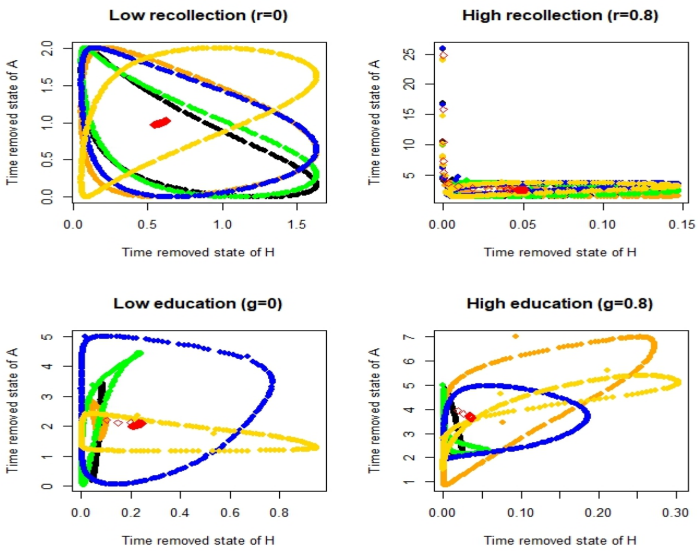

Figure 2: Phase portraits of $H(.)$ and $A(.)$ under a non-interacted $g(.) - r(.)$ system

educating people, changes seem to be reflected in their recollections, we define a matched-phase scenario (3.2 and 3.3 above represent the crossed-phase scenario) where the periods of the oscillations coincide: $r(t) = cos(t + kU(-\frac{\pi}{2}, \frac{\pi}{2}))$ and $g(t) = cos(t + kU(-\frac{\pi}{2}, \frac{\pi}{2}))$ We notice the inverse connection between the states is more clear when the amount of input volatility in the g and r parameters reduces, while the stability of these states increases when those parameters become uncertain. While these scenarios are documented under an interaction framework, i.e., when both $g(.)$ and $r(.)$ vary (sinusoidally) simultaneously, in Figure 2, we document the non-interacted case: when one of the parameters vary sinusoidally, while the other - to highlight its impact - is held fixed at the values shown. We observe lower average values of habitat under intense recollection or education, with the recollection parameter - at least, with this variety of chosen randomness - exerting larger control in reducing the size of habitats overall.

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