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ROBUST LAYER RESOLVING SCHEME FOR A SYSTEM OF TWO SINGULARLY PERTURBED TIME-DEPENDENT DELAY INITIAL VALUE PROBLEMS WITH ROBIN INITIAL CONDITIONS

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ABSTRACT. This paper aimed at proving first order convergence for system of two singularly perturbed time-dependent initial value problems with delay in spatial variable and robin initial conditions. A Classical layer resolving finite difference scheme is developed by implementing uniform mesh for time discretization; Shishkin-mesh, a piecewise uniform mesh for spatial discretization. Shishkin-mesh is constructed is such way it captures the intricacies behavior of the layers. The interior layer is induced by the presence of a delay term in the space term. Error estimate is carried out to prove first order convergence with the help of maximum principle, stability analysis, solution bounds and sharper estimates of the singular components of the solutions. Finally, the numerical illustration is computed for the problem to bolster the scheme.

Key words and phrases: Singularly perturbed problems; Time-dependent; Delay; Layer; Shishkin-mesh; Sharper estimates.

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1. INTRODUCTION

The numerical study of Singularly Perturbed Problems (SPPs) has piqued interest among researchers due to the intricate boundary layer phenomena inherent in their solutions. These boundary layers emerged due to the outcome of the perturbation parameter $(\varepsilon,\ 0 < \varepsilon << 1)$, multiples term involving the highest-order derivative, which causes the solution to exhibit rapid transitions in confined regions, either at the boundary or in the interior, where the width is of $\mathcal{O}(\varepsilon)$. These regions are predominantly positioned near the boundaries of the domain or at specific interior points where the solution encounters a drastic and intense variation. This complex behavior necessitates the development of specialized numerical methods to precisely capture and resolve the sharp transitions that characterize SPPs. Over the past few years, a broad spectrum of innovative techniques has been meticulously developed by many researchers to tackle the difficulties inherent in obtaining approximate solutions of singularly perturbed problems.

Some articles related to time delay are as follows: A fitted Numerov method [7] is devised for singularly perturbed 1D parabolic PDEs with a small negative shift in the temporal variable term. A Crank-Nicolson and central difference method on Shishkin meshes [6] is formulated for singularly perturbed time delayed parabolic problems, achieving second-order uniform convergence and its accuracy is then enhanced to fourth-order using Richardson extrapolation. An efficient method combining reproducing kernel spaces and collocation [13] is proposed for solving Singularly Perturbed delay parabolic PDEs, demonstrating uniform convergence, with an accuracy scaling of T/n. A fitted operator-based approach [8] employing cubic B-splines and implicit Euler is designed for solving SPPs of parabolic reaction-diffusion type with a general time delay and improved accuracy in the temporal direction is achieved through Richardson extrapolation.

Some articles related to space delay are as follows: A computational method [1] integrating implicit Euler for time dimensions and cubic-spline in compression methods for space dimensions is used solve second-order parabolic SPPs with large negative shift and discontinuous coefficients. Finite difference scheme [2] is formulated using Shishkin mesh to solve Singularly Perturbed Parabolic PDEs with spatial delay and integral boundary conditions, establishing almost second order convergence. A novel numerical method [12] implementing a finite difference technique on Shishkin mesh was developed to solve linear system of parabolic second order SPDDEs of reaction-diffusion type.

The article [5] meticulously explores in-depth analysis of 1-D hyperbolic delay differential equations (HDDEs), demonstrating the efficacy of numerical techniques such as, the method of lines and Runge-Kutta methods to solve the problem by ensuring the stability and convergence. In [4] , the authors have employed Restrictive Pade' Approximation to solve the singularly perturbed first-order hyperbolic PDE where small perturbation parameter is multiplied with the term involving time derivative.

The present paper differs from [4] in treatment of small perturbation parameter, multiplied with the term involving time derivative and numerical approach. In this present work we have crafted classical layer resolving finite difference scheme to tackle a related but distinct problem. In our case, we focus on a system of two equations where the perturbation parameters ($\varepsilon_1 \& \varepsilon_2$ are distinct) multiply spatial term involving highest order of derivative and the equations further includes a delay terms in spatial variables. This divergence in problem structure, combined with the robustness of the classical layer resolving numerical scheme, allows our method to effectively handle the complexities introduced by the perturbation parameters and delay terms. As a result, our approach demonstrates superior computational efficiency and accuracy, offering a simpler yet highly effective solutions.

The following outlines are the structure of the present article: In Section 2, the problem is stated and some preliminaries related to the problem are presented. In Section 3, uniqueness and stability of the solution are firmly validated. In Section 4, we prove bounds for the solution and its derivatives. In Section 5, singular and smooth components of the solutions were analysed. In Section 6, Shishkin-mesh is constructed according to the requirement of the problem. Section 7, explores finite difference scheme and elucidates the discrete maximum principle, as well as the stability result. Section 8 discusses Discrete Shishkin decomposition; Section 9 explains about local truncation error. Error estimates are articulated in Section 10. Finally, the numerical illustration is computed for the problem to bolster the scheme in Section 11.

2. THE PROBLEM STATEMENT

The problem stated for all $(\varkappa, \mathfrak{t}) \in \Psi$,

$$\vec{L}\vec{u}(\varkappa,\mathfrak{t}) = \frac{\partial \vec{u}}{\partial \mathfrak{t}}(\varkappa,\mathfrak{t}) + E\,\frac{\partial \vec{u}}{\partial \varkappa}(\varkappa,\mathfrak{t}) + \Re(\varkappa,\mathfrak{t})\vec{u}(\varkappa,\mathfrak{t}) + \Im(\varkappa,\mathfrak{t})\vec{u}(\varkappa-1,\mathfrak{t}) = \vec{\mathfrak{f}}(\varkappa,\mathfrak{t}) \text{ on } \Psi \tag{2.1}$$

with

$$\vec{\xi}\vec{u}(\varkappa,\mathfrak{t}) = \vec{u}(\varkappa,t) - E\,\frac{\partial \vec{u}}{\partial \varkappa}(\varkappa,\mathfrak{t}) = \vec{\varphi}_L(\varkappa,\mathfrak{t}) \ on\ \Psi^* \tag{2.2}$$

$$\vec{u}(\varkappa,0) = \vec{\varphi}_B(\varkappa) \ on\ \Delta_B \tag{2.3}$$

where $\Psi = \{(\varkappa,\mathfrak{t}) : 0 < \varkappa \le 2, 0 < \mathfrak{t} \le T\}$, $\Psi^* = \{(\varkappa,\mathfrak{t}) : -1 \le \varkappa \le 0, 0 < \mathfrak{t} \le T\}$, $\overline{\Psi} = \Psi \cup \Delta, \Delta = \Delta_L \cup \Delta_B$ where $\Delta_L = \{(0,\mathfrak{t}) : 0 \le \mathfrak{t} \le T\}$ and $\Delta_B = \{(\varkappa,0) : 0 \le \varkappa \le 2\}$,$\Psi^- = \{(\varkappa,\mathfrak{t}) : 0 < \varkappa < 1, 0 < \mathfrak{t} \le T\}$, $\Psi^+ = \{(\varkappa,\mathfrak{t}) : 1 < \varkappa \le 2, 0 < \mathfrak{t} \le T\}$ and $\overline{\Psi}^- = \{(\varkappa,\mathfrak{t}) : 0 \le \varkappa \le 1, 0 \le \mathfrak{t} \le T\}$, $\overline{\Psi}^+ = \{(\varkappa,\mathfrak{t}) : 1 \le \varkappa \le 2, 0 \le \mathfrak{t} \le T\}$. For all $(\varkappa,\mathfrak{t}) \in \overline{\Psi}$, $\vec{u}(\varkappa,\mathfrak{t}) = (u_1(\varkappa,\mathfrak{t}), u_2(\varkappa,\mathfrak{t}))^T$ and $\vec{\mathfrak{f}}(\varkappa,\mathfrak{t}) = (\mathfrak{f}_1(\varkappa,\mathfrak{t}), \mathfrak{f}_2(\varkappa,\mathfrak{t}))^T$. $\Re(\varkappa,\mathfrak{t}), \Im(\varkappa,\mathfrak{t})$ are 2×2 matrices. Distinct perturbation parameters are defined in problem as 2×2 matrix, $E = diag(\vec{\varepsilon}), \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ with the relation $0 < \varepsilon_1 < \varepsilon_2 \le 1$ and $\varepsilon_\iota, \iota = 1, 2$. $\Im = diag(\vec{\beta}), \vec{\beta} = (\beta_1(\varkappa,t), \beta_2(\varkappa,t))$. Here

$$\vec{L}\vec{u} = \vec{\mathfrak{f}} \text{ on } \Psi, \tag{2.4}$$

$$\vec{L} = \frac{\partial}{\partial \mathfrak{t}} + E\frac{\partial}{\partial \varkappa} + \Re + \Im. \tag{2.5}$$

Remark 2.1. For all $(\varkappa,\mathfrak{t}) \in \overline{\Psi}$, the components $\mathfrak{r}_{\iota\jmath}(\varkappa,\mathfrak{t})$ of $\Re(\varkappa,\mathfrak{t})$ and $\beta_\iota(\varkappa,\mathfrak{t})$ of $\Im(\varkappa,\mathfrak{t})$ satisfy the inequalities

$$\begin{cases} \mathfrak{r}_{\iota\iota}(\varkappa,\mathfrak{t}) > \sum\limits_{\substack{\jmath\neq\iota \\ \jmath=1}}^{2} |\mathfrak{r}_{\iota\jmath}(\varkappa,\mathfrak{t}) + \beta_\iota(\varkappa,\mathfrak{t})|, \iota = 1, 2 \\ \beta_\iota(\varkappa,\mathfrak{t}), \mathfrak{r}_{\iota\jmath}(\varkappa,\mathfrak{t}) \le 0 \quad \text{for } 0 \le \iota \neq \jmath \le 2. \end{cases} \tag{2.6}$$

Remark 2.2. α is considered as a positive number and it satisfy

$$0 < \alpha < \min_{\substack{\varkappa\in\overline{\Psi} \\ 0\le\iota\le 2}} \left\{ \sum_{\jmath=1}^{2} \mathfrak{r}_{\iota\jmath}(\varkappa,\mathfrak{t}) + \beta_\iota(\varkappa,\mathfrak{t}) \right\}. \tag{2.7}$$

The problem (2.1) is re-written as,

$$\vec{L}_1\vec{u}(\varkappa,\mathfrak{t}) = (\frac{\partial\vec{u}}{\partial\mathfrak{t}} + E\frac{\partial\vec{u}}{\partial\varkappa} + \Re\vec{u})(\varkappa,\mathfrak{t}) = \vec{\Lambda}(\varkappa,\mathfrak{t}) \text{ on } \Psi^- \tag{2.8}$$

$$\vec{L}_2\vec{u}(\varkappa,\mathfrak{t}) = (\frac{\partial\vec{u}}{\partial\mathfrak{t}} + E\frac{\partial\vec{u}}{\partial\varkappa} + \Re\vec{u})(\varkappa,\mathfrak{t}) + \Im(\varkappa,\mathfrak{t})\vec{u}(\varkappa-1,\mathfrak{t}) = \vec{\mathfrak{f}}(\varkappa,\mathfrak{t}) \text{ on } \Psi^+ \tag{2.9}$$

where $\vec{\Lambda}(\varkappa,\mathfrak{t}) = \vec{\mathfrak{f}}(\varkappa,\mathfrak{t}) - \Im(\varkappa,\mathfrak{t})\vec{\varphi}_L(\varkappa-1,\mathfrak{t})$.

The reduced problem of (2.8) and (2.9) are given by

$$\frac{\partial\vec{u}_0}{\partial\mathfrak{t}}(\varkappa,\mathfrak{t}) + \Re(\varkappa,\mathfrak{t})\vec{u}_0(\varkappa,\mathfrak{t}) = \vec{\Lambda}(\varkappa,\mathfrak{t}) \text{ for all } \varkappa \in \Psi^- \tag{2.10}$$

$$\frac{\partial\vec{u}_0}{\partial\mathfrak{t}}(\varkappa,\mathfrak{t}) + \Re(\varkappa,\mathfrak{t})\vec{u}_0(\varkappa,\mathfrak{t}) + \Im(\varkappa,\mathfrak{t})\vec{u}_0(\varkappa-1,\mathfrak{t}) = \vec{\mathfrak{f}}(\varkappa,\mathfrak{t}) \text{ for all } \varkappa \in \Psi^+. \tag{2.11}$$

The arbitrary initial conditions cannot be imposed to the equations (2.10), (2.11) since it has an unique solution for each value of $(\varkappa,\mathfrak{t})$ which gives rise to the initial layers by the solution components in the neighborhood of $(\varkappa,\mathfrak{t}) = 0$. The solution components $\vec{u}(\varkappa,\mathfrak{t})$ features an interior layer at $\varkappa = 1$, due the presence of delay term at this point and $\vec{u}_0(1-,\mathfrak{t})$ needs not be equal to $\vec{u}_0(1+,\mathfrak{t})$ in general. The following layer pattern is exhibited by the components u_1 and u_2, where both components have layers of width $O(\varepsilon_2)$ and in addition to this the component u_1 has a sublayer of width $O(\varepsilon_1)$.

3. Analytical Results

In the succeeding lemma, the operator $\vec{L}$ satisfies maximum principle .

Lemma 3.1. *Let* $\Re(\varkappa,\mathfrak{t})$ *and* $\Im(\varkappa,\mathfrak{t})$ *satisfy* (2.6) *and* (2.7). *Let* $\vec{\aleph}(\varkappa,\mathfrak{t})$ *be any function in the domain of* $\vec{L}$, *such that it satifies* $\vec{\xi}\,\vec{\aleph}(\varkappa,\mathfrak{t}) \geq \vec{0}$ *on* Ψ^* *and* $\vec{\aleph}(\varkappa,0) \geq \vec{0}$ *on* Δ_B. *Then* $\vec{L}_1\vec{\aleph}(\varkappa,\mathfrak{t}) \geq \vec{0}$ *on* Ψ^- *and* $\vec{L}_2\vec{\aleph}(\varkappa,\mathfrak{t}) \geq \vec{0}$ *on* Ψ^+ *implies that* $\vec{\aleph}(\varkappa,\mathfrak{t}) \geq \vec{0}$ *on* $\overline{\Psi}$.

Proof. For $\iota^*,\varkappa^*,\mathfrak{t}^*$, let us consider $\aleph_{\iota^*}(\varkappa^*,\mathfrak{t}^*) = \min_{\iota=1,2}\min_{\overline{\Psi}}\aleph_\iota(\varkappa,\mathfrak{t})$. If $\aleph_{\iota^*}(\varkappa^*,\mathfrak{t}^*) \geq 0$, there is nothing to prove. Suppose that $\aleph_{\iota^*}(\varkappa^*,\mathfrak{t}^*) < 0$. For $(\varkappa^*,\mathfrak{t}^*) \in \Psi^*$, then

$$(\vec{\xi}\vec{\aleph})_{\iota^*}(\varkappa^*,\mathfrak{t}^*) = \aleph_{\iota^*}(\varkappa^*,\mathfrak{t}^*) - \varepsilon_{\iota^*}\frac{\partial\aleph_{\iota^*}}{\partial\varkappa}(\varkappa^*,\mathfrak{t}^*) < 0,$$

which contradicts our assumption and for $\mathfrak{t}^* = 0, \aleph_{\iota^*}(\varkappa^*,0) < 0$, which also contradicts our assumption. Therefore $(\varkappa^*,\mathfrak{t}^*) \notin \Delta$.

Also, $\frac{\partial\aleph_{\iota^*}}{\partial\varkappa}(\varkappa^*,\mathfrak{t}^*) \leq 0$ and $\frac{\partial\aleph_{\iota^*}}{\partial\mathfrak{t}}(\varkappa^*,\mathfrak{t}^*) \leq 0$.

Thus for $(\varkappa^*,\mathfrak{t}^*) \in \Psi^-$, we have

$$\begin{aligned}(\vec{L}\vec{\aleph})_{\iota^*}(\varkappa^*,\mathfrak{t}^*) = (\vec{L}_1\vec{\aleph})_{\iota^*}(\varkappa^*,\mathfrak{t}^*)&\\ &= \frac{\partial\aleph_{\iota^*}}{\partial\mathfrak{t}}(\varkappa^*,\mathfrak{t}^*) + \varepsilon_{\iota^*}\frac{\partial\aleph_{\iota^*}}{\partial\varkappa}(\varkappa^*,\mathfrak{t}^*) + \mathfrak{r}_{\iota^*1}(\varkappa^*,\mathfrak{t}^*)\aleph_1(\varkappa^*,\mathfrak{t}^*)\\ &\quad + \mathfrak{r}_{\iota^*2}(\varkappa^*,\mathfrak{t}^*)\aleph_2(\varkappa^*,\mathfrak{t}^*)\\ &\leq \sum_{\jmath=1}^{2}\mathfrak{r}_{\iota^*\jmath}(\varkappa^*,\mathfrak{t}^*)\aleph_{\iota^*}(\varkappa^*,\mathfrak{t}^*) < 0,\end{aligned}$$

which contradicts our assumption. For $(\varkappa^*, \mathfrak{t}^*) \in \Psi^+$, we get

$$\begin{aligned}(\vec{L}\vec{\aleph})_{\iota^*}(\varkappa^*, \mathfrak{t}^*) &= (\vec{L}_2\vec{\aleph})_{\iota^*}(\varkappa^*, \mathfrak{t}^*)\\ &= \frac{\partial \aleph_{\iota^*}}{\partial \mathfrak{t}}(\varkappa^*, \mathfrak{t}^*) + \varepsilon_{\iota^*}\frac{\partial \aleph_{\iota^*}}{\partial \varkappa}(\varkappa^*, \mathfrak{t}^*) + \mathfrak{r}_{\iota^*1}(\varkappa^*, \mathfrak{t}^*)\aleph_1(\varkappa^*, \mathfrak{t}^*)\\ &\quad + \mathfrak{r}_{\iota^*2}(\varkappa^*, \mathfrak{t}^*)\aleph_2(\varkappa^*, \mathfrak{t}^*) + \beta_{\iota^*}(\varkappa^*, \mathfrak{t}^*)\aleph_{\iota^*}(\varkappa^* - 1, \mathfrak{t}^*)\\ &\leq \sum_{\jmath=1}^{2} \mathfrak{r}_{\iota^*\jmath}(\varkappa^*, \mathfrak{t}^*)\aleph_{\iota^*}(\varkappa^*, \mathfrak{t}^*) + \beta_{\iota^*}(\varkappa^*, \mathfrak{t}^*)\aleph_{\iota^*}(\varkappa^*, \mathfrak{t}^*) < 0,\end{aligned}$$

which contradicts our assumption. This indicates that our assumption is not vaild. It follows that $\aleph_{\iota^*}(\varkappa^*, \mathfrak{t}^*) \geq 0$ and for all $(\varkappa, \mathfrak{t}) \in \overline{\Psi}$, $\vec{\aleph}(\varkappa, \mathfrak{t}) \geq \vec{0}$, which proves the lemma.

The stability result is established in the next lemma as an immediate consequence of the preceding lemma.

Lemma 3.2. *Let $\Re(\varkappa, \mathfrak{t}), \Im(\varkappa, \mathfrak{t})$ satisfy (2.6) and (2.7). Let $\vec{\aleph}(\varkappa, \mathfrak{t})$ be any function in the domain of $\vec{L}$, such that for each $(\varkappa, \mathfrak{t}) \in \overline{\Psi}$, then*

$$|\vec{\aleph}(\varkappa, \mathfrak{t})| \leq \max\left\{\| \vec{\xi}\,\vec{\aleph}(0, \mathfrak{t}) \|, \| \vec{\aleph}(\varkappa, 0) \|,\ \frac{1}{\alpha} \| \vec{L}_1\vec{\aleph} \|,\ \frac{1}{\alpha} \| \vec{L}_2\vec{\aleph} \|\right\}$$

Proof. Consider the barrier functions

$$\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t}) = \max\left\{\| \vec{\xi}\,\vec{\aleph}(0, \mathfrak{t}) \|,\ \| \vec{\aleph}(\varkappa, 0) \|,\ \frac{1}{\alpha} \| \vec{L}_1\vec{\aleph} \|,\ \frac{1}{\alpha} \| \vec{L}_2\vec{\aleph} \|\right\} \pm \vec{\aleph}(\varkappa, \mathfrak{t})$$
$$\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t}) = M \pm \vec{\aleph}(\varkappa, \mathfrak{t})$$

where $M = \max\left\{\| \vec{\xi}\,\vec{\aleph}(0, \mathfrak{t}) \|,\ \| \vec{\aleph}(\varkappa, 0) \|,\ \frac{1}{\alpha} \| \vec{L}_1\vec{\aleph} \|,\ \frac{1}{\alpha} \| \vec{L}_2\vec{\aleph} \|\right\}$.

$$\begin{aligned}\vec{\xi}\vec{\vartheta}^{\pm}(0, \mathfrak{t}) &= \vec{\vartheta}^{\pm}(0, \mathfrak{t}) - E\frac{\partial \vec{\vartheta}^{\pm}}{\partial \varkappa}(0, \mathfrak{t}) \pm \vec{\xi}\,\vec{\aleph}(0, \mathfrak{t})\\ &= \vec{\vartheta}^{\pm}(0, \mathfrak{t}) - E(0) \pm \vec{\varphi}_L(0, \mathfrak{t}) \geq \vec{0}.\\ \vec{\vartheta}^{\pm}(\varkappa, 0) &= M \pm \vec{\aleph}(\varkappa, 0) = M \pm \vec{\varphi}_B(\varkappa, 0)\\ \vec{\vartheta}^{\pm}(\varkappa, 0) &\geq \vec{0}.\end{aligned}$$

If $(\varkappa, \mathfrak{t}) \in \Psi^-$, then

$$\begin{aligned}\vec{L}\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t}) &= \vec{L}_1\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t})\\ &= \frac{\partial \vec{\vartheta}^{\pm}}{\partial \mathfrak{t}}(\varkappa, \mathfrak{t}) + E\frac{\partial \vec{\vartheta}^{\pm}}{\partial \varkappa}(\varkappa, \mathfrak{t}) + \Re(\varkappa, \mathfrak{t})\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t}) \pm \vec{L}_1\vec{\aleph}(\varkappa, \mathfrak{t}) \geq \vec{0}.\end{aligned}$$

If $(\varkappa, \mathfrak{t}) \in \Psi^+$, then

$$\begin{aligned}\vec{L}\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t}) &= \vec{L}_2\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t})\\ &= \frac{\partial \vec{\vartheta}^{\pm}}{\partial \mathfrak{t}}(\varkappa, \mathfrak{t}) + E\frac{\partial \vec{\vartheta}^{\pm}}{\partial \varkappa}(\varkappa, \mathfrak{t}) + \Re(\varkappa, \mathfrak{t})\vec{\vartheta}^{\pm}(\varkappa, \mathfrak{t})\\ &\quad + \Im(\varkappa, \mathfrak{t})\vec{\vartheta}^{\pm}(\varkappa - 1, \mathfrak{t}) \pm \vec{L}_2\vec{\aleph}(\varkappa, \mathfrak{t}) \geq \vec{0}.\end{aligned}$$

It follows from Lemma 3.1 that, $\vec{\vartheta}^{\pm}(\varkappa,\mathfrak{t}) \geq \vec{0}$. Then,

$$|\vec{\aleph}(\varkappa,\mathfrak{t})| \leq \max\left\{\| \vec{\xi}\,\vec{\aleph}(0,\mathfrak{t}) \|, \| \vec{\aleph}(\varkappa,0) \|,\ \frac{1}{\alpha} \| \vec{L}_1\vec{\aleph} \|,\ \frac{1}{\alpha} \| \vec{L}_2\vec{\aleph} \|\right\}.$$

The proof of the lemma is complete.

4. Estimates of derivatives

Theorem 4.1. *Let* $\Re(\varkappa,\mathfrak{t}), \Im(\varkappa,\mathfrak{t})$ *satisfy* (2.6) *and* (2.7). *Let* $\vec{u}$ *represents the solution of the problem* (2.1), (2.2) *and* (2.3). *Then for each* $\iota, \iota = 1,2$ *and for all* $(\varkappa,\mathfrak{t}) \in \overline{\Psi}$, *there exist a constant* C *such that*

$$|u_\iota(\varkappa,\mathfrak{t})| \leq C\Big\{ \| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \| \vec{\mathfrak{f}} \| \Big\}$$

$$|\frac{\partial^\ell u_\iota}{\partial \mathfrak{t}^\ell}(\varkappa,\mathfrak{t})| \leq C\left\{\| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \sum_{q=0}^{\ell} \| \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \|\right\}, \ell = 0,1,2$$

$$|\frac{\partial u_\iota}{\partial \varkappa}(\varkappa,\mathfrak{t})| \leq C\varepsilon_\iota^{-1}\left\{\| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \sum_{q=0}^{1} \| \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \|\right\}$$

$$|\frac{\partial^2 u_\iota}{\partial \varkappa \partial \mathfrak{t}}(\varkappa,\mathfrak{t})| \leq C\varepsilon_\iota^{-1}\left\{\| \vec{\varphi}_L(t) \| + \| \vec{\varphi}_B(\varkappa) \| + \sum_{q=0}^{2} \| \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \|\right\}$$

$$|\frac{\partial^2 u_\iota}{\partial \varkappa^2}(\varkappa,\mathfrak{t})| \leq C\varepsilon_\iota^{-2}\left\{\| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \| \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} \| + \sum_{q=0}^{2} \| \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \|\right\}.$$

Proof. The bound on the solution components u_ι is an immediate outcome of Lemma 3.2,

$$|u_\iota(\varkappa,\mathfrak{t})| \leq C\Big\{ \| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \| \vec{\mathfrak{f}} \| \Big\}.$$

Now differentiate (2.1) partially with respect to variable $\mathfrak{t}$ once, we get

$$\frac{\partial^2 \vec{u}}{\partial \mathfrak{t}^2} + E\frac{\partial^2 \vec{u}}{\partial \varkappa \partial \mathfrak{t}} + \Re\frac{\partial \vec{u}}{\partial \mathfrak{t}} + \vec{u}\frac{\partial \Re}{\partial \mathfrak{t}} + \Im\frac{\partial \vec{u}}{\partial \mathfrak{t}} + \frac{\partial \Im}{\partial \mathfrak{t}}\vec{u} = \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}}. \tag{4.1}$$

It is observed that,

$$\vec{L}\frac{\partial \vec{u}}{\partial \mathfrak{t}} = \frac{\partial}{\partial \mathfrak{t}}\frac{\partial \vec{u}}{\partial \mathfrak{t}} + E\frac{\partial}{\partial \varkappa}\frac{\partial \vec{u}}{\partial \mathfrak{t}} + \Re\frac{\partial \vec{u}}{\partial \mathfrak{t}} + \Im\frac{\partial \vec{u}}{\partial \mathfrak{t}} = \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}} - \vec{u}\frac{\partial \Re}{\partial \mathfrak{t}} - \frac{\partial \Im}{\partial \mathfrak{t}}\vec{u}$$

$$\vec{L}\frac{\partial \vec{u}}{\partial \mathfrak{t}} = \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}} - \left[\frac{\partial \Re}{\partial \mathfrak{t}} + \frac{\partial \Im}{\partial \mathfrak{t}}\right]\vec{u}$$

and hence

$$\| \vec{L}\frac{\partial \vec{u}}{\partial \mathfrak{t}} \| \leq C \| \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}} \| + \| \vec{u} \|$$

$$\begin{aligned}\| \vec{L}\frac{\partial \vec{u}}{\partial \mathfrak{t}} \| &\leq C\ \Big[\| \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}} \| + \| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \| \vec{\mathfrak{f}} \| \Big] \\ &\leq C\ \Big[\| \vec{\varphi}_L(\mathfrak{t}) \| + \| \vec{\varphi}_B(\varkappa) \| + \sum_{q=0}^{1} \| \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \| \Big].\end{aligned}$$

Also,

$$\frac{\partial \vec{u}}{\partial \mathfrak{t}}(\varkappa,0) = 0 \qquad \text{and} \qquad \vec{\xi}\,\frac{\partial \vec{u}}{\partial \mathfrak{t}}(0,\mathfrak{t}) = \frac{\partial \vec{u}}{\partial \mathfrak{t}}(0,t) - E\frac{\partial^2 \vec{u}}{\partial \mathfrak{t}\partial \varkappa}(0,\mathfrak{t}) = C.$$

Then by Lemma 3.2,

$$|\frac{\partial u_\iota}{\partial \mathfrak{t}}(\varkappa,\mathfrak{t})| \le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \frac{1}{\alpha} \parallel \vec{L}_1\vec{\aleph} \parallel,\ \frac{1}{\alpha} \parallel \vec{L}_2\vec{\aleph} \parallel \Big]$$
$$\le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{1} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big]\Big]$$
$$\le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{1} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big].$$

Now differentiating (4.1) partially with respect to variable $\mathfrak{t}$, it is found that

$$\vec{L}\frac{\partial^2\vec{u}}{\partial\mathfrak{t}^2} = \frac{\partial^2\vec{\mathfrak{f}}}{\partial\mathfrak{t}^2} - \frac{\partial^2\Re}{\partial\mathfrak{t}^2}\vec{u} - \frac{\partial\vec{u}}{\partial\mathfrak{t}}\frac{\partial\Re}{\partial\mathfrak{t}} - \frac{\partial^2\Im}{\partial\mathfrak{t}^2}\vec{u} - \frac{\partial\vec{u}}{\partial\mathfrak{t}}\frac{\partial\Im}{\partial\mathfrak{t}} - \frac{\partial\vec{u}}{\partial\mathfrak{t}}\frac{\partial\Re}{\partial\mathfrak{t}} - \frac{\partial\Im}{\partial\mathfrak{t}}\frac{\partial\vec{u}}{\partial\mathfrak{t}}$$
$$\vec{L}\frac{\partial^2\vec{u}}{\partial\mathfrak{t}^2} = \frac{\partial^2\vec{\mathfrak{f}}}{\partial\mathfrak{t}^2} - \Big[\frac{\partial^2\Re}{\partial\mathfrak{t}^2} + \frac{\partial^2\Im}{\partial\mathfrak{t}^2}\Big]\vec{u} - 2\Big[\frac{\partial\Re}{\partial\mathfrak{t}} + \frac{\partial\Im}{\partial\mathfrak{t}}\Big]\frac{\partial\vec{u}}{\partial\mathfrak{t}}.$$

Therefore,

$$\parallel \vec{L}\frac{\partial^2\vec{u}}{\partial\mathfrak{t}^2} \parallel \le \parallel \frac{\partial^2\vec{\mathfrak{f}}}{\partial\mathfrak{t}^2} \parallel + C \parallel \vec{u} \parallel + C \parallel \frac{\partial\vec{u}}{\partial\mathfrak{t}} \parallel$$
$$\le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \parallel \vec{\mathfrak{f}} \parallel + \parallel \frac{\partial\vec{\mathfrak{f}}}{\partial\mathfrak{t}} \parallel + \parallel \frac{\partial^2\vec{\mathfrak{f}}}{\partial\mathfrak{t}^2} \parallel \Big]$$
$$\le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{2} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big].$$

Also,

$$\frac{\partial^2\vec{u}}{\partial\mathfrak{t}^2}(\varkappa,0) = 0 \qquad \text{and} \qquad \vec{\xi}\,\frac{\partial^2\vec{u}}{\partial\mathfrak{t}^2}(0,\mathfrak{t}) = \frac{\partial^2\vec{u}}{\partial\mathfrak{t}^2}(0,\mathfrak{t}) - E\frac{\partial^3\vec{u}}{\partial x\partial\mathfrak{t}^2}(0,\mathfrak{t}) = C.$$

Then by Lemma 3.2,

$$|\frac{\partial^2 u_\iota}{\partial \mathfrak{t}^2}(\varkappa,\mathfrak{t})| \le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \frac{1}{\alpha} \parallel \vec{L}_1\vec{\aleph} \parallel,\ \frac{1}{\alpha} \parallel \vec{L}_2\vec{\aleph} \parallel \Big]$$
$$\le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{2} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big]\Big]$$
$$\le C\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{2} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big].$$

To obtain bound on $\frac{\partial\vec{u}}{\partial\varkappa}$,

$$\frac{\partial\vec{u}}{\partial\varkappa} = E^{-1}\Big[\vec{\mathfrak{f}} - \Re\vec{u} - \Im\vec{u} - \frac{\partial\vec{u}}{\partial\mathfrak{t}}\Big]$$

from which it is found that for sufficiently large C,

$$|\frac{\partial u_\iota}{\partial \varkappa}(\varkappa,\mathfrak{t})| \le C\varepsilon_\iota^{-1}\Big[\parallel \vec{\mathfrak{f}} \parallel + \parallel \vec{u} \parallel + \parallel \frac{\partial\vec{u}}{\partial\mathfrak{t}} \parallel \Big]$$
$$\le C\varepsilon_\iota^{-1}\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{1} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big]$$

and hence the required estimate of the bound on $\frac{\partial\vec{u}}{\partial\varkappa}$ is obtained.
Now differentiate (2.1) partially with respect to variable $\mathfrak{t}$ once, we get

$$\frac{\partial^2 \vec{u}}{\partial \mathfrak{t}^2} + E\frac{\partial^2 \vec{u}}{\partial \varkappa \partial \mathfrak{t}} + \Re\frac{\partial \vec{u}}{\partial \mathfrak{t}} + \vec{u}\frac{\partial \Re}{\partial \mathfrak{t}} + \Im\frac{\partial \vec{u}}{\partial \mathfrak{t}} + \frac{\partial \Im}{\partial \mathfrak{t}}\vec{u} = \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}}$$

$$\frac{\partial^2 \vec{u}}{\partial \varkappa \partial \mathfrak{t}} = E^{-1}\Big[\frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}} - [\Re + \Im]\frac{\partial \vec{u}}{\partial \mathfrak{t}} - [\frac{\partial \Re}{\partial \mathfrak{t}} + \frac{\partial \Im}{\partial \mathfrak{t}}]\vec{u} - \frac{\partial^2 \vec{u}}{\partial \mathfrak{t}^2}\Big]$$

and hence it is observed that,

$$\begin{aligned}|\frac{\partial^2 u_\iota}{\partial \varkappa \partial \mathfrak{t}}(\varkappa, \mathfrak{t})| &\leq C\varepsilon_i^{-1}\Big[\parallel \frac{\partial \vec{\mathfrak{f}}}{\partial \mathfrak{t}} \parallel + \parallel \vec{u} \parallel + \parallel \frac{\partial \vec{u}}{\partial \mathfrak{t}} \parallel + \parallel \frac{\partial^2 \vec{u}}{\partial \mathfrak{t}^2} \parallel \Big] \\ &\leq C\varepsilon_i^{-1}\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \sum_{q=0}^{2} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big].\end{aligned}$$

Differentiating (2.1) partially with respect to variable $\varkappa$,

$$\frac{\partial^2 \vec{u}}{\partial \varkappa \partial \mathfrak{t}} + E\frac{\partial^2 \vec{u}}{\partial \varkappa^2} + \Re\frac{\partial \vec{u}}{\partial \varkappa} + \frac{\partial \Re}{\partial \varkappa}\vec{u} + \Im\frac{\partial \vec{u}}{\partial \varkappa} + \frac{\partial \Im}{\partial \varkappa}\vec{u} = \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa}$$

$$\frac{\partial^2 \vec{u}}{\partial \varkappa^2} = E^{-1}\Big[\frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} - [\frac{\partial \Re}{\partial \varkappa} + \frac{\partial \Im}{\partial \varkappa}]\vec{u} - [\Re + \Im]\frac{\partial \vec{u}}{\partial \varkappa} - \frac{\partial^2 \vec{u}}{\partial \varkappa \partial \mathfrak{t}}\Big]$$

and hence it is observed that

$$\begin{aligned}|\frac{\partial^2 u_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| &\leq C\varepsilon_\iota^{-1}\Big[\parallel \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} \parallel + \parallel \vec{u} \parallel + \parallel \frac{\partial \vec{u}}{\partial \varkappa} \parallel + \parallel \frac{\partial^2 \vec{u}}{\partial \varkappa \partial \mathfrak{t}} \parallel \Big] \\ &\leq C\varepsilon_\iota^{-2}\Big[\parallel \vec{\varphi}_L(\mathfrak{t}) \parallel + \parallel \vec{\varphi}_B(\varkappa) \parallel + \parallel \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} \parallel + \sum_{q=0}^{2} \parallel \frac{\partial^q \vec{\mathfrak{f}}}{\partial \mathfrak{t}^q} \parallel \Big].\end{aligned}$$

The proof of the theorem is complete.

5. SHISHKIN DECOMPOSITION

The exact solution $\vec{u}$ of (2.1) is decomposed as $\vec{u} = \vec{\gamma} + \vec{\varsigma}$, where the smooth component $\vec{\gamma}$ is the solution of

(5.1) $$\vec{L}_1\vec{\gamma}(\varkappa, \mathfrak{t}) = (\frac{\partial \vec{\gamma}}{\partial \mathfrak{t}} + E\frac{\partial \vec{\gamma}}{\partial \varkappa} + \Re\vec{\gamma})(\varkappa, \mathfrak{t}) = \vec{\mathfrak{f}}(\varkappa, \mathfrak{t}) - \Im(\varkappa, \mathfrak{t})\vec{\varphi}_L(\varkappa - 1, \mathfrak{t}) \text{ on } \Psi^-$$

(5.2) $$\vec{L}_2\vec{\gamma}(\varkappa, \mathfrak{t}) = (\frac{\partial \vec{\gamma}}{\partial \mathfrak{t}} + E\frac{\partial \vec{\gamma}}{\partial \varkappa} + \Re\vec{\gamma})(\varkappa, \mathfrak{t}) + \Im(\varkappa, \mathfrak{t})\vec{\gamma}(\varkappa - 1, \mathfrak{t}) = \vec{\mathfrak{f}}(\varkappa, \mathfrak{t}) \text{ on } \Psi^+$$

(5.3) $$\vec{\xi}\vec{\gamma}(0, \mathfrak{t}) = \vec{\xi}\vec{u}_0(0, \mathfrak{t}),\ \vec{\gamma}(\varkappa, 0) = \vec{u}_o(\varkappa, 0)$$

and the singular component $\vec{\varsigma}$ is the solution of

(5.4) $$\vec{L}_1\vec{\varsigma}(\varkappa, \mathfrak{t}) = (\frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}} + E\frac{\partial \vec{\varsigma}}{\partial \varkappa} + \Re\vec{\varsigma})(\varkappa, \mathfrak{t}) = \vec{0} \text{ on } \Psi^-$$

(5.5) $$\vec{L}_2\vec{\varsigma}(\varkappa, \mathfrak{t}) = (\frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}} + E\frac{\partial \vec{\varsigma}}{\partial \varkappa} + \Re\vec{\varsigma})(\varkappa, \mathfrak{t}) + \Im(\varkappa, \mathfrak{t})\vec{\varsigma}(\varkappa - 1, \mathfrak{t}) = \vec{0} \text{ on } \Psi^+$$

with $\vec{\varsigma} = \vec{u} - \vec{\gamma}$,

(5.6) $$\vec{\xi}\vec{\varsigma}(0, \mathfrak{t}) = \vec{\xi}(\vec{u} - \vec{\gamma})(0, \mathfrak{t}),\ \vec{\varsigma}(\varkappa, 0) = \vec{0}.$$

Next two lemmas deal with finding the bounds on $\vec{\gamma}$ and $\vec{\varsigma}$.

Lemma 5.1. *Let $\Re(\varkappa,\mathfrak{t}),\Im(\varkappa,\mathfrak{t})$ satisfy (2.6), (2.7). Then for each $\iota,\iota=1,2$ and for all $(\varkappa,\mathfrak{t})\in\overline{\Psi}$, there exist a constant C independent of ε_ι such that*

$$|\gamma_\iota(\varkappa,\mathfrak{t})|\le C$$
$$|\frac{\partial^\ell\gamma_\iota}{\partial\mathfrak{t}^\ell}(\varkappa,\mathfrak{t})|\le C,\ \textit{for}\ \ell=1,2$$
$$|\frac{\partial^2\gamma_\iota}{\partial\varkappa\partial\mathfrak{t}}(\varkappa,\mathfrak{t})|\le C$$
$$|\frac{\partial^\ell\gamma_\iota}{\partial\varkappa^\ell}(\varkappa,\mathfrak{t})|\le C\varepsilon_\iota^{1-\ell},\ \textit{for}\ \ell=1,2.$$

Proof. The proof is by the method of steps. The bounds on the interval $\overline{\Psi}^-$ for $\vec{\gamma}$ and its derivatives are derived first and by using these bounds, the bounds on the interval $\overline{\Psi}^+$ are derived.

For $(\varkappa,\mathfrak{t})\in\overline{\Psi}^-$, the bounds on $\vec{\gamma}$ are an immediate result of the defining equations for $\vec{\gamma}$ and Lemma 3.2,

$$|\vec{\gamma}|\le C.$$

Differentiating (5.1) partially with respect to $\mathfrak{t}$ once, we get

$$\vec{L}_1\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}=\frac{\partial\vec{\mathfrak{f}}}{\partial\mathfrak{t}}-\left[\frac{\partial\Re}{\partial\mathfrak{t}}+\frac{\partial\Im}{\partial\mathfrak{t}}\right]\vec{\gamma}.$$

Therefore,

$$\parallel\vec{L}_1\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}\parallel\le\parallel\frac{\partial\vec{\mathfrak{f}}}{\partial\mathfrak{t}}\parallel+C\parallel\vec{\gamma}\parallel\le C.$$

Also,

$$\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}(\varkappa,0)=\vec{0}\qquad\text{and}\qquad\vec{\xi}\,\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}(0,\mathfrak{t})=\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}(0,\mathfrak{t})-E\frac{\partial^2\vec{\gamma}}{\partial\varkappa\partial\mathfrak{t}}(0,\mathfrak{t})=C.$$

Using the stability result for the operator $\vec{L}_1$, we get

$$|\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}|\le C.$$

Similarly, differentiating again the same equation with respect to $\mathfrak{t}$, we get

$$\vec{L}_1\frac{\partial^2\vec{\gamma}}{\partial\mathfrak{t}^2}=\frac{\partial^2\vec{\mathfrak{f}}}{\partial\mathfrak{t}^2}-\Big[\frac{\partial^2\Re}{\partial\mathfrak{t}^2}+\frac{\partial^2\Im}{\partial\mathfrak{t}^2}\Big]\vec{\gamma}-2\Big[\frac{\partial\Re}{\partial\mathfrak{t}}+\frac{\partial\Im}{\partial\mathfrak{t}}\Big]\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}.$$

Therefore,

$$\parallel\vec{L}_1\frac{\partial^2\vec{\gamma}}{\partial\mathfrak{t}^2}\parallel\le\parallel\frac{\partial^2\vec{\mathfrak{f}}}{\partial\mathfrak{t}^2}\parallel+C\parallel\vec{\gamma}\parallel+C\parallel\frac{\partial\vec{\gamma}}{\partial\mathfrak{t}}\parallel\le C.$$

Also,

$$\frac{\partial^2\vec{\gamma}}{\partial\mathfrak{t}^2}(\varkappa,0)=\vec{0}\qquad\text{and}\qquad\vec{\xi}\,\frac{\partial^2\vec{\gamma}}{\partial\mathfrak{t}^2}(0,\mathfrak{t})=\frac{\partial^2\vec{\gamma}}{\partial\mathfrak{t}^2}(0,\mathfrak{t})-E\frac{\partial^3\vec{\gamma}}{\partial\varkappa\partial\mathfrak{t}^2}(0,\mathfrak{t})=C.$$

Using the stability result for the operator $\vec{L}_1$, we get

$$|\frac{\partial^2\vec{\gamma}}{\partial\mathfrak{t}^2}|\le C.$$

To obtain bound on $\frac{\partial\vec{\gamma}}{\partial\varkappa}$, differentiating (5.1) partially with respect to $\varkappa$, we get

$$\vec{L}_1\frac{\partial\vec{\gamma}}{\partial\varkappa}=\frac{\partial\vec{\mathfrak{f}}}{\partial\varkappa}-\left[\frac{\partial\Re}{\partial\varkappa}+\frac{\partial\Im}{\partial\varkappa}\right]\vec{\gamma}.$$

Therefore,

$$\| \vec{L}_1 \frac{\partial \vec{\gamma}}{\partial \varkappa} \| \leq \| \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} \| + C \| \vec{\gamma} \| \leq C.$$

Also,

$$\frac{\partial \vec{\gamma}}{\partial \varkappa}(\varkappa, 0) = C \qquad \text{and} \qquad \vec{\xi}\, \frac{\partial \vec{\gamma}}{\partial \varkappa}(0, \mathfrak{t}) = \frac{\partial \vec{\gamma}}{\partial \varkappa}(0, \mathfrak{t}) - E \frac{\partial^2 \vec{\gamma}}{\partial \varkappa^2}(0, \mathfrak{t}) = \vec{0}.$$

Using the stability result for the operator $\vec{L}_1$, we get

$$|\frac{\partial \vec{\gamma}}{\partial \varkappa}| \leq C.$$

To find the bound on $\frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial t}$, it is easy to check that

$$\vec{L}_1 \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}} = \frac{\partial^2 \vec{\mathfrak{f}}}{\partial \varkappa \partial \mathfrak{t}} - [\frac{\partial^2 \Re}{\partial \varkappa \partial \mathfrak{t}} + \frac{\partial^2 \Im}{\partial \varkappa \partial \mathfrak{t}}]\vec{\gamma} - [\frac{\partial \Re}{\partial \varkappa} + \frac{\partial \Im}{\partial \varkappa}]\frac{\partial \vec{\gamma}}{\partial \mathfrak{t}} - [\frac{\partial \Re}{\partial \mathfrak{t}} + \frac{\partial \Im}{\partial \mathfrak{t}}]\frac{\partial \vec{\gamma}}{\partial \varkappa}.$$

Therefore,

$$\| \vec{L}_1 \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}} \| \leq \| \frac{\partial^2 \vec{\mathfrak{f}}}{\partial \varkappa \partial \mathfrak{t}} \| + C \| \vec{\gamma} \| + C \| \frac{\partial \vec{\gamma}}{\partial \mathfrak{t}} \| + C \| \frac{\partial \vec{\gamma}}{\partial \varkappa} \| \leq C.$$

Further,

$$\frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}}(\varkappa, 0) = \frac{\partial}{\partial \mathfrak{t}}(\frac{\partial \vec{\phi}_B}{\partial \varkappa}(\varkappa)) = \vec{0} \text{ and } \vec{\xi} \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}}(0, \mathfrak{t}) = \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}}(0, \mathfrak{t}) - E \frac{\partial^3 \vec{\gamma}}{\partial \mathfrak{t} \partial \varkappa^2}(0, \mathfrak{t}) = \vec{0}.$$

Using stability result for the operator $\vec{L}_1$, we get

$$\| \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}} \| \leq C.$$

Differentiating (5.1) partially with respect to $\varkappa$, we get

$$\frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}} + E \frac{\partial^2 \vec{\gamma}}{\partial \varkappa^2} + \Re \frac{\partial \vec{\gamma}}{\partial \varkappa} + \frac{\partial \Re}{\partial \varkappa}\vec{\gamma} + \Im \frac{\partial \vec{\gamma}}{\partial \varkappa} + \frac{\partial \Im}{\partial \varkappa}\vec{\gamma} = \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa}$$

$$\frac{\partial^2 \vec{\gamma}}{\partial \varkappa^2} = E^{-1} \Big[\frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} - [\frac{\partial \Re}{\partial \varkappa} + \frac{\partial \Im}{\partial \varkappa}]\vec{\gamma} - [\Re + \Im]\frac{\partial \vec{\gamma}}{\partial \varkappa} - \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}} \Big]$$

and hence it is observed that

$$|\frac{\partial^2 \gamma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| \leq C \varepsilon_\iota^{-1} \Big[\| \frac{\partial \vec{\mathfrak{f}}}{\partial \varkappa} \| + \| \vec{\gamma} \| + \| \frac{\partial \vec{\gamma}}{\partial \varkappa} \| + \| \frac{\partial^2 \vec{\gamma}}{\partial \varkappa \partial \mathfrak{t}} \| \Big] \leq C \varepsilon_\iota^{-1}.$$

Therefore,

$$|\frac{\partial^2 \gamma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| \leq C \varepsilon_\iota^{-1}.$$

Using analogous procedures, the bounds of $\vec{\gamma}$ on the interval $\overline{\Psi}^+$ follow.
By incorporating results for $\vec{\gamma}$ in $\overline{\Psi}^-$ and $\overline{\Psi}^+$, we obtain the required bounds of $\vec{\gamma}$ in the whole of $\overline{\Psi}$. The proof of the lemma is complete.

We introduce the layer functions $\Upsilon_{0,\iota}, \Upsilon_{1,\iota}, \iota = 1, 2$, which are associated with the solution u_ι, defined by

$$\Upsilon_{\wp,\iota}(\varkappa) = e^{-\alpha(\varkappa - \wp)/\varepsilon_\iota}, \ \wp = 0, 1, \ \iota = 1, 2 \ \text{ on } (\varkappa, \mathfrak{t}) \in \overline{\Psi} \tag{5.7}$$

where

$$\Upsilon_{0,\iota}(\varkappa) = e^{-\alpha\varkappa/\varepsilon_\iota} \qquad \text{on } \overline{\Psi}^-$$

$$\Upsilon_{1,\iota}(\varkappa) = e^{-\alpha(\varkappa - 1)/\varepsilon_\iota} \quad \text{on } \overline{\Psi}^+.$$

It has to be noted that for $\iota = 1, 2$

$$\Upsilon_{0,\iota}(\varkappa - 1) = \Upsilon_{1,\iota}(\varkappa).$$

Lemma 5.2. *Let* $\Re(\varkappa, \mathfrak{t}), \Im(\varkappa, \mathfrak{t})$ *satisfy* (2.6) *and* (2.7). *Then there exist a constant C, such that for* $(\varkappa, \mathfrak{t}) \in \overline{\Psi}^-$ *and* $\iota = 1, 2$*,*

$$|\frac{\partial^\ell \varsigma_\iota}{\partial \mathfrak{t}^\ell}(\varkappa, \mathfrak{t})| \leq C\,\Upsilon_{0,2}(\varkappa), \text{ for } \ell = 0, 1, 2$$

$$|\frac{\partial \varsigma_\iota}{\partial x}(\varkappa, \mathfrak{t})| \leq C \sum_{q=\iota}^{2} \frac{\Upsilon_{0,q}(\varkappa)}{\varepsilon_q}$$

$$|\frac{\partial^2 \varsigma_\iota}{\partial x^2}(\varkappa, \mathfrak{t})| \leq C \varepsilon_\iota^{-1} \sum_{q=1}^{2} \frac{\Upsilon_{0,q}(\varkappa)}{\varepsilon_q}$$

and for $(\varkappa, \mathfrak{t}) \in \overline{\Psi}^+$ *and* $\iota = 1, 2$*,*

$$|\frac{\partial^\ell \varsigma_\iota}{\partial \mathfrak{t}^\ell}(\varkappa, \mathfrak{t})| \leq C\,\Upsilon_{1,2}(\varkappa), \text{ for } \ell = 0, 1, 2$$

$$|\frac{\partial \varsigma_\iota}{\partial \varkappa}(\varkappa, \mathfrak{t})| \leq C \sum_{q=\iota}^{2} \frac{\Upsilon_{1,q}(\varkappa)}{\varepsilon_q}$$

$$|\frac{\partial^2 \varsigma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| \leq C \varepsilon_\iota^{-1} \sum_{q=1}^{2} \frac{\Upsilon_{1,q}(\varkappa)}{\varepsilon_q}.$$

Proof. The bound for $\vec{\varsigma}$ and its derivatives are derived first in the interval $\overline{\Psi}^-$. To acquire the bound of $\vec{\varsigma}$, the barrier functions are defined as

$$\aleph_\iota^\pm(\varkappa, \mathfrak{t}) = C\,\Upsilon_{0,2} \pm \varsigma_\iota(\varkappa, \mathfrak{t}),\ \ \iota = 1, 2.$$

It is clear that, for $(\varkappa, \mathfrak{t}) \in \overline{\Psi}^-$, $(\vec{\xi}\vec{\aleph}^\pm)_\iota(0, \mathfrak{t}) \geq 0$, $\aleph_\iota^\pm(\varkappa, 0) \geq 0$, and $(\vec{L}_1\vec{\aleph}^\pm)_\iota(\varkappa, \mathfrak{t}) \geq 0$ on $(\varkappa, \mathfrak{t}) \in \Psi^-$. Then by Lemma 3.2, $\aleph_\iota^\pm(\varkappa, \mathfrak{t}) \geq 0$ for $(\varkappa, \mathfrak{t}) \in \overline{\Psi}^-$. It follows that,

$$|\varsigma_\iota(\varkappa, \mathfrak{t})| \leq C\,\Upsilon_{0,2}(\varkappa).$$

Now differentiate (5.4) partially with respect to variable $\mathfrak{t}$ once and using Lemma 3.2, it is not hard to see that

$$\left(\vec{L}_1 \frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}}\right)_\iota = -\left[\frac{\partial \Re}{\partial \mathfrak{t}}\right]\varsigma_\iota.$$

Therefore,

$$\left\|\left(\vec{L}_1 \frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}}\right)_\iota\right\| \leq C \parallel \varsigma_\iota \parallel$$

$$\left\|\left(\vec{L}_1 \frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}}\right)_\iota\right\| \leq C\,\Upsilon_{0,2}(\varkappa).$$

Also,

$$\frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}}(\varkappa, 0) = \vec{0} \qquad \text{and} \qquad \frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}}(0, \mathfrak{t}) = \frac{\partial \vec{u}}{\partial \mathfrak{t}}(0, \mathfrak{t}) - \frac{\partial \vec{\gamma}}{\partial \mathfrak{t}}(0, \mathfrak{t}) = C.$$

Using the Lemma 3.2 for the operator $\vec{L}_1$, we get

$$|\frac{\partial \varsigma_\iota}{\partial \mathfrak{t}}| \leq C\,\Upsilon_{0,2}(\varkappa).$$

Similarly, differentiating again the same equation with respect to variable $\mathfrak{t}$, we get

$$\vec{L}_1 \frac{\partial^2 \vec{\varsigma}}{\partial \mathfrak{t}^2} = -\left[\frac{\partial^2 \Re}{\partial \mathfrak{t}^2}\right]\vec{\varsigma} - 2\left[\frac{\partial \Re}{\partial \mathfrak{t}}\right]\frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}}.$$

Therefore,

$$\| \vec{L}_1 \frac{\partial^2 \vec{\varsigma}}{\partial \mathfrak{t}^2} \| \leq C \| \vec{\varsigma} \| + C \| \frac{\partial \vec{\varsigma}}{\partial \mathfrak{t}} \|$$

$$\| \vec{L}_1 \frac{\partial^2 \vec{\varsigma}}{\partial \mathfrak{t}^2} \| \leq C\, \Upsilon_{0,2}(\varkappa).$$

Also,

$$\frac{\partial^2 \vec{\varsigma}}{\partial \mathfrak{t}^2}(\varkappa, 0) = 0 \text{ and } \frac{\partial^2 \vec{\varsigma}}{\partial \mathfrak{t}^2}(0, \mathfrak{t}) = \frac{\partial^2 \vec{u}}{\partial \mathfrak{t}^2}(0, \mathfrak{t}) - \frac{\partial^2 \vec{\gamma}}{\partial \mathfrak{t}^2}(0, \mathfrak{t}) = C.$$

Using the Lemma 3.2 for the operator $\vec{L}_1$, we get

$$|\frac{\partial^2 \varsigma_\iota}{\partial \mathfrak{t}^2}| \leq C\, \Upsilon_{0,2}(\varkappa).$$

Now differentiate (5.4) partially with respect to variable $\mathfrak{t}$ once, for $\iota = 1, 2$ we get

$$\frac{\partial^2 \varsigma_\iota}{\partial \mathfrak{t}^2} + \varepsilon_\iota \frac{\partial^2 \varsigma_\iota}{\partial \varkappa \partial \mathfrak{t}} + \sum_{\jmath=1}^{2} \mathfrak{r}_{\iota\jmath} \frac{\partial \varsigma_\jmath}{\partial \mathfrak{t}} + \sum_{\jmath=1}^{2} \frac{\partial \mathfrak{r}_{\iota\jmath}}{\partial \mathfrak{t}} \varsigma_\jmath = 0$$

$$\frac{\partial^2 \varsigma_\iota}{\partial \varkappa \partial \mathfrak{t}} = \varepsilon_\iota^{-1}[-\sum_{\jmath=1}^{2} \mathfrak{r}_{\iota\jmath} \frac{\partial \varsigma_\jmath}{\partial \mathfrak{t}} - \sum_{\jmath=1}^{2} \frac{\partial \mathfrak{r}_{\iota\jmath}}{\partial \mathfrak{t}} \varsigma_\jmath - \frac{\partial^2 \varsigma_\iota}{\partial \mathfrak{t}^2}].$$

Therefore,

$$|\frac{\partial^2 \varsigma_\iota}{\partial \varkappa \partial \mathfrak{t}}| \leq C \varepsilon_\iota^{-1} \Upsilon_{0,2}(\varkappa).$$

To obtain bound on $\frac{\partial \varsigma_1}{\partial \varkappa}$, let us consider $(\vec{L}_1 \vec{\varsigma})_1 = 0$.

$$\frac{\partial \varsigma_1}{\partial \mathfrak{t}}(\varkappa, \mathfrak{t}) + \varepsilon_1 \frac{\partial \varsigma_1}{\partial \varkappa}(\varkappa, \mathfrak{t}) + \mathfrak{r}_{11}(\varkappa, \mathfrak{t})\varsigma_1(\varkappa, \mathfrak{t}) + \mathfrak{r}_{12}(\varkappa, \mathfrak{t})\varsigma_2(\varkappa, \mathfrak{t}) = 0$$

$$\begin{aligned} \varepsilon_1 \frac{\partial \varsigma_1}{\partial \varkappa} &= -\mathfrak{r}_{11}\varsigma_1 - \mathfrak{r}_{12}\varsigma_2 - \frac{\partial \varsigma_1}{\partial \mathfrak{t}} \\ \frac{\partial \varsigma_1}{\partial \varkappa} &= \varepsilon_1^{-1}[-\mathfrak{r}_{11}\varsigma_1 - \mathfrak{r}_{12}\varsigma_2 - \frac{\partial \varsigma_1}{\partial \mathfrak{t}}] \\ |\frac{\partial \varsigma_1}{\partial \varkappa}| &\leq C\varepsilon_1^{-1}[|\varsigma_1| + |\varsigma_2| + |\frac{\partial \varsigma_1}{\partial \mathfrak{t}}|] \\ &\leq C\varepsilon_1^{-1}\Upsilon_{0,2}(\varkappa). \end{aligned}$$

Hence,

$$|\frac{\partial \varsigma_1}{\partial \varkappa}| \leq C\varepsilon_1^{-1}\Upsilon_{0,2}(\varkappa).$$

Similarly,

$$|\frac{\partial \varsigma_2}{\partial \varkappa}| \leq C\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa).$$

To find Sharper estimates of $\frac{\partial \varsigma_1}{\partial \varkappa}$, consider the equation

$$\left(\vec{L}_1 \frac{\partial \vec{\varsigma}}{\partial \varkappa}\right)_1 = \frac{\partial}{\partial \mathfrak{t}}\frac{\partial \varsigma_1}{\partial \varkappa} + \varepsilon_1 \frac{\partial}{\partial \varkappa}\frac{\partial \varsigma_1}{\partial \varkappa} + \mathfrak{r}_{11}\frac{\partial \varsigma_1}{\partial \varkappa} + \mathfrak{r}_{12}\frac{\partial \varsigma_2}{\partial \varkappa} = -\frac{\partial \mathfrak{r}_{11}}{\partial \varkappa}\varsigma_1 - \frac{\partial \mathfrak{r}_{12}}{\partial \varkappa}\varsigma_2.$$

Clearly,

$$\left|\left(\vec{L}_1\frac{\partial\vec{\varsigma}}{\partial\varkappa}\right)_1\right| \le C\,[|\varsigma_1|+|\varsigma_2|] \le C\,\Upsilon_{0,2}(\varkappa).$$

Define the barrier functions

$$(\vec{\vartheta}^{\pm})_1 = C[\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)] \pm \frac{\partial\varsigma_1}{\partial\varkappa}.$$

It is clear that, for $(\varkappa,t)\in\Psi^-$ and C sufficiently large, $(\vec{\xi}\vec{\vartheta}^{\pm})_1\ge 0$, $(\vec{\vartheta}^{\pm})_1(\varkappa,0)\ge 0$, and

$$\begin{aligned}(\vec{L}_1\vec{\vartheta}^{\pm})_1 &= C[\varepsilon_1(\frac{-\alpha}{\varepsilon_1})\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_1(\frac{-\alpha}{\varepsilon_2})\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)]\\ &\quad+\mathfrak{r}_{11}[\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)]\pm|L_1\frac{\partial\varsigma_1}{\partial\varkappa}|\\ &\ge C(-\alpha+\mathfrak{r}_{11})[\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)]\pm C\,\Upsilon_{0,2}(\varkappa)\ge 0.\end{aligned}$$

Then by maximum principle for the operator $\vec{L}_1$, $(\vec{\vartheta}^{\pm})_1 \ge 0$ for $(\varkappa,\mathfrak{t})\in\overline{\Psi}^-$. Hence the required bound for $|\frac{\partial\varsigma_1}{\partial\varkappa}|$ is

$$|\frac{\partial\varsigma_1}{\partial\varkappa}| \le C[\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)].$$

Differentiating $(\vec{L}_1\vec{\varsigma})_1 = 0$ and $(\vec{L}_1\vec{\varsigma})_2 = 0$ partially with respect to $\varkappa$ once and using the estimates of $|\frac{\partial\varsigma_\iota}{\partial\varkappa}|$, $|\frac{\partial^2\varsigma_\iota}{\partial\varkappa\partial\mathfrak{t}}|$, we get

$$\begin{aligned}|\frac{\partial^2\varsigma_1}{\partial\varkappa^2}| &\le C\varepsilon_1^{-1}[\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)],\\ |\frac{\partial^2\varsigma_2}{\partial\varkappa^2}| &\le C\varepsilon_2^{-1}[\varepsilon_1^{-1}\Upsilon_{0,1}(\varkappa)+\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa)].\end{aligned}$$

The bounds of $\vec{\varsigma}$ and its derivatives on $\overline{\Psi}^+$ are analogous. By incorporating the results for $\vec{\varsigma}$ and its derivatives in the intervals $\overline{\Psi}^-$ and $\overline{\Psi}^+$, we arrive at required bounds of $\vec{\varsigma}$ and its derivatives in the whole of $\overline{\Psi}$. The proof of the lemma is complete.

Definition. As in [11], we have for each $1\le\iota\ne\jmath\le 2$, define an unique point $g+\varkappa_{\iota,\jmath}$ in $(g,g+1], g=0,1$ as

$$\frac{\Upsilon_{g,\iota}\,(g+\varkappa_{\iota,\jmath})}{\varepsilon_\iota} = \frac{\Upsilon_{g,\jmath}\,(g+\varkappa_{\iota,\jmath})}{\varepsilon_\jmath} \tag{5.8}$$

Uniqueness and ordering of the points $g+\varkappa_{\iota,\jmath}$ in $(g,g+1], g=0,1$ are shown in succeeding lemma.

Lemma 5.3. *For all $\iota,\jmath$ such that $1\le\iota<\jmath\le 2$, the points $g+\varkappa_{\iota,\jmath}, g=0,1$ are uniquely defined and satisfy the following inequalities.* [11]

$$\frac{\Upsilon_{g,\iota}\,(\varkappa)}{\varepsilon_\iota} > \frac{\Upsilon_{g,\jmath}\,(\varkappa)}{\varepsilon_\jmath}, \qquad \varkappa\in[g,g+\varkappa_{\iota,\jmath}), \tag{5.9}$$

$$\frac{\Upsilon_{g,\iota}\,(\varkappa)}{\varepsilon_\iota} < \frac{\Upsilon_{g,\jmath}\,(\varkappa)}{\varepsilon_\jmath}, \qquad \varkappa\in(g+\varkappa_{\iota,\jmath},g+1]. \tag{5.10}$$

In addition to this, ordering of the points also established as,

$$\varkappa_{\iota,\jmath} < \varkappa_{\iota+1,\jmath} \quad \text{if} \quad \iota+1<\jmath, \tag{5.11}$$

$$\varkappa_{\iota,\jmath} < \varkappa_{\iota,\jmath+1} \quad \text{if} \quad \iota<\jmath. \tag{5.12}$$

6. Shishkin Mesh

A piecewise uniform Shishkin mesh is established in the domain of problem (2.1) with $M \times N$ mesh intervals.

Let $\Psi_{\mathfrak{t}}^M = \{\mathfrak{t}_\kappa\}_{\kappa=1}^M$, $\overline{\Psi}_{\mathfrak{t}}^M = \{\mathfrak{t}_\kappa\}_{\kappa=0}^M$, $\Psi_{\varkappa}^N = \{\varkappa_\jmath\}_{\jmath=1}^N$, $\overline{\Psi}_{\varkappa}^N = \{\varkappa_\jmath\}_{\jmath=0}^N$. $\Psi^{M,N} = \Psi_{\mathfrak{t}}^M \times \Psi_{\varkappa}^N$, $\overline{\Psi}^{M,N} = \overline{\Psi}_{\mathfrak{t}}^M \times \overline{\Psi}_{\varkappa}^N$ and $\Delta^{M,N} = \Delta \cap \overline{\Psi}^{M,N}$.

$\overline{\Psi}_{\mathfrak{t}}^M$: uniform mesh comprising M mesh intervals on Δ_L.

$\overline{\Psi}_{\varkappa}^N$: piecewise-uniform mesh comprising N mesh intervals on Δ_B.

$\overline{\Psi}_{\varkappa}^N = \overline{\Psi}_{\varkappa}^{-N} \cup \overline{\Psi}_{\varkappa}^{+N}$ where $\overline{\Psi}_{\varkappa}^{-N} = \{\varkappa_\jmath\}_{\jmath=0}^{\frac{N}{2}}$, $\overline{\Psi}_{\varkappa}^{+N} = \{\varkappa_\jmath\}_{\jmath=\frac{N}{2}}^{N}$.

$\Psi_{\varkappa}^{-N} = \{\varkappa_\jmath\}_{\jmath=1}^{\frac{N}{2}-1}$, $\Psi_{\varkappa}^{+N} = \{\varkappa_\jmath\}_{\jmath=\frac{N}{2}+1}^{N}$, $\Psi^{-M,N} = \Psi_{\mathfrak{t}}^M \times \Psi_{\varkappa}^{-N}$, $\overline{\Psi}^{-M,N} = \overline{\Psi}_{\mathfrak{t}}^M \times \overline{\Psi}_{\varkappa}^{-N}$, $\Psi^{+M,N} = \Psi_{\mathfrak{t}}^M \times \Psi_{\varkappa}^{+N}$, $\overline{\Psi}^{+M,N} = \overline{\Psi}_{\mathfrak{t}}^M \times \overline{\Psi}_{\varkappa}^{+N}$.

The domain Δ_B is partitioned into 6 sub-intervals as follows

$$[0,\eta_1) \cup [\eta_1,\eta_2) \cup [\eta_2,1] \cup [1,1+\eta_1) \cup [1+\eta_1,1+\eta_2) \cup [1+\eta_2,2]$$

where η_1, η_2 are the transition parameters, defined as:

$$\eta_\iota = \min\left\{\frac{\eta_{\iota+1}}{2}, \frac{\varepsilon_\iota}{\alpha}\ln N\right\}, \iota = 1,2 \ \text{ and } \ \eta_3 = 1. \tag{6.1}$$

Clearly,

$$0 < \eta_1 < \eta_2 \leq \frac{1}{2}. \tag{6.2}$$

Then, on the subinterval $[\eta_2, 1]$ and $[1+\eta_2, 2]$, a uniform mesh with $\frac{N}{4}$ mesh points is deployed and on each of the sub-intervals $[0,\eta_1)$, $[\eta_1,\eta_2)$, $[1,1+\eta_1)$ and $[1+\eta_1, 1+\eta_2)$ a uniform mesh of $\frac{N}{8}$ mesh points is deployed. In particular, when both the transition parameters $\eta_s, s = 1,2$ take on its left-hand value, the Shishkin mesh $\overline{\Psi}_x^N$ becomes classical uniform mesh in the interval $[0,2]$. We introduce the following notations:

$$\mathfrak{H}_\jmath = \varkappa_\jmath - \varkappa_{\jmath-1} \tag{6.3}$$

If $\varkappa_\jmath = \eta$, then $\mathfrak{H}_\jmath^- = \varkappa_\jmath - \varkappa_{\jmath-1}$, $\mathfrak{H}_\jmath^+ = \varkappa_{\jmath+1} - \varkappa_\jmath$, $J = \{\varkappa_\jmath : \mathfrak{H}_\jmath^- \neq \mathfrak{H}_\jmath^+\}$ and $\mu_\kappa = \mathfrak{t}_\kappa - \mathfrak{t}_{\kappa-1}$. This framework generates a collection of four possible Shishkin piecewise meshes $\mathfrak{G}_{\mathfrak{b}_\iota}$ where $\mathfrak{b}_\iota = 0$ if $\eta_\iota = \frac{\eta_{\iota+1}}{2}$ and $\mathfrak{b}_\iota = 1$ otherwise.

7. The discrete problem

In this section, to construct the discrete problem with the help of finite difference operator, we discretized (2.1) with the help of uniform and piecewise uniform mesh, $\overline{\Psi}_{\mathfrak{t}}^M$ and $\overline{\Psi}_{\varkappa}^N$ respectively.

$$\vec{L}^{M,N}\vec{U} = (D_{\mathfrak{t}}^-\vec{U} + ED_{\varkappa}^-\vec{U} + \Re\vec{U})(\varkappa_\jmath, \mathfrak{t}_\kappa) + \Im(\varkappa_\jmath, \mathfrak{t}_\kappa)\vec{U}(\varkappa_\jmath - 1, \mathfrak{t}_\kappa) = \vec{\mathfrak{f}}(\varkappa_\jmath, \mathfrak{t}_\kappa) \text{ on } \Psi^{M,N}, \tag{7.1}$$

and

$$\vec{\xi}^{M,N}\vec{U} = (\vec{U} - ED_{\varkappa}^+\vec{U})(\varkappa_\jmath, \mathfrak{t}_\kappa) = \vec{\varphi}_L(\varkappa_\jmath, \mathfrak{t}_\kappa), \ \text{on } \Psi^{*M,N}, \quad \vec{U}(\varkappa_\jmath, 0) = \vec{\varphi}_B(\varkappa_\jmath). \tag{7.2}$$

The problem (7.1), (7.2) is rewritten as

$$\vec{L}_1^{M,N}\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) = (D_\mathfrak{t}^-\vec{U} + ED_\varkappa^-\vec{U} + \Re\vec{U})(\varkappa_\jmath, \mathfrak{t}_\kappa) = \vec{\Lambda}(\varkappa_\jmath, \mathfrak{t}_\kappa) \text{ on } \Psi^{-M,N}, \tag{7.3}$$

and on $\Psi^{+M,N}$,

$$\vec{L}_2^{M,N}\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) = (D_\mathfrak{t}^-\vec{U} + ED_\varkappa^-\vec{U} + \Re\vec{U})(\varkappa_\jmath, \mathfrak{t}_\kappa) + \Im(\varkappa_\jmath, \mathfrak{t}_\kappa)\vec{U}(\varkappa_\jmath - 1, \mathfrak{t}_\kappa) = \vec{\mathfrak{f}}(\varkappa_\jmath, \mathfrak{t}_\kappa) \tag{7.4}$$

$$\vec{\xi}^{M,N}\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) = \vec{\varphi}_L(\varkappa_\jmath, \mathfrak{t}_\kappa), \qquad \vec{U}(\varkappa_\jmath, 0) = \vec{\varphi}_B(\varkappa_\jmath),$$

where

$$D_\mathfrak{t}^-\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) = \frac{\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) - \vec{U}(\varkappa_\jmath, \mathfrak{t}_{\kappa-1})}{\mathfrak{t}_\kappa - \mathfrak{t}_{\kappa-1}}$$
$$D_\varkappa^-\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) = \frac{\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) - \vec{U}(\varkappa_{\jmath-1}, \mathfrak{t}_\kappa)}{\varkappa_\jmath - \varkappa_{\jmath-1}}$$
$$D_\varkappa^+\vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa) = \frac{\vec{U}(\varkappa_{\jmath+1}, \mathfrak{t}_\kappa) - \vec{U}(\varkappa_\jmath, \mathfrak{t}_\kappa)}{\varkappa_{\jmath+1} - \varkappa_\jmath}.$$

Lemma 7.1. *Let $\Re(\varkappa_\jmath, \mathfrak{t}_\kappa)$ and $\Im(\varkappa_\jmath, \mathfrak{t}_\kappa)$ satisfy* (2.6) *and* (2.7). *Then for any mesh function $\vec{\Xi}$, the inequalities $\vec{\xi}^{M,N}\vec{\Xi}(0, \mathfrak{t}_\kappa) \geq \vec{0}$, $\vec{\Xi}(\varkappa_\jmath, 0) \geq \vec{0}$ and $\vec{L}_1^{M,N}\vec{\Xi} \geq \vec{0}$ on $\Psi^{-M,N}$, $\vec{L}_2^{M,N}\vec{\Xi} \geq \vec{0}$ on $\Psi^{+M,N}$ imply that $\vec{\Xi} \geq \vec{0}$ on $\overline{\Psi}^{M,N}$.*

Proof. For $\iota^*, \jmath^*, \kappa^*$, let us consider $\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) = \min\limits_{\iota,\jmath,\kappa} \Xi_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)$ and assume that the lemma does not hold. Then $\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) < 0$. Using the hypotheses, it is simple to establish that, $(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \notin \Delta^{M,N}$, $\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) - \Xi_{\iota^*}(\varkappa_{\jmath^*-1}, \mathfrak{t}_{\kappa^*}) \leq 0$ and $\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) - \Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*-1}) \leq 0$. It follows that, for $(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \in \Psi^{-M,N}$,

$$\begin{aligned}
(\vec{L}_1^{M,N}\vec{\Xi})_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) &= D_\mathfrak{t}^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \varepsilon_{\iota^*}D_\varkappa^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \sum_{l=1}^{2}\mathfrak{r}_{\iota^* l}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*})\Xi_l(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \\
&= D_\mathfrak{t}^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \varepsilon_{\iota^*}D_\varkappa^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \sum_{l=1}^{2}\mathfrak{r}_{\iota^* l}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*})\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \\
&< 0
\end{aligned}$$

which contradicts our assumption and for $(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \in \Psi^{+M,N}$,

$$\begin{aligned}
(\vec{L}_2^{M,N}\vec{\Xi})_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) &= D_\mathfrak{t}^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \varepsilon_{\iota^*}D_\varkappa^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \sum_{l=1}^{2}\mathfrak{r}_{\iota^* l}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*})\Xi_l(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \\
&\quad + \beta_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*})\Xi_{\iota^*}(\varkappa_{\jmath^*} - 1, t_{\kappa^*}) \\
&= D_\mathfrak{t}^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \varepsilon_{\iota^*}D_\varkappa^-\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) + \sum_{l=1}^{2}\mathfrak{r}_{\iota^* l}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*})\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \\
&\quad + \beta_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*})\Xi_{\iota^*}(\varkappa_{\jmath^*}, \mathfrak{t}_{\kappa^*}) \\
&< 0
\end{aligned}$$

which contradicts our assumption.
If $\varkappa_{\jmath^*} = 0$, then

$$\vec{\xi}^{M,N}\vec{\Xi}(0, \mathfrak{t}_{\kappa^*}) = \vec{\Xi}(0, \mathfrak{t}_{\kappa^*}) - ED_{\varkappa}^{+}\vec{\Xi}(0, \mathfrak{t}_{\kappa^*}) < 0,$$

a contradiction. Therefore $\varkappa_{\jmath^*} \neq 0$. For $\mathfrak{t}_{\kappa^*} = 0, \vec{\Xi}(\varkappa_{\jmath^*}, 0) < 0$, this proves that our assumption is contradictory. Therefore, $\mathfrak{t}_{\kappa^*} \neq 0$. The proof of the lemma is complete.

The stability result is established in the next lemma as an immediate result of the preceding lemma.

Lemma 7.2. *Let* $\Re(\varkappa_\jmath, \mathfrak{t}_\kappa)$ *and* $\Im(\varkappa_\jmath, \mathfrak{t}_\kappa)$ *satisfy* (2.6) *and* (2.7). *Then for any mesh function* $\vec{\Xi}$ *on* $\overline{\Psi}^{M,N}$

$$|\vec{\Xi}(\varkappa_\jmath, \mathfrak{t}_\kappa)| \leq \max\{\| \vec{\xi}^{M,N}\vec{\Xi}(0, \mathfrak{t}_\kappa), ||\vec{\Xi}(\varkappa_\jmath, 0)||, \ \frac{1}{\alpha} \parallel \vec{L}_1^{M,N}\vec{\Xi} \parallel, \ \frac{1}{\alpha} \parallel \vec{L}_2^{M,N}\vec{\Xi} \parallel\}.$$

Proof. Define barrier functions

$$\vec{\Theta}^{\pm}(\varkappa_\jmath, \mathfrak{t}_\kappa) = \max\{\| \vec{\xi}^{M,N}\vec{\Xi}(0, \mathfrak{t}_\kappa), ||\vec{\Xi}(\varkappa_\jmath, 0)||, \ \frac{1}{\alpha} \parallel \vec{L}_1^{M,N}\vec{\Xi} \parallel, \ \frac{1}{\alpha} \parallel \vec{L}_2^{M,N}\vec{\Xi} \parallel\} \pm \vec{\Xi}(\varkappa_\jmath, \mathfrak{t}_\kappa).$$

It is simple to demonstrate that $\vec{\xi}^{M,N}\vec{\Theta}^{\pm}(0, \mathfrak{t}_\kappa) \geq \vec{0}, \ \vec{\Theta}^{\pm}(\varkappa_\jmath, 0) \geq \vec{0}$ and also $\vec{L}_1^{M,N}\vec{\Theta}^{\pm} \geq \vec{0}$ on $\Psi^{-M,N}$, $\vec{L}_2^{M,N}\vec{\Theta}^{\pm} \geq \vec{0}$ on $\Psi^{+M,N}$. From Lemma 7.1 we get the result $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Psi}^{M,N}$.

8. DISCRETE SHISHKIN DECOMPOSITION

Just as in the continuous case, $\vec{U} = \vec{\Gamma} + \vec{\zeta}$ be the Shishkin decomposition of the discrete solution $\vec{U}$ of (7.1) and (7.2), where $\vec{\Gamma}$ is the smooth component of the solution

(8.1) $$\vec{L}_1^{M,N}\vec{\Gamma}(\varkappa_\jmath, \mathfrak{t}_\kappa) = (D_{\mathfrak{t}}^{-}\vec{\Gamma} + ED_{\varkappa}^{-}\vec{\Gamma} + A\vec{\Gamma})(\varkappa_\jmath, \mathfrak{t}_\kappa) = \vec{\Lambda} \text{ on } \Psi^{-M,N},$$

and on $\Psi^{+M,N}$,

(8.2) $$\vec{L}_2^{M,N}\vec{\Gamma}(\varkappa_\jmath, \mathfrak{t}_\kappa) = (D_{\mathfrak{t}}^{-}\vec{\Gamma} + ED_{\varkappa}^{-}\vec{\Gamma} + A\vec{\Gamma})(\varkappa_\jmath, \mathfrak{t}_\kappa) + B(\varkappa_\jmath, \mathfrak{t}_\kappa)\vec{\Gamma}(\varkappa_\jmath - 1, \mathfrak{t}_\kappa) = \vec{\mathfrak{f}}(\varkappa_\jmath, \mathfrak{t}_\kappa)$$

(8.3) $$\vec{\xi}^{M,N}\vec{\Gamma}(0, \mathfrak{t}_\kappa) = \vec{\xi}\vec{\gamma}(0, \mathfrak{t}_\kappa), \qquad \vec{\Gamma}(\varkappa_\jmath, 0) = \vec{\gamma}(\varkappa_\jmath, 0)$$

and the singular component $\vec{\zeta}$ is the solution of

(8.4) $$\vec{L}_1^{M,N}\vec{\zeta} = \vec{0} \ \text{ on } \ \Psi^{-M,N}, \qquad \vec{L}_2^{M,N}\vec{\zeta} = \vec{0} \ \text{ on } \ \Psi^{+M,N},$$

(8.5) $$\vec{\xi}^{M,N}\vec{\zeta}(0, \mathfrak{t}_\kappa) = \vec{\xi}\vec{\varsigma}(0, \mathfrak{t}_\kappa), \quad \vec{\zeta}(\varkappa_\jmath, 0) = \vec{\varsigma}(\varkappa_\jmath, 0).$$

9. THE LOCAL TRUNCATION ERROR

From Lemma 7.2, it's evident that in order to bound the error $\vec{U} - \vec{u}$, it is enough to bound $\vec{\xi}^{M,N}(\vec{U} - \vec{u})(0, \mathfrak{t}_\kappa), (\vec{U} - \vec{u})(\varkappa_\jmath, 0)$ and $\vec{L}^{M,N}(\vec{U} - \vec{u})$. Note that, for $(\varkappa_\jmath, \mathfrak{t}_\kappa) \in \Psi^{M,N}$

$$\begin{aligned}\vec{L}^{M,N}(\vec{U} - \vec{u}) &= \vec{L}^{M,N}\vec{U} - \vec{L}^{M,N}\vec{u} \\ &= \vec{f} - \vec{L}^{M,N}\vec{u} \\ &= \vec{L}\vec{u} - \vec{L}^{M,N}\vec{u} \\ &= (\vec{L} - \vec{L}^{M,N})\vec{u}.\end{aligned}$$

Then

$$(\vec{L}^{M,N}(\vec{U} - \vec{u}))_\iota = (\frac{\partial}{\partial t} - D_t^{-})u_\iota + \varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})u_\iota$$

$$(\vec{L}^{M,N}(\vec{U} - \vec{u}))_\iota = (\vec{L}^{M,N}(\vec{\Gamma} - \vec{\gamma}))_\iota + (\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota$$

which denotes that the local truncation error is associated with the first derivative. Then the triangular inequality is given below,

$$|(\vec{L}^{M,N}(\vec{U}-\vec{u}))_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| \le |(\vec{L}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| + |(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)|.$$

Let $\vec{\Gamma}$ and $\vec{\zeta}$ be the discrete analogous of $\vec{\gamma}$ and $\vec{\varsigma}$ respectively. Then,

$$(\vec{L}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota = (\frac{\partial}{\partial \mathsf{t}} - D_{\mathsf{t}}^-)\gamma_\iota + \varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\gamma_\iota \tag{9.1}$$

$$(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota = (\frac{\partial}{\partial \mathsf{t}} - D_{\mathsf{t}}^-)\varsigma_\iota + \varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_\iota. \tag{9.2}$$

Therefore

$$|(\vec{L}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| \le |(\frac{\partial}{\partial \mathsf{t}} - D_{\mathsf{t}}^-)\gamma_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| + \varepsilon_\iota|(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\gamma_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| \tag{9.3}$$

$$|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| \le |(\frac{\partial}{\partial \mathsf{t}} - D_{\mathsf{t}}^-)\varsigma_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)| + \varepsilon_\iota|(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath,\mathsf{t}_\kappa)|. \tag{9.4}$$

Further,

$$|(\vec{\xi}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota(0,\mathsf{t}_\kappa)| = |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^+)\gamma_\iota(0,\mathsf{t}_\kappa)| \tag{9.5}$$

$$|(\vec{\xi}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(0,\mathsf{t}_\kappa)| = |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^+)\varsigma_\iota(0,\mathsf{t}_\kappa)|. \tag{9.6}$$

Thus, the local truncation error corresponding to the smooth and singular components is handled separately. It is important to highlight that the following distinct estimates of the local truncation error of its first order partial derivatives are valid for any function $\vec{\aleph}$.
For each $(\varkappa_\jmath,\mathsf{t}_\kappa) \in \Psi^{M,N}$

$$|(\frac{\partial}{\partial \mathsf{t}} - D_{\mathsf{t}}^-)\aleph(\varkappa_\jmath,\mathsf{t}_\kappa)| \le C(\mathsf{t}_\kappa - \mathsf{t}_{\kappa-1}) \max_{s\in[\mathsf{t}_{\kappa-1},\,\mathsf{t}_\kappa]} |\frac{\partial^2\aleph}{\partial \mathsf{t}^2}(\varkappa_\jmath\,,s)| \tag{9.7}$$

$$|(\frac{\partial}{\partial \mathsf{t}} - D_{\mathsf{t}}^+)\aleph(\varkappa_\jmath,\mathsf{t}_\kappa)| \le C(\mathsf{t}_{\kappa+1} - \mathsf{t}_\kappa) \max_{s\in[\mathsf{t}_\kappa,\,\mathsf{t}_{\kappa+1}]} |\frac{\partial^2\aleph}{\partial \mathsf{t}^2}(\varkappa_\jmath\,,s)| \tag{9.8}$$

$$|(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\aleph(\varkappa_\jmath,\mathsf{t}_\kappa)| \le C(\varkappa_\jmath - \varkappa_{\jmath-1}) \max_{s\in[\varkappa_{\jmath-1},\,\varkappa_\jmath]} |\frac{\partial^2\aleph}{\partial\varkappa^2}(s,\,\mathsf{t}_\kappa)| \tag{9.9}$$

$$|(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\aleph(\varkappa_\jmath,\mathsf{t}_\kappa)| \le C \max_{s\in[\varkappa_{\jmath-1},\,\varkappa_\jmath]} |\frac{\partial\aleph}{\partial\varkappa}(s,\,\mathsf{t}_\kappa)| \tag{9.10}$$

$$|(\frac{\partial}{\partial\varkappa} - D_\varkappa^+)\aleph(\varkappa_\jmath,\mathsf{t}_\kappa)| \le C(\varkappa_{\jmath+1} - \varkappa_\jmath) \max_{s\in[\varkappa_\jmath,\,\varkappa_{\jmath+1}]} |\frac{\partial^2\aleph}{\partial\varkappa^2}(s,\,\mathsf{t}_\kappa)| \tag{9.11}$$

$$|(\frac{\partial}{\partial\varkappa} - D_\varkappa^+)\aleph(\varkappa_\jmath,\mathsf{t}_\kappa)| \le C \max_{s\in[\varkappa_\jmath,\,\varkappa_{\jmath+1}]} |\frac{\partial\aleph}{\partial\varkappa}(s,\,\mathsf{t}_\kappa)|. \tag{9.12}$$

10. ERROR ESTIMATE

The succeeding theorem estimates the error estimates of the smooth component of the local truncation error.

Theorem 10.1. *Let* $\Re(\varkappa,\mathsf{t})$ *and* $\Im(\varkappa,\mathsf{t})$ *satisfy* (2.6) *and* (2.7). *Let* $\vec{\gamma}$ *represents the smooth component of the solution of* (5.1), (5.2) *and* (5.3) *and let* $\vec{\Gamma}$ *represents the smooth component of the solution of* (8.1),(8.2) *and* (8.3) *then*

$$\parallel \vec{\Gamma} - \vec{\gamma} \parallel \le C(M^{-1} + N^{-1}).$$

Proof. Based on the expression (9.5) and (9.11),

$$\begin{aligned}|(\vec{\xi}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota(0,\mathfrak{t}_\kappa)| &= |\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D^+_\varkappa)\gamma_\iota(0,\,\mathfrak{t}_\kappa)|\\ &\leq C\varepsilon_\iota(\varkappa_1-\varkappa_0)\max_{s\in[\varkappa_0,\varkappa_1]}|\frac{\partial^2\gamma_\iota}{\partial\varkappa^2}(s,\mathfrak{t}_\kappa)|\\ &\leq C\varepsilon_\iota N^{-1}C\varepsilon_\iota^{-1}\leq CN^{-1}.\end{aligned}$$

It is easy to demonstrate that

$$\begin{aligned}|(\vec{L}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota(\varkappa_\jmath,\,\mathfrak{t}_\kappa)| &\leq |(\frac{\partial}{\partial\mathfrak{t}}-D^-_\mathfrak{t})\gamma_\iota(\varkappa_\jmath,\,\mathfrak{t}_\kappa)|+|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D^-_\varkappa)\gamma_\iota(\varkappa_\jmath,\,\mathfrak{t}_\kappa)|\\ &\leq C\mu_\kappa\max_{s\in[\mathfrak{t}_{\kappa-1},\mathfrak{t}_\kappa]}|\frac{\partial^2\gamma_\iota}{\partial\mathfrak{t}^2}(\varkappa_\jmath,s)|+C\varepsilon_\iota\mathfrak{H}_\jmath\max_{s\in[\varkappa_{\jmath-1},\varkappa_\jmath]}|\frac{\partial^2\gamma_\iota}{\partial\varkappa^2}(s,\,\mathfrak{t}_\kappa)|\\ &\leq CM^{-1}+CN^{-1}\varepsilon_\iota C\varepsilon_\iota^{-1},\text{ using Lemma 5.1}\\ &\leq C(M^{-1}+N^{-1})\end{aligned}$$

The proof of the theorem is complete.
Prior to estimating the singular part of the error, the following lemmas are established.

Lemma 10.2. *Let* $\Re(\varkappa,\mathfrak{t})$ *and* $\Im(\varkappa,\mathfrak{t})$ *satisfy* (2.6) *and* (2.7). *Let* $\vec{\varsigma}$ *represents the singular component of the solution of* (5.4), (5.5) *and* (5.6) *and let* $\vec{\zeta}$ *represents the singular component of the solution of* (8.4), (8.5) *then*

$$|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|\leq C\frac{\mathfrak{H}_\jmath}{\varepsilon_1}\qquad\text{and}\tag{10.1}$$

$$|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|\leq C(M^{-1}+\frac{\mathfrak{H}_\jmath}{\varepsilon_1}).\tag{10.2}$$

Proof.

$$\begin{aligned}|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\leq C\varepsilon_\iota\mathfrak{H}_\jmath\max_{s\in[\varkappa_{\jmath-1},\,\varkappa_\jmath]}|\frac{\partial^2\varsigma_\iota}{\partial\varkappa^2}(s,\,\mathfrak{t}_\kappa)|\\ &\leq C\varepsilon_\iota\mathfrak{H}_\jmath C\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q},\text{ using Lemma 5.2}\\ &\leq C\mathfrak{H}_\jmath\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q}\end{aligned}$$

$$
\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\leq C\frac{\mathfrak{H}_\jmath}{\varepsilon_1}.\\
|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\leq |(\frac{\partial}{\partial \mathfrak{t}} - D_\mathfrak{t}^-)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)|\\
&\leq C\mu_\kappa \max_{s\in[\mathfrak{t}_{\kappa-1},\mathfrak{t}_\kappa]}|\frac{\partial^2\varsigma_\iota}{\partial \mathfrak{t}^2}(\varkappa_\jmath, s)|\\
&\quad + C\varepsilon_\iota\mathfrak{H}_\jmath \max_{s\in[\varkappa_{\jmath-1},\varkappa_\jmath]}|\frac{\partial^2\varsigma_\iota}{\partial \varkappa^2}(s, \mathfrak{t}_\kappa)|\\
&\leq CM^{-1} + C\varepsilon_\iota\mathfrak{H}_\jmath\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q}, \text{using Lemma 5.2}\\
&\leq CM^{-1} + C\frac{\mathfrak{H}_\jmath}{\varepsilon_1}\\
|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\leq C(M^{-1} + \frac{\mathfrak{H}_\jmath}{\varepsilon_1}).
\end{aligned}
$$

The proof of the lemma is complete.

Lemma 10.3. *Let* $\Re(\varkappa,\mathfrak{t})$ *and* $\Im(\varkappa,t)$ *satisfy* (2.6) *and* (2.7)*. Then for each* $\iota = 1,2$ *and for any mesh* $M_{\vec{b}}$ *with* $b_1 = 1$*, there exist a decomposition*

$$\varsigma_\iota(\varkappa,\mathfrak{t}) = \varsigma_{\iota,1}(\varkappa,\mathfrak{t}) + \varsigma_{\iota,2}(\varkappa,\mathfrak{t})$$

for which the following estimates hold. For $(\varkappa,\mathfrak{t}) \in \overline{\Psi}^-$

$$
\begin{aligned}
|\frac{\partial\varsigma_{\iota,1}}{\partial\varkappa}(\varkappa,\mathfrak{t})| &\leq C\varepsilon_\iota^{-1}\Upsilon_{0,1}(\varkappa)\\
|\frac{\partial^2\varsigma_{\iota,2}}{\partial\varkappa^2}(\varkappa,\mathfrak{t})| &\leq C\varepsilon_\iota^{-1}\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa).
\end{aligned}
$$

For $(\varkappa,\mathfrak{t}) \in \overline{\Psi}^+$,

$$
\begin{aligned}
|\frac{\partial\varsigma_{\iota,1}}{\partial\varkappa}(\varkappa,\mathfrak{t})| &\leq C\varepsilon_\iota^{-1}\Upsilon_{1,1}(\varkappa)\\
|\frac{\partial^2\varsigma_{\iota,2}}{\partial\varkappa^2}(\varkappa,\mathfrak{t})| &\leq C\varepsilon_\iota^{-1}\varepsilon_2^{-1}\Upsilon_{1,2}(\varkappa).
\end{aligned}
$$

Proof. Since $\varepsilon_1 < \frac{\varepsilon_2}{2}$, define a function $\varsigma_{\iota,2}(\varkappa,\mathfrak{t})$, $\iota = 1,2$ for $(\varkappa,\mathfrak{t}) \in \overline{\Psi}^-$ as follows

$$
\begin{aligned}
\varsigma_{\iota,2}(\varkappa,\mathfrak{t}) &= \begin{cases} \varsigma_\iota(\varkappa,\mathfrak{t}) & \text{if } \varkappa \in [\varkappa_{\iota,\jmath}, 1]\\ \sum_{\ell=0}^{2}\frac{\partial^\ell\varsigma_\iota}{\partial\varkappa^\ell}(\varkappa_{\iota,\jmath},\mathfrak{t})\frac{(\varkappa-\varkappa_{\iota,\jmath})^\ell}{\ell!} & \text{if } \varkappa \in [0,\varkappa_{\iota,\jmath})\end{cases}\\
\varsigma_{\iota,1}(\varkappa,\mathfrak{t}) &= \varsigma_\iota(\varkappa,\mathfrak{t}) - \varsigma_{\iota,2}(\varkappa,\mathfrak{t}) \qquad \text{for } \varkappa \in [0,1].
\end{aligned}
$$

For $\varkappa \in [0,\varkappa_{\iota,\jmath})$ and by using Lemma 5.2, we get

$$
\begin{aligned}
|\varepsilon_\iota\frac{\partial^2\varsigma_{\iota,2}}{\partial\varkappa^2}(\varkappa,\mathfrak{t})| = \varepsilon_\iota|\frac{\partial^2\varsigma_\iota}{\partial\varkappa^2}(\varkappa_{\iota,\jmath},\mathfrak{t})| &\leq \varepsilon_\iota C\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{0,q}(\varkappa_{\iota,\jmath})}{\varepsilon_q},\\
&\leq C\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa_{\iota,\jmath})\\
&\leq C\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa).
\end{aligned}
$$

For $\varkappa \in [\varkappa_{\iota,\jmath}, 1]$ and by using Lemma 5.2, we get

$$
\begin{aligned}
|\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,2}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| = \varepsilon_\iota |\frac{\partial^2 \varsigma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| &\leq \varepsilon_\iota C \varepsilon_\iota^{-1} \sum_{q=1}^{2} \frac{\Upsilon_{0,q}(\varkappa_{\iota,\jmath})}{\varepsilon_q}, \\
&\leq C\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa).
\end{aligned}
$$

On the interval $[\varkappa_{\iota,\jmath}, 1]$,

$$\varsigma_{\iota,1} = \varsigma_\iota - \varsigma_{\iota,2} = 0$$

and hence

$$\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa} = \frac{\partial^2 \varsigma_{\iota,1}}{\partial \varkappa^2} = 0.$$

On the interval $[0, \varkappa_{\iota,\jmath})$ and using Lemma 5.2 it follows that

$$
\begin{aligned}
|\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,1}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| &= |\varepsilon_\iota \frac{\partial^2 \varsigma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| + |\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,2}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| \\
&\leq C \sum_{q=1}^{2} \frac{\Upsilon_{0,q}(\varkappa)}{\varepsilon_q} + C\varepsilon_2^{-1}\Upsilon_{0,2}(\varkappa) \leq C\frac{\Upsilon_{0,1}}{\varepsilon_1}(\varkappa).
\end{aligned}
$$

Integrating over $(\varkappa, \varkappa_{\iota,\jmath})$,

$$
\begin{aligned}
\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(\varkappa_{\iota,\jmath}) - \frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(\varkappa) &= \int_{\varkappa}^{\varkappa_{\iota,\jmath}} \frac{\partial^2 \varsigma_{\iota,1}(s)}{\partial \varkappa^2} ds \\
|\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(\varkappa)| &\leq \int_{\varkappa}^{\varkappa_{\iota,\jmath}} |\frac{\partial^2 \varsigma_{\iota,1}(s)}{\partial \varkappa^2}|\, ds \\
&\leq \int_{\varkappa}^{\varkappa_{\iota,\jmath}} \varepsilon_\iota^{-1} C \frac{\Upsilon_{0,1}(s)}{\varepsilon_1} ds \\
&\leq C\varepsilon_\iota^{-1}\varepsilon_1^{-1} \int_{\varkappa}^{\varkappa_{\iota,\jmath}} \Upsilon_{0,1}(s) ds \\
&\leq C\varepsilon_\iota^{-1}\varepsilon_1^{-1}[\frac{-\varepsilon_1}{\alpha}][\Upsilon_{0,1}(s)]_{\varkappa}^{\varkappa_{\iota,\jmath}} \\
&\leq C\varepsilon_\iota^{-1}[\Upsilon_{0,1}(\varkappa) - \Upsilon_{0,1}(\varkappa_{\iota,\jmath})] \\
&\leq C\varepsilon_\iota^{-1}\Upsilon_{0,1}(\varkappa).
\end{aligned}
$$

For the interval $(\varkappa_\jmath, \mathfrak{t}_\kappa) \in \overline{\Psi}^+$, define the functions $\varsigma_{\iota,2}(\varkappa, \mathfrak{t})$, $\iota = 1, 2$ as follows

$$
\begin{aligned}
\varsigma_{\iota,2}(\varkappa, \mathfrak{t}) &= \begin{cases} \varsigma_\iota(\varkappa, \mathfrak{t}) & \text{if } \varkappa \in [1 + \varkappa_{\iota,\jmath}, 2] \\ \sum_{\ell=0}^{2} \frac{\partial^\ell \varsigma_\iota}{\partial \varkappa^\ell}(1 + \varkappa_{\iota,\jmath}, \mathfrak{t}) \frac{(\varkappa - (1 + \varkappa_{\iota,\jmath}))^\ell}{\ell!} & \text{if } \varkappa \in [1, 1 + \varkappa_{\iota,\jmath}) \end{cases} \\
\varsigma_{\iota,1}(\varkappa, \mathfrak{t}) &= \varsigma_\iota(\varkappa, \mathfrak{t}) - \varsigma_{\iota,2}(\varkappa, \mathfrak{t}) \qquad \text{for } \varkappa \in [1, 2].
\end{aligned}
$$

For $\varkappa \in [1, 1 + \varkappa_{\iota,\jmath})$ and by using Lemma 5.2, we get

$$
\begin{aligned}
|\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,2}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| &= \varepsilon_\iota |\frac{\partial^2 \varsigma_\iota}{\partial \varkappa^2}(1 + \varkappa_{\iota,\jmath}, \mathfrak{t})| \\
&\leq \varepsilon_\iota C \varepsilon_\iota^{-1} \sum_{q=1}^{2} \frac{\Upsilon_{1,q}(1 + \varkappa_{\iota,\jmath})}{\varepsilon_q} \\
&\leq C\varepsilon_2^{-1}\Upsilon_{1,2}(1 + \varkappa_{\iota,\jmath}) \\
&\leq C\varepsilon_2^{-1}\Upsilon_{1,2}(\varkappa).
\end{aligned}
$$

For $\varkappa \in [1+\varkappa_{\iota,\jmath}, 2]$ and by using Lemma 5.2, we get

$$\begin{aligned}|\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,2}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| &= \varepsilon_\iota |\frac{\partial^2 \varsigma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| \\ &\leq \varepsilon_\iota C \varepsilon_\iota^{-1} \sum_{q=1}^{2} \frac{\Upsilon_{1,q}(\varkappa)}{\varepsilon_q} \\ &\leq C\varepsilon_2^{-1}\Upsilon_{1,2}(\varkappa).\end{aligned}$$

On the interval $[1+\varkappa_{\iota,\jmath}, 2]$,

$$\varsigma_{\iota,1} = \varsigma_\iota - \varsigma_{\iota,2} = 0$$

and hence

$$\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa} = \frac{\partial^2 \varsigma_{\iota,1}}{\partial \varkappa^2} = 0.$$

On the interval $[1, 1+\varkappa_{\iota,\jmath})$ and using Lemma 5.2, it follows that

$$\begin{aligned}|\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,1}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| &= |\varepsilon_\iota \frac{\partial^2 \varsigma_\iota}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| + |\varepsilon_\iota \frac{\partial^2 \varsigma_{\iota,2}}{\partial \varkappa^2}(\varkappa, \mathfrak{t})| \\ &\leq C \sum_{q=1}^{2} \frac{\Upsilon_{1,q}}{\varepsilon_q}(\varkappa) + C\varepsilon_2^{-1}\Upsilon_{1,2}(\varkappa) \\ &\leq C\frac{\Upsilon_{1,1}}{\varepsilon_1}(\varkappa).\end{aligned}$$

Integrating over $(\varkappa, 1+\varkappa_{\iota,\jmath})$

$$\begin{aligned}\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(1+\varkappa_{\iota,\jmath}) - \frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(\varkappa) &= \int_{\varkappa}^{1+\varkappa_{\iota,\jmath}} \frac{\partial^2 \varsigma_{\iota,1}(s)}{\partial \varkappa^2} ds \\ |\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(\varkappa)| &\leq \int_{\varkappa}^{1+\varkappa_{\iota,\jmath}} |\frac{\partial^2 \varsigma_{\iota,1}(s)}{\partial \varkappa^2}|\, ds \\ &\leq \int_{\varkappa}^{1+\varkappa_{\iota,\jmath}} \varepsilon_\iota^{-1} C \frac{\Upsilon_{1,1}(s)}{\varepsilon_1} ds \\ &\leq C\varepsilon_\iota^{-1}\varepsilon_1^{-1} \int_{\varkappa}^{1+\varkappa_{\iota,\jmath}} \Upsilon_{1,1}(s) ds \\ &\leq C\varepsilon_\iota^{-1}\varepsilon_1^{-1} [\frac{-\varepsilon_1}{\alpha}][\Upsilon_{1,1}(s)]_{\varkappa}^{1+\varkappa_{\iota,\jmath}} \\ &\leq C\varepsilon_\iota^{-1}[\Upsilon_{1,1}(\varkappa) - \Upsilon_{1,1}(\varkappa_{\iota,\jmath})] \\ &\leq C\varepsilon_\iota^{-1}\Upsilon_{1,1}(\varkappa).\end{aligned}$$

The proof of the lemma is complete.

Lemma 10.4. *Let* $\Re(\varkappa, \mathfrak{t})$ *and* $\Im(\varkappa, \mathfrak{t})$ *satisfy* (2.6) *and* (2.7). *Then for each* $\iota = 1, 2$ *and* $\jmath = 1, 2, \ldots, N$, $\kappa = 1, 2, \ldots, M$ *on each mesh* $M_{\vec{b}}$, *the following estimates*

$$|\varepsilon_\iota (D_\varkappa^- - \frac{\partial}{\partial \varkappa})\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| \leq C\, \Upsilon_{\wp,2}(\varkappa_{\jmath-1}) \quad \textit{where} \;\; \wp = 0, 1$$

and

$$|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| \leq C(M^{-1} + \Upsilon_{\wp,2}(\varkappa_{\jmath-1}))$$

hold.

Proof. From the standard estimates for first derivatives of Lemma 5.2, for all $\iota = 1,2$ and $\jmath = 1,2,\dots,N$, $\kappa = 1,2,\dots,M$ it's not hard to verify that

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\le C\varepsilon_\iota \max_{s\in[\varkappa_{\jmath-1},\,\varkappa_\jmath]} |\frac{\partial\varsigma_\iota}{\partial\varkappa}(s,\, \mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota \sum_{q=\iota}^{2} \frac{\Upsilon_{\wp,q}(\varkappa_{\jmath-1})}{\varepsilon_q} \\
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\le C\,\Upsilon_{\wp,2}(\varkappa_{\jmath-1}) \\
|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}} - D^-_\mathfrak{t})\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[\mathfrak{t}_{\kappa-1},\,\mathfrak{t}_\kappa]} |\frac{\partial^2\varsigma}{\partial\mathfrak{t}^2}(\varkappa_\jmath, s)| + C\varepsilon_\iota \max_{s\in[\varkappa_{\jmath-1},\,\varkappa_\jmath]} |\frac{\partial\varsigma}{\partial\varkappa}(s, \mathfrak{t}_\kappa)| \\
&\le CM^{-1} + C\varepsilon_\iota \sum_{q=\iota}^{2} \frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q} \\
|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\le C(M^{-1} + \Upsilon_{\wp,2}(\varkappa_{\jmath-1})).
\end{aligned}$$

The proof of the lemma is complete.

Theorem 10.5. *Let* $\Re(\varkappa, \mathfrak{t})$ *and* $\Im(\varkappa, \mathfrak{t})$ *satisfy* (2.6) *and* (2.7). *Then for each* $\iota = 1,2$ *and* $\jmath = 1,2,\dots,N$, $\kappa = 1,2,\dots,M$ *on each mesh* $\mathfrak{G}_{\vec{\mathfrak{b}}}$*, the following estimates*

$$|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| \le C(M^{-1} + N^{-1}\ln N)$$

hold.

Proof. The lemma is proved for four cases.
Case (i): On mesh $\mathfrak{G}_{\vec{\mathfrak{b}}}$ with $\vec{\mathfrak{b}} = (0,0)$.
Here the mesh is uniform and hence $\mathfrak{H}_\jmath = \varkappa_\jmath - \varkappa_{\jmath-1} = N^{-1}$.
Since $\eta_1 = \frac{1}{4}, \frac{\varepsilon_1}{\alpha}\ln N \ge \frac{1}{4}$ or $\varepsilon_1^{-1} \le C\ln N$, similarly $\varepsilon_2^{-1} \le C\ln N$.
From (9.6) and Lemma 5.2, it's not hard to verify that

$$\begin{aligned}
|(\vec{\xi}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(0, \mathfrak{t}_\kappa)| &\le \varepsilon_\iota|(\frac{\partial}{\partial\varkappa} - D^+_\varkappa)\varsigma_\iota(0,\, \mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota(\varkappa_1 - \varkappa_0) \max_{s\in[\varkappa_0,\varkappa_1]} |\frac{\partial^2\varsigma_\iota}{\partial\varkappa^2}(s, \mathfrak{t}_k)| \\
&\le C\varepsilon_\iota\eta_1 N^{-1} C\varepsilon_\iota^{-1} \sum_{q=1}^{2} \frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q} \\
&\le CN^{-1}\ln N. \\
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\le C\frac{\mathfrak{H}_\jmath}{\varepsilon_1}, \text{ using (10.1)} \\
&\le CN^{-1}\ln N. \\
|(\vec{L}^{M,N}(\vec{\zeta} - \vec{\varsigma}))_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}} - D^-_\mathfrak{t})\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath, \mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[\mathfrak{t}_{\kappa-1},\,\mathfrak{t}_\kappa]} |\frac{\partial^2\varsigma}{\partial\mathfrak{t}^2}(\varkappa_\jmath, s)| + CN^{-1}\ln N \\
&\le C(M^{-1} + N^{-1}\ln N).
\end{aligned}$$

Case (ii): On mesh $\mathfrak{G}_{\vec{\mathfrak{b}}}$ with $\vec{\mathfrak{b}} = (0, 1)$.
In this scenario, the mesh is piecewise uniform and the following conditions hold $\eta_1 = \frac{\eta_2}{2}$, $\eta_2 = \frac{\varepsilon_2}{\alpha}\ln N$. Hence $\frac{\eta_2}{2} < \frac{\varepsilon_1}{\alpha}\ln N$ or $\varepsilon_1 > \frac{\varepsilon_2}{2}$. Also $\eta_2 - \eta_1 = \eta_1$.
On the interval $(g, g+\eta_1]$, where $g = 0, 1$,

$$\begin{aligned}
|(\vec{\xi}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(0,\mathfrak{t}_\kappa)| &\le \varepsilon_\iota|(\frac{\partial}{\partial\varkappa} - D^+_\varkappa)\varsigma_\iota(0,\,\mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota(\varkappa_1-\varkappa_0)\max_{s\in[\varkappa_0,\varkappa_1]}|\frac{\partial^2\varsigma_\iota}{\partial\varkappa^2}(s,\mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota\eta_1 N^{-1}C\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q} \\
&\le CN^{-1}(\frac{\varepsilon_2}{2\alpha}\ln N)\varepsilon_1^{-1}\Upsilon_{\wp,2}(\varkappa) \\
&\le CN^{-1}\ln N. \\
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le C\frac{\mathfrak{H}_\jmath}{\varepsilon_1} \le C\frac{\eta_1}{\varepsilon_1}N^{-1}, \text{ using (10.1)} \\
&\le CN^{-1}\ln N.
\end{aligned}$$

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}} - D^-_{\mathfrak{t}})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|, \text{using (9.7)} \\
&\le C\mu_\kappa\max_{s\in[g,\,g+\eta_1]}|\frac{\partial^2\varsigma}{\partial t^2}(\varkappa_\jmath,s)| + CN^{-1}\ln N \\
&\le C(M^{-1}+N^{-1}\ln N).
\end{aligned}$$

On the interval $(g+\eta_1, g+\eta_2]$, where $g = 0, 1$,

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le C\frac{\mathfrak{H}_\jmath}{\varepsilon_1} \le C\frac{(\eta_2-\eta_1)}{\varepsilon_1}N^{-1} \le C\frac{\eta_1}{\varepsilon_1}N^{-1}, \text{ using (10.1)} \\
&\le CN^{-1}\ln N, \text{ as } \eta_2-\eta_1=\eta_1 \\
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}} - D^-_{\mathfrak{t}})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa\max_{s\in[g+\eta_1,\,g+\eta_2]}|\frac{\partial^2\varsigma}{\partial\mathfrak{t}^2}(\varkappa_\jmath,s)| + CN^{-1}\ln N \\
&\le C(M^{-1}+N^{-1}\ln N).
\end{aligned}$$

On the interval $(g+\eta_2, g+1]$, where $g = 0, 1$,

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D^-_\varkappa)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le C\varepsilon_\iota\max_{s\in[\varkappa_{\jmath-1},\,\varkappa_\jmath]}|\frac{\partial\varsigma_\iota}{\partial\varkappa}(s,\,\mathfrak{t}_\kappa)|, \text{ using (9.10)} \\
&\le C\varepsilon_\iota\sum_{q=\iota}^{2}\frac{\Upsilon_{\wp,q}(\varkappa_{\jmath-1})}{\varepsilon_q}, \text{ using Lemma 5.2} \\
&\le C\varepsilon_2\varepsilon_2^{-1}\Upsilon_{\wp,2}(\varkappa_{\jmath-1}) \text{ since } \varepsilon_1\le\varepsilon_2 \\
&\le CN^{-1}.
\end{aligned}$$

Since $\varkappa_\jmath > g+\eta_2,\ \varkappa_{\jmath-1} \geq g+\eta_2$ and hence $\Upsilon_{\wp,2}(\varkappa_{\jmath-1}) \leq N^{-1}$.

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\leq |(\frac{\partial}{\partial \mathfrak{t}} - D_{\mathfrak{t}}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\leq C\mu_\kappa \max_{s\in[g+\eta_2,\,g+1]} |\frac{\partial^2\varsigma}{\partial \mathfrak{t}^2}(\varkappa_\jmath,s)| + CN^{-1} \\
&\leq C(M^{-1}+N^{-1}).
\end{aligned}$$

Case (iii): On the mesh $\mathfrak{G}_{\vec{\mathfrak{b}}}$ with $\vec{\mathfrak{b}}=(1,0)$.
In this scenario, the mesh is piecewise uniform and following conditions hold
$\eta_2=\frac{1}{2}$, $\varepsilon_2^{-1}\leq C\ln N$ and as $\eta_1=\frac{\varepsilon_1}{\alpha}\ln N$, $\varepsilon_1<\frac{\varepsilon_2}{2}$.
On the intervals $(g,g+\eta_1]$, where $g=0,1$,

$$\begin{aligned}
|(\vec{\xi}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(0,\mathfrak{t}_\kappa)| &\leq \varepsilon_\iota|(\frac{\partial}{\partial\varkappa} - D_\varkappa^+)\varsigma_\iota(0,\ \mathfrak{t}_\kappa)| \\
&\leq C\varepsilon_\iota(\varkappa_1-\varkappa_0)\max_{s\in[\varkappa_0,\varkappa_1]}|\frac{\partial^2\varsigma_\iota}{\partial\varkappa^2}(s,\mathfrak{t}_\kappa)|,\ \text{using (9.11)} \\
&\leq C\varepsilon_\iota\eta_1 N^{-1}C\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q},\ \text{using Lemma 5.2} \\
&\leq CN^{-1}(\frac{\varepsilon_1}{\alpha}\ln N)\varepsilon_1^{-1}\Upsilon_{\wp,2}(\varkappa) \\
&\leq CN^{-1}\ln N. \\
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\leq C\frac{\mathfrak{H}_\jmath}{\varepsilon_1} \leq C\frac{\eta_1}{\varepsilon_1}N^{-1},\ \text{using (10.1)} \\
&\leq CN^{-1}\ln N \\
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\leq |(\frac{\partial}{\partial \mathfrak{t}} - D_{\mathfrak{t}}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\leq C\mu_\kappa\max_{s\in[g,\,g+\eta_1]}|\frac{\partial^2\varsigma}{\partial \mathfrak{t}^2}(\varkappa_\jmath,s)| + CN^{-1}\ln N \\
&\leq C(M^{-1}+N^{-1}\ln N).
\end{aligned}$$

On the interval $(g+\eta_1,g+\eta_2]$,where $g=0,1$,

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\leq |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)[\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa)+\varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)]| \\
&\leq |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial\varkappa} - D_\varkappa^-)\varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\leq C\varepsilon_\iota\max_{s\in[g+\eta_1,g+\eta_2]}|\frac{\partial\varsigma_{\iota,1}}{\partial\varkappa}(s,\ \mathfrak{t}_\kappa)| \\
&\quad + C\mathfrak{H}_\jmath\varepsilon_\iota\max_{s\in[g+\eta_1,g+\eta_2]}|\frac{\partial^2\varsigma_{\iota,2}}{\partial\varkappa^2}(s,\ \mathfrak{t}_\kappa)|,\ \text{using (9.9) and (9.10)} \\
&\leq CN^{-1}\ln N, \text{using Lemma 10.3.}
\end{aligned}$$

Since $\varkappa_\jmath > g+\eta_\iota,\ \ \varkappa_{\jmath-1}\geq g+\eta_\iota$ and hence for $\iota=1,2,\ \Upsilon_{\wp,\iota}(\varkappa_{\jmath-1})\leq N^{-1}$.

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial \mathfrak{t}} - D_{\mathfrak{t}}^{-})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[g+\eta_1,\, g+\eta_2]} |\frac{\partial^2\varsigma}{\partial \mathfrak{t}^2}(\varkappa_\jmath, s)| + CN^{-1}\ln N \\
&\le CM^{-1} + CN^{-1}\ln N \\
&\le C(M^{-1} + N^{-1}\ln N).
\end{aligned}$$

On the interval $(g+\eta_2, g+1]$, where $g = 0, 1$,

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})[\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa) + \varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)]| \\
&\le |\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})\varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota \max_{s\in[g+\eta_2, g+1]} |\frac{\partial \varsigma_{\iota,1}}{\partial \varkappa}(s,\ \mathfrak{t}_\kappa)| \\
&\quad + C\varepsilon_\iota \mathfrak{H}_\jmath \max_{s\in[g+\eta_2, g+1]} |\frac{\partial^2 \varsigma_{\iota,2}}{\partial \varkappa^2}(s,\ \mathfrak{t}_\kappa)|, \text{ using (9.9) and (9.10)} \\
&\le CN^{-1}\ln N, \text{using Lemma 10.3.}
\end{aligned}$$

Since $\varkappa_\jmath > g+\eta_\iota$, $\varkappa_{\jmath-1} \ge g+\eta_\iota$ and hence for $\iota = 1, 2$, $\Upsilon_{\wp,\iota}(\varkappa_{\jmath-1}) \le N^{-1}$.

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial \mathfrak{t}} - D_{\mathfrak{t}}^{-})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| + |\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[g+\eta_2, g+1]} |\frac{\partial^2\varsigma}{\partial \mathfrak{t}^2}(\varkappa_\jmath, s)| + CN^{-1}\ln N \\
&\le C(M^{-1} + N^{-1}\ln N).
\end{aligned}$$

Case (iv): On the mesh $\mathfrak{G}_{\vec{\mathfrak{b}}}$ with $\vec{\mathfrak{b}} = (1,1)$.
Here the mesh is piecewise uniform as $\eta_1 = \frac{\varepsilon_1}{\alpha}\ln N$, $\eta_2 = \frac{\varepsilon_2}{\alpha}\ln N$.
On the intervals $(g, g+\eta_1]$, where $g = 0, 1$,

$$\begin{aligned}
|(\vec{\xi}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(0,\mathfrak{t}_k)| &\le \varepsilon_\iota|(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{+})\varsigma_\iota(0,\ \mathfrak{t}_\kappa)|, \text{ using (9.6)} \\
&\le C\varepsilon_\iota(\varkappa_1 - \varkappa_0) \max_{s\in[\varkappa_0,\varkappa_1]} |\frac{\partial^2\varsigma_\iota}{\partial \varkappa^2}(s,\mathfrak{t}_\kappa)|, \text{ using (9.11)} \\
&\le C\varepsilon_\iota\eta_1 N^{-1} C\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q}, \text{ using Lemma 5.2} \\
&\le CN^{-1}(\frac{\varepsilon_1}{\alpha}\ln N)\varepsilon_1^{-1}\Upsilon_{\wp,2}(\varkappa) \\
&\le CN^{-1}\ln N.
\end{aligned}$$

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial \varkappa} - D_{\varkappa}^{-})\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le C\varepsilon_\iota\mathfrak{H}_\jmath \max_{s\in[g,g+\eta_1]} |\frac{\partial^2\varsigma}{\partial \varkappa^2}(s,\ \mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota\eta_1 N^{-1} C\varepsilon_\iota^{-1}\sum_{q=1}^{2}\frac{\Upsilon_{\wp,q}(\varkappa)}{\varepsilon_q}, \text{ using Lemma 5.2} \\
&\le C\varepsilon_\iota\eta_1 N^{-1}\varepsilon_\iota^{-1}\varepsilon_1^{-1}\Upsilon_{\wp,2}(\varkappa_{\jmath-1}) \\
&\le CN^{-1}\ln N.
\end{aligned}$$

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}}-D_{\mathfrak{t}}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|+|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[g,g+\eta_1]}|\frac{\partial^2\varsigma}{\partial\mathfrak{t}^2}(\varkappa_\jmath,s)|+CN^{-1}\ln N \\
&\le C(M^{-1}+N^{-1}\ln N).
\end{aligned}$$

On the interval $(g+\eta_1,g+\eta_2]$, where $g=0,1$,

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa)|+|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota \max_{s\in[g+\eta_1,g+\eta_2]}|\frac{\partial\varsigma_{\iota,1}}{\partial\varkappa}(s,\ \mathfrak{t}_\kappa)| \\
&+C\mathfrak{H}_\jmath\varepsilon_\iota \max_{s\in[g+\eta_1,g+\eta_2]}|\frac{\partial^2\varsigma_{\iota,2}}{\partial\varkappa^2}(s,\ \mathfrak{t}_\kappa)|,\ \text{using (9.9) and (9.10)} \\
&\le CN^{-1}\ln N, \text{using Lemma 10.3.}
\end{aligned}$$

Since $\varkappa_\jmath > g+\eta_\iota,\ \ \varkappa_{\jmath-1}\ge g+\eta_\iota$ and hence for $\iota=1,2,\ \Upsilon_{\wp,\iota}(\varkappa_{\jmath-1})\le N^{-1}$.

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}}-D_{\mathfrak{t}}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|+|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[g+\eta_1,g+\eta_2]}|\frac{\partial^2\varsigma}{\partial\mathfrak{t}^2}(\varkappa_\jmath,s)|+CN^{-1}\ln N \\
&\le C(M^{-1}+N^{-1}\ln N).
\end{aligned}$$

On the interval $(g+\eta_2,g+1]$, where $g=0,1$,

$$\begin{aligned}
|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)[\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa)+\varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)]| \\
&\le |\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_{\iota,1}(\varkappa_\jmath,\mathfrak{t}_\kappa)|+|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_{\iota,2}(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\varepsilon_\iota \max_{s\in[g+\eta_2,g+1]}|\frac{\partial\varsigma_{\iota,1}}{\partial\varkappa}(s,\ \mathfrak{t}_\kappa)| \\
&+C\varepsilon_\iota\mathfrak{H}_\jmath \max_{s\in[g+\eta_2,g+1]}|\frac{\partial^2\varsigma_{\iota,2}}{\partial\varkappa^2}(s,\ \mathfrak{t}_k)|,\ \text{using (9.9) and (9.10)} \\
&\le CN^{-1}\ln N, \text{using Lemma 10.3.}
\end{aligned}$$

Since $\varkappa_\jmath > g+\eta_\iota,\ \varkappa_{\jmath-1}\ge g+\eta_\iota$ and hence for $\iota=1,2,\ \Upsilon_{\wp,\iota}(\varkappa_{\jmath-1})\le N^{-1}$.

$$\begin{aligned}
|(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| &\le |(\frac{\partial}{\partial\mathfrak{t}}-D_{\mathfrak{t}}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|+|\varepsilon_\iota(\frac{\partial}{\partial\varkappa}-D_{\varkappa}^-)\varsigma_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)| \\
&\le C\mu_\kappa \max_{s\in[g+\eta_2,g+1]}|\frac{\partial^2\varsigma}{\partial\mathfrak{t}^2}(\varkappa_\jmath,s)|+CN^{-1}\ln N \\
&\le C(M^{-1}+N^{-1}\ln N).
\end{aligned}$$

The proof of the theorem is complete..

Theorem 10.6. *Let $\vec{u}$ represents the solution of the continuous problem* (2.1),(2.2) *and* (2.3)*, while $\vec{U}$ represents the solution of the discrete problem* (7.1)*,* (7.2)*. Then,*

$$||\vec{U}(\varkappa_\jmath,\mathfrak{t}_\kappa)-\vec{u}(\varkappa_\jmath,\mathfrak{t}_\kappa)||\le CN^{-1}\ln N.$$

Proof. Based on Lemma 7.2, it is evident that to prove the above theorem, it suffices to prove that $||(\vec{L}^{M,N}(\vec{U}-\vec{u}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|| \leq CN^{-1}\ln N$. But $||(\vec{L}^{M,N}(\vec{U}-\vec{u}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|| \leq ||(\vec{L}^{M,N}(\vec{\Gamma}-\vec{\gamma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)|| + ||(\vec{L}^{M,N}(\vec{\zeta}-\vec{\varsigma}))_\iota(\varkappa_\jmath,\mathfrak{t}_\kappa)||$. Hence using theorem 10.1 and theorem 10.5 the aforementioned result is derived.

11. NUMERICAL ILLUSTRATION

An example presented in this section exemplifies the numerical scheme proposed in this paper.

Example 11.1. *Let us consider the problem*

$$\left.\begin{aligned}\frac{\partial u_1}{\partial \mathfrak{t}}(\varkappa,\mathfrak{t}) + \varepsilon_1\frac{\partial u_1}{\partial \varkappa}(\varkappa,\mathfrak{t}) + (5+\varkappa)\,u_1(\varkappa,\mathfrak{t}) - u_2(\varkappa,\mathfrak{t}) - u_1(\varkappa-1,\mathfrak{t}) = 3,\\ \frac{\partial u_2}{\partial \mathfrak{t}}(\varkappa,\mathfrak{t}) + \varepsilon_2\frac{\partial u_2}{\partial \varkappa}(\varkappa,\mathfrak{t}) - u_1(\varkappa,\mathfrak{t}) + 5\,u_2(\varkappa,\mathfrak{t}) - 1.5\,u_2(\varkappa-1,\mathfrak{t}) = 2\end{aligned}\right\} \forall\ (\varkappa,\mathfrak{t}) \in \Psi$$

with

$$u_1(0,\mathfrak{t}) - \varepsilon_1\frac{\partial u_1}{\partial \varkappa}(0,\mathfrak{t}) = 1, u_1(\varkappa,0) = 1$$
$$u_2(0,\mathfrak{t}) - \varepsilon_2\frac{\partial u_2}{\partial \varkappa}(0,\mathfrak{t}) = 1, u_2(\varkappa,0) = 1.$$

In Table 11.1, the order of convergence and the error constant for example 11.1 are shown, calculated using a variant of Two mesh algorithm [12]. In this table

$\mathfrak{Y}^N_\varepsilon$: maximum point-wise two-mesh differences,

$\mathfrak{Y}^N$: ε - uniform maximum point-wise two-mesh differences,

$\mathfrak{p}^N$: ε - uniform order of local convergence,

$\mathfrak{p}^*$: ε - uniform order of convergence and

$\mathfrak{C}^N_{\mathfrak{p}^*}$: ε - uniform error constant.

Table 11.1: Values of $\mathfrak{Y}^N_\varepsilon$, $\mathfrak{Y}^N$, $\mathfrak{p}^N$, $\mathfrak{p}^*$ *and* $\mathfrak{C}^N_{\mathfrak{p}^*}$ *generated for* $\varepsilon_1 = \frac{\chi}{16}, \varepsilon_2 = \frac{\chi}{8}$.

χ	N: Number of mesh points			
	64	128	256	512
0.250E+00	0.143E+00	0.112E+00	0.793E-01	0.515E-01
0.625E-01	0.143E+00	0.112E+00	0.793E-01	0.514E-01
0.156E-01	0.143E+00	0.112E+00	0.793E-01	0.514E-01
0.391E-02	0.143E+00	0.112E+00	0.793E-01	0.514E-01
0.977E-03	0.143E+00	0.112E+00	0.793E-01	0.514E-01
$\mathfrak{Y}^N$	0.143E+00	0.112E+00	0.793E-01	0.515E-01
$\mathfrak{p}^N$	0.348E+00	0.500E+00	0.624E+00	
$\mathfrak{C}^N_{\mathfrak{p}}$	0.283E+01	0.283E+01	0.255E+01	0.211E+01
$\mathfrak{p}^* = 0.3482349E+00$				
$\mathfrak{C}^N_{\mathfrak{p}^*} = 0.2833228E+01$				

12. CONCLUSION

First order convergence was proved in this paper using a classical layer resolving scheme, with the construction of a Shishkin Mesh for spatial discretization and a uniform mesh for time discretization, applied to a system of two singularly perturbed time-dependent delay differential equations with robin initial condition. Solutions to the problem was further classified as singular components and smooth components for in-depth analysis of the layer behavior of the solutions. Layer functions was constructed. Spacial discretization is carried with Shishkin mesh, since it allows us to capture layer pattern induced by the solutions. Our method is effective since it provides better order of convergence and also it capture the solution's layer behavior due the presence of perturbation parameters and delay terms. To bolster our scheme, a numerical illustration was carried out for an example problem, and the resulting table shows the order of convergence and error constant, demonstrating the effectiveness of our scheme. The future work will focus on solving semi-linear problems and also the problems with discontinuous source terms.

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