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NONLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. Let $h(u) > 0$ be a smooth monotonically growing function, $lim_{u\to\infty}h(u)|u|^{-1} = +\infty$, $f(x) \geq 0$ is bounded. Consider a nonlinear elliptic problem $-\Delta u = h(u)+f$ in $D \subset \mathbb{R}^3$, D is a bounded domain with a smooth boundary S, $u = 0$ on S. We prove the existence of a solution to this problem.

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1. INTRODUCTION

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary S. Consider a nonlinear elliptic problem:

$$-\Delta u = h(u) + f, \ in \ D, \quad u|_S = 0, \tag{1.1}$$

where

$$h(u) > 0, \quad h'(u) > 0, \tag{1.2}$$

$h(u)$ is a smooth function,

$$\lim_{u\to\infty} h(u)|u|^{-1} = +\infty, \tag{1.3}$$

so that $h(u) = b(u)u$ when $u \to \infty$, $\lim_{u\to\infty} b(u) = \infty$, and

$$f(y) \geq 0, \tag{1.4}$$

where $f(y) \in C(D)$.

We want to find out if problem (1.1) has a solution. There is a large literature on the subject, see, for example, [2], [3], and on the Leray-Schauder method, but the estimates we use seem new.

Theorem 1. Problem (1.1) **has a solution under our assumptions.**

2. PROOFS

Since D is a bounded domain with a smooth boundary, there exists the Green's funcion $G(x, y)$:

$$-\Delta G = \delta(x - y), \tag{2.1}$$

$$G|_S = 0, \tag{2.2}$$

where $G(x, y) = G(y, x)$, and

$$0 \leq G(x, y) < \frac{1}{4\pi|x - y|}. \tag{2.3}$$

Therefore, problem (1.1) is equivalent to the integral equation:

$$u(x) = \int_D G(x, y)h(u(y))dy + \int_D G(x, y)f(y)dy. \tag{2.4}$$

To prove the existence of a solution to equation (2.4), we use the Leray-Schauder principle (see [1], pp. 205-207).

Proposition 1. Consider the equation in a Banach space X:

$$u = \lambda A(u), \quad 0 \leq \lambda \leq 1, \tag{2.5}$$

where A is a compact nonlinear operator in X. If

$$\sup_{0\leq\lambda\leq1} \|u(\lambda)\| \leq c, \tag{2.6}$$

where $\|u\|$ is the norm in X of a solution to equation (2.5) **and $c > 0$ is a constant independent of λ, then equation $u = A(u)$ has a solution in X.**

Choose $C(D)$, the space of continuous functions in D with the standard norm as X. Let us check that condition (2.6) is satisfied for equation (2.4) if $M = \infty$.

Let

$$D_\eta := \{x \in D, M - \eta < |u(x)|\},$$

where $\eta > 0$ is a small number, $M = \sup_{x \in D} |u(x)| := \|u\|$, and

$$A(u) := \int_D G(x,y)h(u(y))dy + \int_D G(x,y)f(y)dy.$$

Consider the following estimate for the u which satisfies equation (2.4):

$$M \geq \int_{D_\eta} G(x,y)dy \min_{y \in D_\eta} h(u(y)) + c_f \geq \int_{D_\eta} G(x,y)dy \min_{y \in D_\eta} h(u(y)), \tag{2.7}$$

where

$$c_f := \min_{x \in D_\eta} \int_{D_\eta} G(x,y)dy \min_{y \in D_\eta} f(y) > 0.$$

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We have assumed that $f \geq 0$ and we know that $G(x,y) \geq 0$. Therefore $c_\eta := \min_{x \in D_\eta} \int_{D_\eta} G(x,y)dy > 0$, and.

$$M \geq c_\eta \min_{y \in D_\eta} h(u(y)). \tag{2.8}$$

Using assumption (1.2), one gets $h(u) \geq h(M - \eta)$, and from equation (2.8) one derives the inequality:

$$M \geq c_\eta \min_{y \in D_\eta} h(u(y)) \geq c_\eta h(M - \eta). \tag{2.9}$$

Let us take into account that by assumption (1.2) one has:

$$M \geq b(M - \eta)(M - \eta). \tag{2.10}$$

Since $\lim_{u \to \infty} b(u) = \infty$ and $\lim_{M \to \infty} \frac{M}{M - \eta} = 1$, we have a contradiction in inequality (2.10) if $M = \infty$. Therefore, $M < \infty$.

Theorem 1 is proved. □

Example 1. Consider equation

$$-\Delta u = u^{10} + e^{|x|}, \quad u|_S = 0. \tag{2.11}$$

The domain D is a ball $D = \{x : |x| < 1\}$. The assumptions of Theorem 1 are satisfied. Therefore, problem (2.11) has a solution.

3. CONCLUSION

Sufficient conditions are given for the nonlinear elliptic boundary value problem in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary S, $-\Delta u = h(u) + f, \quad u|_S = 0$, to have a solution. We assume that $h(u) > 0,\ h'(u) > 0,$ is a smooth function and $f(x) \in C(D)$, $f \geq 0$. Under these assumptions the existence of a solution is proved.

4. DISCLOSURE STATEMENT.

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