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SOLVING FIXED POINT PROBLEMS AND VARIATIONAL INCLUSIONS USING VISCOSITY APPROXIMATIONS

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ABSTRACT. This paper proposes a new algorithm to find a common element of the fixed point set of a finite family of demimetric mappings and the set of solutions of a general split variational inclusion problem in Hilbert spaces. The algorithm is based on the viscosity approximation method, which is a powerful tool for solving fixed point problems and variational inclusion problems. Under some conditions, we prove that the sequence generated by the algorithm converges strongly to this common solution.

Key words and phrases: Split variational inclusion problem; Variational inequality; Demimetric mapping.

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1. INTRODUCTION

In 1976, Rockafellar [16] studied the inclusion problem of finding

$$\eta^\dagger \in \Upsilon^{-1}(0), \tag{1.1}$$

where Υ is a maximal monotone set-valued mapping, the author has devised a method called the proximal point method to tackle the inclusion problem (1.1) in a Hilbert space Σ. Over the years, due to its practical uses in various fields such as science, engineering, management, and social sciences, the inclusion problem has been extended and generalized in many ways, as seen in references [3, 4, 15, 17, 11, 2, 1, 18, 12, 14, 13]. Suppose Σ_1 and Σ_2 be two real Hilbert spaces, and let $\chi_1 : \Sigma_1 \to 2^{\Sigma_1}$ and $\chi_2 : \Sigma_2 \to 2^{\Sigma_2}$ be maximal monotone mappings. Then the split variational inclusion problem is to find a point $\mu \in \Sigma_1$ in such a way that

$$0 \in \chi_1(\mu) \text{ and } 0 \in \chi_2 \Upsilon(\mu),$$

where $\Upsilon : \Sigma_1 \to \Sigma_2$ is a bounded linear mapping.

There are a number of iterative algorithms available to for finding the solution of split variation inclusion problem. In 2012, Byrne *et. al* introduced a one step algorithm as

$$\zeta_{n+1} = J_\lambda^{\chi_1}\left[\zeta_n + \varepsilon \Upsilon^* \left(J_\lambda^{\chi_2} - I\right) \Upsilon(\zeta_n)\right],$$

where $\varepsilon \in \left(0, \frac{2}{\|\Upsilon^*\Upsilon\|}\right)$ and proves some convergence results to solve split variational inclusion problem. In the context of a Hilbert space, Kazmi and Rizvi [5] presented a two-step algorithm aimed at discovering a shared solution for both the split variational inclusion problem and the fixed point of nonexpansive mappings, as follows:

$$\begin{cases} \xi_n = J_\lambda^{\chi_1}\left[\zeta_n + \epsilon \Upsilon^* \left(J_\lambda^{\chi_2} - I\right) \Upsilon(\zeta_n)\right], \\ \zeta_{n+1} = \delta_n \Gamma(\zeta_n) + (1-\delta_n)\Lambda(\xi_n). \end{cases}$$

Where $\Upsilon : \Sigma_1 \to \Sigma_2$ is a bounded linear mapping, $\Lambda : \Sigma_1 \to \Sigma_1$ a nonexpansive mapping and $\Gamma : \Sigma_1 \to \Sigma_1$ is a contraction mapping. In 2023, Pan and Wang [10] introduced general split variational inclusion problem for finding a point $\zeta \in \Sigma_1$ in such a way that

$$\zeta \in \bigcap_{i=1}^{N} \chi_i^{-1}(0), \text{ and } \Upsilon(\zeta) \in \bigcap_{i=1}^{N} \Xi_i^{-1}(0),$$

where $\chi_i : \Sigma_1 \to \Sigma_1$, $\Xi_i : \Sigma_2 \to \Sigma_2$, $i = 1, 2, \dots N$ are two families of maximal monotone mappings. They introduced an inertial viscosity iterative approach for addressing the general split variational inclusion problem and the fixed point problem associated with nonexpansive mappings. More recently, in the year 2023, Mehra *et.al* [9] introduced a new mapping termed the ξ-demimetric mapping, defined as follows:

Definition 1.1. A mapping $\Lambda : \Sigma_1 \to \Sigma_1$ is said to be ξ-demimetric with respect to M-norm, where $\xi \in (-\infty, 1)$ if $F(\Lambda) \neq \emptyset$ such that

$$\langle \zeta - \zeta^\dagger, (I-\Lambda)\zeta \rangle_M \geq \frac{1}{2}(1-\xi)\|(I-\Lambda)\zeta\|_M^2, \forall \zeta \in \Sigma_1, \zeta^\dagger \in F(\Lambda).$$

Building upon the research conducted in [5, 9, 10], we introduce an algorithm and establish the strong convergence of the generated sequence to a common solution for the general split variational inclusion problem and the set of fixed points associated with a finite family of ξ-demimetric mappings.

2. PRELIMINARIES

Let $\mathcal{E}$ be a nonempty closed convex subset of a real Hilbert space $(\Sigma, \langle ., . \rangle)$ and $\Lambda : \mathcal{E} \to \mathcal{E}$ a mapping. A point $\zeta^\dagger \in \mathcal{E}$ is said to be a fixed point of Λ if $\Lambda(\zeta^\dagger) = \zeta^\dagger$. The set of all fixed points of Λ will be denoted by $F(\Lambda)$.

Lemma 2.1. [8] *Let $\chi : \Sigma_1 \to 2^{\Sigma_1}$ be a set valued maximal monotone mapping and $\beta > 0$, the following hold:*

(i) *for each $\beta > 0$, the resolvent mapping J_β^χ is a single valued and firmly nonexpansive mapping;*
(ii) $D(J_\beta^\chi) = \Sigma_1$, $F(J_\beta^\chi) = \chi^{-1}(0) = \{\zeta \in D(\chi), 0 \in \chi(\zeta)\}$*;*
(iii) $(I - J_\beta^\chi)$ *is a firmly nonexpansive mapping;*
(iv) *suppose that $\chi^{-1}(0) \neq \emptyset$, then for all $\zeta \in \Sigma_1, \eta \in \chi^{-1}(0)$ and $\left\| J_\beta^\chi(\zeta) - \eta \right\|^2 \leq \|\zeta - \eta\|^2 - \left\| J_\beta^\chi(\zeta) - \zeta \right\|^2$;*
(v) *suppose that $\chi^{-1}(0) \neq \emptyset$, then for all $\zeta \in \Sigma_1, \eta \in \chi^{-1}(0)$ and $\left\langle \zeta - J_\beta^\chi(\zeta), J_\beta^\chi(\zeta) - \eta \right\rangle \geq 0$.*

Lemma 2.2. [10] *Assume that Σ_1 and Σ_2 are two Hilbert spaces. Let $\Upsilon : \Sigma_1 \to \Sigma_2$ be a linear and bounded mapping with its adjoint Υ^*. let $\chi_i : \Sigma_1 \to \Sigma_1$, $\Xi_i : \Sigma_2 \to \Sigma_2$, $i = 1, 2, \ldots N$ are two families of maximal monotone mappings. Let $J_{\beta_i}^{\chi_i}$ and $J_{\beta_i}^{\Xi_i}$ be the resolvent mappings of χ_i and Ξ_i, respectively. Suppose that $\Theta \neq \emptyset$ and $\beta_i > 0, \lambda_i > 0$. Then for any $\eta \in \Sigma_1$, η is a solution of general split variational inclusion problem if and only if $J_{\beta_i}^{\chi_i}\left[\eta - \lambda_i \Upsilon^* \left(I - J_{\beta_i}^{\Xi_i}\right)\Upsilon(\eta)\right] = \eta$.*

Lemma 2.3. [9] *Let $\Lambda : \Sigma_1 \to \Sigma_1$ is ξ-demimetric mapping with respect to M-norm, where $\xi \in (-\infty, 1)$ and $F(\Lambda) \neq \emptyset$. Let $P = (1-\gamma)I + \gamma\Lambda$, where $\gamma \in (-\infty, \infty)$ with $\gamma \in (0, 1-\xi]$, then $P : \Sigma_1 \to \Sigma_1$ is a quasi nonexpansive mapping.*

Lemma 2.4. [20]. *Let $\{\Pi_n\}$ be a sequence of nonnegative real numbers such that*

$$\Pi_{n+1} \leq (1 - \delta_n)\Pi_n + \delta_n \Psi_n$$

where $\{\delta_n\} \subseteq (0, 1)$, $\{\Psi_n\}$ is sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(b) $\limsup_{n\to\infty} \Psi_n \leq 0$.

Then $\Pi_n \to 0$.

Lemma 2.5. [7]. *Suppose $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}$ defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$ and, for all $n \geq n_0$, it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and we have $\Gamma_n \leq \Gamma_{\tau(n)+1}$.

Lemma 2.6. [19] *Let $\{\zeta_n\}$ and $\{\eta_n\}$ be bounded sequences in a Banach space E such that*

$$\zeta_{n+1} = (1 - \beta_n)\eta_n + \beta_n\zeta_n \ \ \forall n \geq 1,$$

where $\{\beta_n\}$ is a real sequence in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. If $\limsup_{n\to\infty}(\|\eta_{n+1} - \eta_n\| - \|\zeta_{n+1} - \zeta_n\|) \leq 0$, then $\lim_{n\to\infty} \|\eta_n - \zeta_n\| = 0$.

Lemma 2.7. [6] *Suppose that the* $\{\tau_n\}$ *is a sequence of nonnegative real numbers satisfying* $\tau_{n_i} < \tau_{n_i+1}\ \forall\, i \in \mathbb{N}$, *where* $\{n_j\}$ *is a subsequence of* $\{n\}$. *Then,* $\exists$ *a nondecreasing sequence* $\{l_j\} \subset \mathbb{N}$ *in such a way that* $l_j \to \infty$, $j \in \mathbb{N}$, *we have*

$$\tau_{l_j} < \tau_{l_j+1} \text{ and } \tau_j < \tau_{l_j+1}.$$

In fact, $l_j = \max\{k \leq j : \tau_k < \tau_{k+1}\}$.

Lemma 2.8. [10] *Assume that* Σ_1 *and* Σ_2 *are two real Hilbert spaces. Suppose* $\Upsilon : \Sigma_1 \to \Sigma_2$ *be a linear and bounded operator with its adjoint* Υ^*. *Let* $\chi_i : \Sigma_1 \to \Sigma_1$, $\Xi_i : \Sigma_2 \to \Sigma_2$, $i = 1, 2, \ldots, N$ *are two families of maximal monotone mappings. Suppose* $J_{\beta_i}^{\chi_i}$ *and* $J_{\beta_i}^{\Xi_i}$ *be the resolvent mapping of* χ_i *and* Ξ_i, *respectively. Suppose that the solution set* Θ *is nonempty and* $\beta_i > 0, \lambda_i > 0$. *Then for any* $\zeta^\dagger \in \Sigma_1$, $\zeta^\dagger$ *is a solution of general split variational inclusion problem if and only if* $J_{\beta_i}^{\chi_i}\left[\zeta^\dagger - \lambda_i \Upsilon^*\left(I - J_{\beta_i}^{\Xi_i}\right)\Upsilon(\zeta^\dagger)\right] = \zeta^\dagger$.

3. MAIN RESULT

Now, we consider the following general split variational inclusion problem of finding a point $\zeta \in \Sigma_1$ such that

$$\zeta \in \bigcap_{i=1}^{N} \chi_i^{-1}(0), \text{ and } \Upsilon(\zeta) \in \bigcap_{i=1}^{N} \Xi_i^{-1}(0),$$

where $\chi_i : \Sigma_1 \to \Sigma_1$ and $\Xi_i : \Sigma_2 \to \Sigma_2$, $i = 1, 2, \ldots N$ are two families of maximal monotone mappings. We denote the solution set of general split variational inclusion problem by Θ.

Theorem 3.1. *Let* Σ_1, Σ_2 *be Hilbert spaces and* $\Upsilon : \Sigma_1 \to \Sigma_1$ *a bounded linear mapping with its adjoint* Υ^*. *Suppose* $\chi_i : \Sigma_1 \to \Sigma_1$ *and* $\Xi_i : \Sigma_2 \to \Sigma_2$, $i = 1, 2, \ldots N$ *are two families of maximal monotone mappings. Let* $J_{\beta_i}^{\chi_i}$ *and* $J_{\beta_i}^{\Xi_i}$ *be the resolvent mappings of* χ_i *and* Ξ_i, *respectively. Let* $F : \Sigma_1 \to \Sigma_1$ *be a contraction mapping with coefficient* $\rho \in (0,1)$ *and* $\Lambda_i : \Sigma_1 \to \Sigma_1$ *a finite family of* ξ*-demimetric mappings with* $\xi \in (-\infty, 1)$ *such that* $I - \Lambda_i$ *is demiclosed at origin for all* $i = 1, 2, \ldots N$ *and* $\bigcap_{i=1}^{n} F(\Lambda_i) \bigcap \Theta \neq \emptyset$. *Suppose* $\{\alpha_n\}$, $\{\delta_n\}$, $\{\gamma_{i,n}\} \subset (0,1)$ *and* $\{\beta_{i,n}\}$, $\{\lambda_{i,n}\}$ *are sequences of positive real numbers. For any given* $\zeta_0, \zeta_1 \in \Sigma_1$ *we define sequence as follows*

$$\begin{cases} \eta_n = \zeta_n + \theta_n(\zeta_n - \zeta_{n-1}), \\ \varpi_n = \sum\limits_{i=1}^{n} \gamma_{i,n} J_{\beta_{i,n}}^{\chi_i}\left[\eta_n - \lambda_{i,n}\Upsilon^*\left(I - J_{\beta_{i,n}}^{\Xi_i}\right)\Upsilon(\eta_n)\right], \\ \zeta_{n+1} = \alpha_n F(\zeta_n) + (1-\alpha_n)[\delta_n(\varpi_n) + (1-\delta_n)\Lambda_n(\varpi_n)]. \end{cases} \tag{3.1}$$

where, $\Lambda_n = \frac{1}{N}\sum\limits_{i=1}^{N}(1-q_n)I + q_n\Lambda_i$. *If the sequence defined by* (3.1) *satisfying the following conditions:*

(i) *Let the parameter* θ_n *chosen as*

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|\zeta_n - \zeta_{n-1}\|}\right\} \text{ if } \zeta_n \neq \zeta_{n-1}, \\ \theta \text{ otherwise }, \end{cases}$$

where $\theta > 0$, ε_n *is a positive real sequence such that* $\varepsilon_n = o(\alpha_n)$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\delta_n \subset [a,b] \subset (0,1)$;

(iii) $\sum_{n=0}^{\infty} \gamma_{i,n} = 1, \gamma_{i,n} \subset [c,d] \subset (0,1), \lambda_{i,n} \in \left(0, \frac{2}{\|\Upsilon\|^2}\right)$,

then the sequence $\{\zeta_n\}$ *converges strongly to an element* $\zeta^\dagger \in \bigcap_{i=1}^{n} F(\Lambda_i) \bigcap \Theta$*, where* $\zeta^\dagger = P_{\bigcap_{i=1}^{n} F(\Lambda_i) \bigcap \Theta} F(\zeta^\dagger)$.

Proof. Let $\zeta^\dagger \in \bigcap_{i=1}^{n} F(\Lambda_i) \bigcap \Theta$, then we have $\zeta^\dagger = J^{\chi_i}_{\beta_{i,n}}(\zeta^\dagger)$, $\Upsilon(\zeta^\dagger) = J^{\Xi_i}_{\beta_{i,n}} \Upsilon(\zeta^\dagger)$ and $\Lambda_i(\zeta^\dagger) = \zeta^\dagger$. Now

$$\begin{aligned} \|\varpi_n - \zeta^\dagger\| &= \left\| \sum_{i=1}^{N} \gamma_{i,n} J^{\chi_i}_{\beta_{i,n}} \left[\eta_n - \lambda_{i,n} \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon(\eta_n) \right] - \zeta^\dagger \right\|^2 \\ &\leq \sum_{i=1}^{N} \gamma_{i,n} \left\| J^{\chi_i}_{\beta_{i,n}} \left[\eta_n - \lambda_{i,n} \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon(\eta_n) \right] - \zeta^\dagger \right\|^2 . \end{aligned} \tag{3.2}$$

From Lemma 2.1 we can say that $J^{\chi_i}_{\beta_{i,n}} \left[I - \lambda_{i,n} \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon \right]$ is nonexpansive, and hence

$$\begin{aligned} \left\| J^{\chi_i}_{\beta_{i,n}} \left[\eta_n - \lambda_{i,n} \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon(\eta_n) \right] - \zeta^\dagger \right\|^2 &\leq \left\| \eta_n - \lambda_{i,n} \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon(\eta_n) - \zeta^\dagger \right\|^2 \\ &= \|\eta_n - \zeta^\dagger\|^2 + \lambda_{i,n}^2 \left\| \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon(\eta_n) \right\|^2 \\ &+ 2\lambda_{i,n} \left\langle \eta_n - \zeta^\dagger, \Upsilon^* \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\rangle \\ &= \|\eta_n - \zeta^\dagger\|^2 + \lambda_{i,n}^2 \left\langle \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n), \Upsilon\Upsilon^* \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\rangle \\ &+ 2\lambda_{i,n} \left\langle \Upsilon(\eta_n - \zeta^\dagger), \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\rangle \end{aligned}$$

$$\begin{aligned} &\left\| J^{\chi_i}_{\beta_{i,n}} \left[\eta_n - \lambda_{i,n} \Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}} \right) \Upsilon(\eta_n) \right] - \zeta^\dagger \right\|^2 \\ &\leq \|\eta_n - \zeta^\dagger\|^2 + \lambda_{i,n}^2 \|\Upsilon\|^2 \left\| \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\|^2 \\ &+ 2\lambda_{i,n} \left\langle \Upsilon(\eta_n - \zeta^\dagger) + \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) - \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n), \right. \\ &\left. \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\rangle \\ &= \|\eta_n - \zeta^\dagger\|^2 + \lambda_{i,n}^2 \|\Upsilon\|^2 \left\| \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\|^2 - 2\lambda_{i,n} \left\| \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\|^2 \\ &+ 2\lambda_{i,n} \left\langle J^{\Xi_i}_{\beta_{i,n}} \Upsilon(\eta_n) - \Upsilon(\zeta^\dagger), \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\rangle \\ &\leq \|\eta_n - \zeta^\dagger\|^2 - \left(2\lambda_{i,n} - \lambda_{i,n}^2 \|\Upsilon\|^2 \right) \left\| \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\|^2 \\ &= \|\eta_n - \zeta^\dagger\|^2 + \lambda_{i,n} \left(\lambda_{i,n} \|\Upsilon\|^2 - 2 \right) \left\| \left(J^{\Xi_i}_{\beta_{i,n}} - I \right) \Upsilon(\eta_n) \right\|^2 . \end{aligned}$$

Now

$$\|\varpi_n - \zeta^\dagger\|^2 \leq \sum_{i=1}^{N} \gamma_{i,n} \left\| J^{\chi_i}_{\beta_{i,n}} \left[\eta_n - \lambda_{i,n}\Upsilon^* \left(I - J^{\Xi_i}_{\beta_{i,n}}\right)\Upsilon(\eta_n)\right] - \zeta^\dagger \right\|^2$$

$$\leq \|\eta_n - \zeta^\dagger\|^2 + \sum_{i=1}^{N} \gamma_{i,n}\lambda_{i,n}\left(\lambda_{i,n}\|\Upsilon\|^2 - 2\right)\left\|\left(J^{\Xi_i}_{\beta_{i,n}} - I\right)\Upsilon(\eta_n)\right\|^2. \tag{3.3}$$

$$\begin{aligned}\|\eta_n - \zeta^\dagger\| &= \|\zeta_n + \theta_n(\zeta_n - \zeta_{n-1}) - \zeta^\dagger\| \\ &\leq \|\zeta_n - \zeta^\dagger\| + \theta_n\|\zeta_n - \zeta_{n-1}\| \\ &= \|\zeta_n - \zeta^\dagger\| + \alpha_n\frac{\theta_n}{\alpha_n}\|\zeta_n - \zeta_{n-1}\| \\ &\leq \|\zeta_n - \zeta^\dagger\| + \alpha_n C,\end{aligned}$$

where C is a constant and $C > 0$. Now define $\vartheta_n = \delta_n\varpi_n + (1-\delta_n)\Lambda_n(\varpi_n)$, then we have

$$\begin{aligned}\|\vartheta_n - \zeta^\dagger\| &\leq \delta_n\|\varpi_n - \zeta^\dagger\| + (1-\delta_n)\|\Lambda_n(\varpi_n) - \zeta^\dagger\| \\ &\leq \delta_n\|\varpi_n - \zeta^\dagger\| + (1-\delta_n)\|\varpi_n - \zeta^\dagger\| = \|\varpi_n - \zeta^\dagger\|.\end{aligned} \tag{3.4}$$

Now

$$\begin{aligned}\|\zeta_{n+1} - \zeta^\dagger\| &= \|\alpha_n F(\zeta_n) + (1-\alpha_n)\vartheta_n - \zeta^\dagger\| \\ &\leq \alpha_n\|F(\zeta_n) - \zeta^\dagger\| + (1-\alpha_n)\|\vartheta_n - \zeta^\dagger\| \\ &\leq \alpha_n\rho\|\zeta_n - \zeta^\dagger\| + (1-\alpha_n)\|\varpi_n - \zeta^\dagger\| \\ &= \alpha_n\rho\|\zeta_n - \zeta^\dagger\| + (1-\alpha_n)\|\eta_n - \zeta^\dagger\| \\ &\leq \alpha_n\rho\|\zeta_n - \zeta^\dagger\| + (1-\alpha_n)\{\|\zeta_n - \zeta^\dagger\| + \alpha_n C\} \\ &\leq [1 - \alpha_n(1-\rho)]\|\zeta_n - \zeta^\dagger\| + \alpha_n C \\ &\leq \max\left\{\|\zeta_n - \zeta^\dagger\|, \frac{C}{1-\rho}\right\} \leq \cdots \leq \max\left\{\|\zeta_0 - \zeta^\dagger\|, \frac{C}{1-\rho}\right\}.\end{aligned}$$

It implies the sequence $\{\zeta_n\}$ is bounded and hence the sequences $\{\Lambda_n(\varpi_n)\}$, $\{\vartheta_n\}$, $\{\eta_n\}$, $\{\varpi_n\}$ are bounded as well. Since the sequence $\{\zeta_n\}$ is bounded and $\|\eta_n - \zeta^\dagger\| \leq \|\zeta_n - \zeta^\dagger\| + \alpha_n C$, there exists a constant C_1, such that

$$\|\eta_n - \zeta^\dagger\|^2 \leq \|\zeta_n - \zeta^\dagger\|^2 + \alpha_n C_1. \tag{3.5}$$

Now, using (3.3), we get

$$\|\varpi_n - \zeta^\dagger\|^2 \leq \|\eta_n - \zeta^\dagger\|^2 + \sum_{i=1}^{N} \gamma_{i,n}\lambda_{i,n}(\lambda_{i,n}\|\Upsilon\|^2 - 2)\left\|\left(J^{\Xi_i}_{\beta_{i,n}} - I\right)\Upsilon(\eta_n)\right\|^2$$

$$\leq \|\zeta_n - \zeta^\dagger\|^2 + \sum_{i=1}^{N} \gamma_{i,n}\lambda_{i,n}(\lambda_{i,n}\|\Upsilon\|^2 - 2)\left\|\left(J^{\Xi_i}_{\beta_{i,n}} - I\right)\Upsilon(\eta_n)\right\|^2 + \alpha_n C_1. \tag{3.6}$$

Now

$$
\begin{aligned}
&\left\|J^{\chi_i}_{\beta_{i,n}}\left[\eta_n-\lambda_{i,n}\Upsilon^*\left(I-J^{\Xi_i}_{\beta_{i,n}}\right)\Upsilon(\eta_n)\right]-\zeta^\dagger\right\|^2\\
&\le\left\|\eta_n-\lambda_{i,n}\Upsilon^*\left(I-J^{\Xi_i}_{\beta_{i,n}}\right)\Upsilon(\eta_n)-\zeta^\dagger\right\|^2\\
&\le\left\langle\varpi_n-\zeta^\dagger,\eta_n+\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)-\zeta^\dagger\right\rangle\\
&=\frac{1}{2}\left\|\eta_n+\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)-\zeta^\dagger\right\|^2+\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2\\
&-\frac{1}{2}\left\|\varpi_n-\zeta^\dagger-\eta_n-\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)+\zeta^\dagger\right\|^2\\
&=\frac{1}{2}\left\|\eta_n-\zeta^\dagger+\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|^2+\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2\\
&-\frac{1}{2}\left\|\varpi_n-\eta_n-\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|^2\\
&=\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2+\frac{1}{2}\|\eta_n-\zeta^\dagger\|^2+\frac{1}{2}\lambda_{i,n}^2\left\|\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|^2\\
&+\left\langle\eta_n-\zeta^\dagger,\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\rangle-\frac{1}{2}\|\varpi_n-\eta_n\|^2\\
&-\frac{1}{2}\lambda_{i,n}^2\left\|\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|^2+\left\langle\varpi_n-\eta_n,\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\rangle\\
&=\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2+\frac{1}{2}\|\eta_n-\zeta^\dagger\|^2-\frac{1}{2}\|\varpi_n-\eta_n\|^2+\left\langle\varpi_n-\zeta^\dagger,\lambda_{i,n}\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\rangle\\
&\le\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2+\frac{1}{2}\|\eta_n-\zeta^\dagger\|^2-\frac{1}{2}\|\varpi_n-\eta_n\|^2+\lambda_{i,n}\|\varpi_n-\zeta^\dagger\|\left\|\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|.
\end{aligned}
$$

Now we get

$$
\begin{aligned}
\|\varpi_n-\zeta^\dagger\|^2&\le\sum_{i=1}^N\gamma_{i,n}\left\|J^{\chi_i}_{\beta_{i,n}}\left[\eta_n-\lambda_{i,n}\Upsilon^*\left(I-J^{\Xi_i}_{\beta_{i,n}}\right)\Upsilon(\eta_n)\right]-\zeta^\dagger\right\|^2\\
&\le\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2+\frac{1}{2}\|\eta_n-\zeta^\dagger\|^2-\frac{1}{2}\|\varpi_n-\eta_n\|^2\\
&+\sum_{i=1}^N\gamma_{i,n}\lambda_{i,n}\|\varpi_n-\zeta^\dagger\|\left\|\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|.
\end{aligned}
$$

Simplifying above equation and applying (3.5) we get

$$
\begin{aligned}
\|\varpi_n-\zeta^\dagger\|^2&\le\|\zeta_n-\zeta^\dagger\|^2-\|\varpi_n-\eta_n\|^2\\
&+2\sum_{i=1}^N\gamma_{i,n}\lambda_{i,n}\|\varpi_n-\zeta^\dagger\|\left\|\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\|+\alpha_nC_1.
\end{aligned}
\tag{3.7}
$$

Further

$$\begin{aligned}
\|\vartheta_n-\zeta^\dagger\|^2 &\le \|\delta_n\varpi_n+(1-\delta_n)\Lambda_n(\varpi_n)-\zeta^\dagger\|^2\\
&=\langle \vartheta_n-\zeta^\dagger, \delta_n\varpi_n+(1-\delta_n)\Lambda_n(\varpi_n)-\zeta^\dagger\rangle\\
&=\frac{1}{2}\left\|\delta_n\varpi_n+(1-\delta_n)\Lambda_n(\varpi_n)-\zeta^\dagger\right\|^2+\frac{1}{2}\|\vartheta_n-\zeta^\dagger\|^2\\
&-\frac{1}{2}\|\vartheta_n-\zeta^\dagger-\delta_n\varpi_n+(1-\delta_n)\Lambda_n(\varpi_n)+\zeta^\dagger\|^2\\
&=\frac{1}{2}\|\delta_n(\varpi_n-\zeta^\dagger)+(1-\delta_n)(\Lambda_n(\varpi_n)-\zeta^\dagger)\|^2+\frac{1}{2}\|\vartheta_n-\zeta^\dagger\|^2\\
&-\frac{1}{2}\|\delta_n(\vartheta_n-\varpi_n)+(1-\delta_n)(\vartheta_n-\Lambda_n(\varpi_n))\|^2\\
&=\frac{1}{2}\delta_n^2\|\varpi_n-\zeta^\dagger\|^2+\frac{1}{2}(1-\delta_n)^2\|\Lambda_n(\varpi_n)-\zeta^\dagger\|^2\\
&+\delta_n(1-\delta_n)\langle\varpi_n-\zeta^\dagger,\Lambda_n(\varpi_n)-\zeta^\dagger\rangle+\frac{1}{2}\|\vartheta_n-\zeta^\dagger\|^2-\frac{1}{2}\delta_n^2\|\vartheta_n-\varpi_n\|^2\\
&-\frac{1}{2}(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2-\delta_n(1-\delta_n)\langle\vartheta_n-\varpi_n,\vartheta_n-\Lambda_n(\varpi_n)\rangle
\end{aligned}$$

$$\begin{aligned}
\|\vartheta_n-\zeta^\dagger\|^2 &\le \frac{1}{2}\delta_n^2\|\varpi_n-\zeta^\dagger\|^2+\frac{1}{2}(1-\delta_n)^2\|\varpi_n-\zeta^\dagger\|^2+\delta_n(1-\delta_n)\|\varpi_n-\zeta^\dagger\|^2\\
&+\frac{1}{2}\|\vartheta_n-\zeta^\dagger\|^2-\frac{1}{2}\delta_n^2\|\vartheta_n-\varpi_n\|^2-\frac{1}{2}(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2\\
&=\frac{1}{2}\|\vartheta_n-\zeta^\dagger\|^2+\frac{1}{2}\|\varpi_n-\zeta^\dagger\|^2-\frac{1}{2}\delta_n^2\|\vartheta_n-\varpi_n\|^2-\frac{1}{2}(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2.
\end{aligned}$$

It implies

$$\|\vartheta_n-\zeta^\dagger\|^2\le\|\varpi_n-\zeta^\dagger\|^2-\delta_n^2\|\vartheta_n-\varpi_n\|^2-(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2. \tag{3.8}$$

Also

$$\begin{aligned}
\|\eta_n-\zeta^\dagger\|^2 &= \|\zeta_n+\theta_n(\zeta_n-\zeta_{n-1})-\zeta^\dagger\|^2\\
&=\|\zeta_n-\zeta^\dagger+\theta_n(\zeta_n-\zeta_{n-1})\|^2\\
&=\|\zeta_n-\zeta^\dagger\|^2+\theta_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\theta_n\langle\zeta_n-\zeta^\dagger,\zeta_n-\zeta_{n-1}\rangle\\
&\le\|\zeta_n-\zeta^\dagger\|^2+\theta_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\theta_n\|\zeta_n-\zeta^\dagger\|\|\zeta_n-\zeta_{n-1}\|\\
&=\|\zeta_n-\zeta^\dagger\|^2+\theta_n\|\zeta_n-\zeta_{n-1}\|(\theta_n\|\zeta_n-\zeta_{n-1}+2\|\zeta_n-\zeta^\dagger\|)\\
&\le\|\zeta_n-\zeta^\dagger\|^2+\theta_n\|\zeta_n-\zeta_{n-1}\|C_2, \qquad (3.9)
\end{aligned}$$

for some C_2.

Now we have

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &= \|\alpha_n F(\zeta_n)+(1-\alpha_n)\vartheta_n-\zeta^\dagger\|^2\\ &\le \langle \alpha_n F(\zeta_n)+(1-\alpha_n)\vartheta_n-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &= (1-\alpha_n)\langle\vartheta_n-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+\alpha_n\langle F(\zeta_n)-F(\zeta^\dagger),\zeta_{n+1}-\zeta^\dagger\rangle\\ &+\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\le \frac{1-\alpha_n}{2}[\|\vartheta_n-\zeta^\dagger\|^2+\|\zeta_{n+1}-\zeta^\dagger\|^2]+\frac{\alpha_n}{2}[\rho^2\|\zeta_n-\zeta^\dagger\|^2+\|\zeta_{n+1}-\zeta^\dagger\|^2]\\ &+\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\le \frac{1}{2}\|\zeta_{n+1}-\zeta^\dagger\|^2+\frac{1-\alpha_n}{2}\|\vartheta_n-\zeta^\dagger\|^2+\frac{\alpha_n}{2}\rho^2\|\zeta_n-\zeta^\dagger\|^2\\ &+\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle.\end{aligned}$$

After simplifying the above equation and applying (3.3), (3.8) and (3.9) we get

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &\le (1-\alpha_n)\|\vartheta_n-\zeta^\dagger\|^2+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\le (1-\alpha_n)(\|\zeta_n-\zeta^\dagger\|^2+\theta_n\|\zeta_n-\zeta_{n-1}\|C_2)+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2\\ &+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &= [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &+\alpha_n(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|\zeta_n-\zeta_{n-1}\|C_2. \qquad (3.10)\end{aligned}$$

Further, using (3.4) and (3.6) we also have

$$\begin{aligned}&\|\zeta_{n+1}-\zeta^\dagger\|^2 \le (1-\alpha_n)\|\vartheta_n-\zeta^\dagger\|^2+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\le (1-\alpha_n)\left(\|\zeta_n-\zeta^\dagger\|^2+\sum_{i=1}^N\gamma_{i,n}\lambda_{i,n}(\lambda_{i,n}\|\Upsilon\|^2-2)\|(J_{\beta_{i,n}}-I)\Upsilon(\eta_n)\|^2+\alpha_n C_1\right)\\ &+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &= [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &+(1-\alpha_n)\alpha_n C_1+(1-\alpha_n)\sum_{i=1}^N\gamma_{i,n}\lambda_{i,n}\left(\lambda_{i,n}\|\Upsilon\|^2-2\right)\left\|\left(J_{\beta_{i,n}}^{\Xi_i}-I\right)\Upsilon(\eta_n)\right\|^2 \qquad (3.11)\end{aligned}$$

Using (3.4) and (3.7) we can have

$$\begin{aligned}&\|\zeta_{n+1}-\zeta^\dagger\|^2 \le (1-\alpha_n)\|\vartheta_n-\zeta^\dagger\|^2+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\le (1-\alpha_n)\left(\|\zeta_n-\zeta^\dagger\|^2-\|\varpi_n-\eta_n\|^2+2\sum_{i=1}^N\gamma_{i,n}\lambda_{i,n}\|\varpi_n-\zeta^\dagger\|\left\|\Upsilon^*\left(J_{\beta_{i,n}}^{\Xi_i}-I\right)\Upsilon(\eta_n)\right\|\right.\\ &\left.+\alpha_n C_1\right)+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &= [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+(1-\alpha_n)\alpha_n C_1\\ &-(1-\alpha_n)\|\varpi_n-\eta_n\|^2+2(1-\alpha_n)\sum_{i=1}^N\gamma_{i,n}\lambda_{i,n}\|\varpi_n-\zeta^\dagger\|\left\|\Upsilon^*\left(J_{\beta_{i,n}}^{\Xi_i}-I\right)\Upsilon(\eta_n)\right\|. \qquad (3.12)\end{aligned}$$

Using (3.8) and (3.3) and we get

$$
\begin{aligned}
\|\zeta_{n+1}-\zeta^\dagger\|^2 &\leq (1-\alpha_n)\|\vartheta_n-\zeta^\dagger\|^2+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\
&\leq (1-\alpha_n)(\|\varpi_n-\zeta^\dagger\|^2-\delta_n^2\|\vartheta_n-\varpi_n\|^2)+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2\\
&+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\
&\leq (1-\alpha_n)(\|\zeta_n-\zeta^\dagger\|^2+\alpha_nC_1-\delta_n^2\|\vartheta_n-\varpi_n\|^2)+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2\\
&+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\
&=[1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+(1-\alpha_n)\alpha_nC_1\\
&-(1-\alpha_n)\delta_n^2\|\vartheta_n-\varpi_n\|^2, \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\zeta_{n+1}-\zeta^\dagger\|^2 &\leq (1-\alpha_n)\|\vartheta_n-\zeta^\dagger\|^2+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\
&\leq (1-\alpha_n)(\|\varpi_n-\zeta^\dagger\|^2-(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2)\\
&+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\
&\leq (1-\alpha_n)(\|\zeta_n-\zeta^\dagger\|^2+\alpha_nC_1-(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2)+\alpha_n\rho^2\|\zeta_n-\zeta^\dagger\|^2\\
&+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\
&=[1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+(1-\alpha_n)\alpha_nC_1\\
&-(1-\alpha_n)(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2. \qquad (3.14)
\end{aligned}
$$

Now to prove that the sequence $\{\zeta_n\}$ converges to $\zeta^\dagger$ we split the proof in the following two cases.

Case 1: There exists a n_0 such that $\|\zeta_{n+1}-\zeta^\dagger\|\leq\|\zeta_n-\zeta^\dagger\|\ \forall\ n\geq n_0$. This shows that the sequence $\{\|\zeta_n-\zeta^\dagger\|\}$ is convergent. Using condition (ii) on (3.11), (3.13), (3.14) we get

$$
\begin{aligned}
(1-\alpha_n)\sum_{i=1}^{N}\gamma_{i,n}\lambda_{i,n}(2-\lambda_{i,n}\|\Upsilon\|^2)\left\|\left(J_{\beta_{i,n}}^{\Xi_i}-I\right)\Upsilon(\eta_n)\right\|^2&\\
\leq [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2-\|\zeta_{n+1}-\zeta^\dagger\|^2&\\
+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+(1-\alpha_n)\alpha_nC_1\to 0,&
\end{aligned}
$$

$$
\begin{aligned}
(1-\alpha_n)\delta_n^2\|\vartheta_n-\varpi_n\|^2 &= [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2-\|\zeta_{n+1}-\zeta^\dagger\|^2\\
&+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+(1-\alpha_n)\alpha_nC_1\to 0,
\end{aligned}
$$

$$
\begin{aligned}
(1-\alpha_n)(1-\delta_n)^2\|\vartheta_n-\Lambda_n(\varpi_n)\|^2 &= [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2-\|\zeta_{n+1}-\zeta^\dagger\|^2\\
&+2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle+(1-\alpha_n)\alpha_nC_1\to 0.
\end{aligned}
$$

And we get

$$
\left\|\left(J_{\beta_{i,n}}^{\Xi_i}-I\right)\Upsilon(\eta_n)\right\|\to 0,\|\vartheta_n-\varpi_n\|\to 0,\|\vartheta_n-\Lambda_n(\varpi_n)\|\to 0. \qquad (3.15)
$$

From (3.12) we get

$$(1-\alpha_n)\|\varpi_n-\eta_n\|^2 \le [1-\alpha_n(1-\rho^2)]\|\zeta_n-\zeta^\dagger\|^2 - \|\zeta_{n+1}-\zeta^\dagger\|^2 \\ + 2\alpha_n\langle F(\zeta^\dagger)-\zeta^\dagger, \zeta_{n+1}-\zeta^\dagger\rangle + (1-\alpha_n)\alpha_n C_1 \\ + 2(1-\alpha_n)\sum_{i=1}^{N}\gamma_{i,n}\lambda_{i,n}\|\varpi_n-\zeta^\dagger\|\left\|\Upsilon^*\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\| \to 0.$$

And we get

$$\|\varpi_n-\eta_n\| \to 0. \tag{3.16}$$

We also have

$$\|\zeta_n-\eta_n\| = \theta_n\|\zeta_n-\zeta_{n-1}\| \le \alpha_n C \to 0. \tag{3.17}$$

From (3.15) and (3.16) we have

$$\|\zeta_n-\varpi_n\| \le \|\zeta_n-\eta_n\| + \|\eta_n-\varpi_n\| \to 0 \\ \|\zeta_n-\vartheta_n\| \le \|\zeta_n-\varpi_n\| + \|\varpi_n-\vartheta_n\| \to 0 \\ \|\Lambda_n(\varpi_n)-\varpi_n\| \le \|\Lambda_n(\varpi_n)-\vartheta_n\| + \|\vartheta_n-\varpi_n\| \to 0. \tag{3.18}$$

Now using condition (ii) and (3.17) we have

$$\|\zeta_{n+1}-\zeta_n\| \le \|\zeta_{n+1}-\vartheta_n\| + \vartheta_n-\zeta_n\| \\ = \|\alpha_n F(\zeta_n)+(1-\alpha_n)\vartheta_n-\vartheta_n\| + \|\vartheta_n-\zeta_n\| \\ = \alpha_n\|F(\zeta_n)-\vartheta_n\| + \|\vartheta_n-\zeta_n\| \to 0. \tag{3.19}$$

Suppose the sequence $\{\zeta_n\}$ has a subsequence $\{\zeta_{n_j}\}$ such that $\zeta_{n_j} \rightharpoonup \zeta^\dagger$. From (3.17) and (3.18) there exists a subsequence of $\{\eta_n\}$ and $\{\varpi_n\}$ satisfying $\eta_{n_j} \rightharpoonup \zeta^\dagger$ and $\varpi_{n_j} \rightharpoonup \zeta^\dagger$, respectively. Since the mapping Υ is bounded and linear, then $\Upsilon(\eta_n) \rightharpoonup \Upsilon(\zeta^\dagger)$. Moreover we know that $\left\|\left(J^{\Xi_i}_{\beta_{i,n}}-I\right)\Upsilon(\eta_n)\right\| \to 0$, which implies that $\Upsilon(\zeta^\dagger) = J^{\Xi_i}_{\beta_{i,n}}\Upsilon(\zeta^\dagger)$, using Lemma 2.8, we get $\zeta^\dagger \in F(\Lambda_i)$. Hence $\zeta^\dagger \in F(\Lambda_i)\cap\Theta$. Then it follows that

$$\limsup_{n\to\infty}\langle F(\zeta^\dagger)-\zeta^\dagger, \zeta_{n+1}-\zeta^\dagger\rangle = \limsup_{j\to\infty}\langle F(\zeta^\dagger)-\zeta^\dagger, \zeta_{n_j+1}-\zeta^\dagger\rangle \\ = \langle F(\zeta^\dagger)-\zeta^\dagger, \zeta^*-\zeta^\dagger\rangle \le 0.$$

Now applying Lemma 2.4 to (3.10) we get $\zeta_n \to \zeta^\dagger = P_{\bigcap_{i=1}^{n} F(\Lambda_i)\bigcap\Theta} F(\zeta^\dagger)$.

Case 2: If the sequence $\{\|\zeta_{n+1}-\zeta^\dagger\|\}$ is not monotonically decreasing. Then there exists a subsequence n_j such that

$$\left\|\zeta_{n_j}-\zeta^\dagger\right\|^2 \le \left\|\zeta_{n_j+1}-\zeta^\dagger\right\|^2, \text{ for all } j \in \mathbb{N}.$$

By Lemma 2.7, there exists a nondecreasing sequence $\{m_i\} \subset \mathbb{N}$ such that $m_i \to \infty$

$$\|\zeta_{m_i}-\zeta^\dagger\|^2 \le \|\zeta_{m_i+1}-\zeta^\dagger\|^2, \quad \|\zeta_i-\zeta^\dagger\|^2 \le \|\zeta_{m_i+1}-\zeta^\dagger\|^2. \tag{3.20}$$

If we follow similarly the proof in Case (1), we get

$$\|\zeta_{m_i+1}-\zeta^\dagger\|^2 \le [1-\alpha_{m_i}(1-\rho^2)]\|\zeta_{m_i}-\zeta^\dagger\|^2 + 2\alpha_{m_i}\langle F(\zeta^\dagger)-\zeta^\dagger, \zeta_{m_i+1}-\zeta^\dagger\rangle \\ + \alpha_{m_i}(1-\alpha_{m_i})\frac{\theta_{m_i}}{\alpha_{m_i}}\|\zeta_{m_i}-\zeta_{m_i-1}\|C_2,$$

and

$$\limsup_{n\to\infty}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{m_i+1}-\zeta^\dagger\rangle\leq 0,$$

which gives

$$\begin{aligned}0&\leq\|\zeta_{m_i+1}-\zeta^\dagger\|^2-\|\zeta_{m_i}-\zeta^\dagger\|^2\\&=[1-\alpha_{m_i}(1-\rho^2)]\|\zeta_{m_i}-\zeta^\dagger\|^2+2\alpha_{m_i}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{m_i+1}-\zeta^\dagger\rangle\\&+\alpha_{m_i}(1-\alpha_{m_i})\frac{\theta_{m_i}}{\alpha_{m_i}}\|\zeta_{m_i}-\zeta_{m_i-1}\|C_2-\|\zeta_{m_i}-\zeta^\dagger\|^2,\end{aligned}$$

using $\lim\limits_{n\to\infty}\frac{\theta_n}{\alpha_n}\|\zeta_n-\zeta_{n-1}\|\to 0$, we get

$$\|\zeta_{m_i}-\zeta^\dagger\|^2\leq\frac{2}{1-\rho^2}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{m_i+1}-\zeta^\dagger\rangle+\frac{1-\alpha_{m_i}}{1-\rho^2}\frac{\theta_{m_i}}{\alpha_{m_i}}\|\zeta_{m_i}-\zeta_{m_i-1}\|C_2\to 0.$$

By (3.20) we get $\|\zeta_{m_i+1}-\zeta^\dagger\|\to 0$. It follows from $\|\zeta_i-\zeta^\dagger\|^2\leq\|\zeta_{m_i+1}-\zeta^\dagger\|^2$ for all $i\in\mathbb{N}$ that $\|\zeta_i-\zeta^\dagger\|^2\to 0$, applying Lemma 2.7 we get $\zeta_i\to\zeta^\dagger$. Hence sequence $\zeta_n\to\zeta^\dagger$, $n\to\infty$. It completes the proof.

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AUTHORS CONTRIBUTIONS

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

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