Aust. J. Math. Anal. Appl.
Vol. **21** (2024), No. 2, Art. 7, 6 pp.
AJMAA

AUTOMATIC CONTINUITY OF GENERALIZED DERIVATIONS IN CERTAIN *-BANACH ALGEBRAS

M.ABOULEKHLEF[1], Y. TIDLI[2],M.BELAM[2]

Received 9 February, 2024; accepted 2 July, 2024; published 27 August, 2024.

LABORATORY OF APPLIED MATHEMATICS AND INFORMATION AND COMMUNICATION TECHNOLOGY
POLYDISCIPLINARY FACULTY OF KHOURIBGA UNIVERSITY OF SULTAN MOULAY SLIMANE MOROCCO.
aboulekhlef@gmail.com; y.tidli@gmail.com; m.belam@gmail.com

ABSTRACT. Consider the map φ of the Banach algebra $\mathfrak{B}$ in $\mathfrak{B}$, if there exists a derivation δ of $\mathfrak{B}$ in $\mathfrak{B}$ so that for every$x, y \in \mathfrak{B}$, $\varphi(xy) = \varphi(x)y + x\delta(y)$. φ is called a generalized derivation of $\mathfrak{B}$.In [9], Bresar introduced the concept of generalized derivations.
We prove several results about the automatic continuity of generalized derivations on certain Banach algebras.

Key words and phrases: Banach algebras; Automatic continuity; Generalized derivations; Involution.

2010 *Mathematics Subject Classification.* Primary: 46J10, 16Wxx. Secondary: 47B48.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In this paper, the algebras considered are assumed complex, commutative, and not necessarily unitary. In all that follows, $\mathfrak{B}$ can be a Banach algebra, $\mathrm{Jr}(\mathfrak{B})$ denotes the Jacobson radical of $\mathfrak{B}$. The symbols $\mathrm{spm}(y)$ and $\rho(y)$ represent spectrum and the spectral radius of $y \in \mathfrak{B}$, respectively. If for every $x, y \in \mathfrak{B}, \delta(xy) = \delta(x)y + x\delta(y)$ then δ is an additive map from $\mathfrak{B}$ to $\mathfrak{B}$. This map is called a derivation.

Let's briefly discuss the history of our investigation. Singer-Wermer [7] shown in 1955 that if $\mathfrak{B}$ is commutative and δ is continuous, then $\delta(\mathfrak{B}) \subseteq \mathrm{Jr}(\mathfrak{B})$. In particular, $\delta = 0$ when $\mathfrak{B}$ is semisimple. In [8] Bresar and al. showed that $\delta(\mathfrak{B}) \subseteq \mathrm{Jr}(\mathfrak{B})$ if there is $\epsilon \geq 0$ so that $\rho(\delta(x)) \leq \epsilon\rho(x)$ for every $x \in \mathfrak{B}$.

2. PRELIMINARIES

It's convenient to begin by recalling some definitions and known results. A Banach algebra $\mathfrak{B}$ is considered prime for each $x, y \in \mathfrak{B}$ if, for any $x\mathfrak{B}y = 0,$ it implies that either $x =$ 0 or $y = 0$.By a derivation on $\mathfrak{B}$, we mean a linear mapping $\delta : \mathfrak{B} \to \mathfrak{B}$, which satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all x and y in $\mathfrak{B}$.An additive mapping $\tau : \mathfrak{B} \to \mathfrak{B}$ is called a left (resp. right) centralizer of $\mathfrak{B}$ if $\tau(xy) = \tau(x)y$ (resp. $\tau(xy) = x\tau(y)$) for all $x, y \in \mathfrak{B}$.Bresar introduced the concept of generalized derivation in [7].He defines a generalized derivation as follows: If there is a derivation d on $\mathfrak{B}$ such that $\varphi(xy) = \varphi(x)y + x\delta(y)$ for all $x, y \in \mathfrak{B}$, then an additive mapping F is referred to as a generalized derivation.

Lemma 2.1. *Let $\mathfrak{B}$ represent an algebra. Then φ is a generalized derivation of $\mathfrak{B}$ determined by derivation δ if and only if $\varphi = \delta + \tau$, where τ is $\mathfrak{B}$'s left centralizer and δ is $\mathfrak{B}$'s derivation.*

Given a subset E of an algebra $\mathfrak{B}$. The set defined by $G(E) = \{a \in \mathfrak{B}/aE = \{0\}\}$ is known as the left annihilator of E.In a similar manner, the right annihilator is the set of E defined by $D(E) = \{a \in \mathfrak{B}/Ea = \{0\}\}$. We note Ann $(E) = G(E) \cap D$ (E).
If $\mathfrak{B}$ does not have a unit, then we can adjoin one as follows:

Proposition 2.2. *A normed algebra without a unit can be embedded into a unital normed algebra $\mathfrak{B}^{\#}$ as an ideal of codimension one.*

Proof. Let $\mathfrak{B}^{\#} = \mathfrak{B} \oplus \mathbb{C}$ Direct sum of $\mathfrak{B}$ and $\mathbb{C}$(field of complex numbers).
$\mathfrak{B}^{\#}$ represents a vector space under the usual operations :
$+ : \mathfrak{B}^{\#} \times \mathfrak{B}^{\#} \longrightarrow \mathfrak{B}^{\#}$

$((a, \alpha), (b, \beta)) \longrightarrow (a + b, \alpha + \beta)$

$. : \mathfrak{C} \times \mathfrak{B}^{\#} \longrightarrow \mathfrak{B}^{\#}$

$(\lambda, (a, \alpha)) \to (\lambda b, \lambda\alpha)$

In addition to, $\mathfrak{B}^{\#}$ is an algebra when defining a multiplication in $\mathfrak{B}^{\#}$ by :
$\odot : \mathfrak{B}^{\#} \times \mathfrak{B}^{\#}. \longrightarrow \mathfrak{B}^{\#}$

$(a, \alpha), (b, \beta)) \longrightarrow (a, \alpha) \odot (b, \beta)$

$(a, \alpha) \odot (b, \beta) := (a, \alpha)(b, \beta) := (ab + \beta a + \alpha b, \alpha\beta)$

The operation $\odot$ is closed on $\mathfrak{B}^{\#}$, and $\left(\mathfrak{B}^{\#}, +, ., \odot\right)$ is algebra with unit element $(0, 1)$.
Now, define the function $\|\cdot\|$ on $\mathfrak{B}^{\#}$ by :

$\|\cdot\| : \mathfrak{B}^{\#} \longrightarrow \mathbb{R}^{+}$
$(a,\gamma) \longrightarrow \|(a,\gamma)\| = \|a\| + |\gamma|$
then ($\mathfrak{B}^{\#}$,$\|\cdot\|$) constitutes a normed algebra.
Let $B = \{(a,0) : a \in A\}$, and
Identify :
$\phi : A \rightarrow B$

$a \rightarrow (a,0)$
$\|(a,0)\| = \|a\| + |0| = \|a\|$ hence ϕ is isometric isomorphe.
We write $(a,\lambda) = (a,0) + \lambda(0,1)$, since B is an ideal in $A \times \mathbb{C}$ of codimension 1.

∎

Now, define the spectrum and the spectral radius:
Let $\mathfrak{B}$ be an algebra :

(1) If $\mathfrak{B}$ is unital with unit $e_{\mathfrak{B}}$ then the spectrum and the spectral radius of x are defined by:

$$\operatorname{spm}_{\mathfrak{B}}(x) := \{\lambda \in \mathbb{C} : \lambda e_{\mathfrak{B}} - x \notin \operatorname{Inv} \mathfrak{B}\} \tag{2.1}$$

$$\rho_{\mathfrak{B}}(x) := \sup\{|\lambda| : \lambda \in \operatorname{spm}_{\mathfrak{B}}(x)\} \tag{2.2}$$

where Inv $\mathfrak{B}$ is the set of invertible elements of $\mathfrak{B}$.

(2) If $\mathfrak{B}$ is nonunital, we define the quasi-product $\centerdot$ on $\mathfrak{B}$ by

$$x \centerdot y = x + y - xy \quad (x, y \in \mathfrak{B})$$

An element x of $\mathfrak{B}$ is called quasi-invertible if there is $y \in \mathfrak{B}$ such that $x \centerdot y = 0$ and $x \centerdot y = 0$. The set of all quasi-invertible elements of $\mathfrak{B}$ is denoted by $q - \operatorname{Inv}\mathfrak{B}$.
Let $\mathfrak{B}^{\#}$ the Banach algebra obtained by adjoining a unit to $\mathfrak{B}$, called the unitization of $\mathfrak{B}$.
We define spectrum in non-unital Banach algebra :
$\operatorname{spm}_{\mathfrak{B}}(x) = \{0\} \cup \left\{\mu \in \mathbb{C}\backslash\{0\} : \frac{1}{\mu}x \notin q - \operatorname{Inv} \mathfrak{B}\right\}$ and it's clear $spm_{\mathfrak{B}}(x) = sp_{\mathfrak{B}^{\#}}((x,0))$ and $\rho_A(x) = \rho_{A\#}((x,0))$

Lemma 2.3. [10]
If τ is a left centralizer on a Banach algebra $\mathfrak{B}$, then τ is both linear and continuous.

Definition 2.1. The intersection of all maximal left (right) ideals in an algebra $\mathfrak{B}$ is its (Jacobson) radical, represented by $\operatorname{Jr}(\mathfrak{B})$.
If $\operatorname{Jr}(\mathfrak{B}) = 0$, then algebra $\mathfrak{B}$ is considered semisimple.

Definition 2.2. An involution $*$ into $\mathfrak{B}$ is a mapping $* : \mathfrak{B} \rightarrow \mathfrak{B}$ that fulfills these conditions:

$$(a+b)^* = a^* + b^* \qquad (\mu a)^* = \bar{\mu} a^*$$
$$(ab)^* = b^* a^*$$

with involution $*$, $\mathfrak{B}$ is known as the $*$-algebra.

Remark 2.1. In the case that $\mathfrak{B}$ is involutive, define an involution on $\mathfrak{B}^{\#}$ as follows: $(a,\mu)^* := (a^*, \bar{\mu}), \forall (a,\mu) \in \mathfrak{B}^{\#}$

Given J a non-zero *-ideal of $\mathfrak{B}$. Then, * induces an involution on the quotient algebra $\mathfrak{B}$/J, denoted as *,defined by: $(a+J)^* = (a)^* + J$, for every $a \in \mathfrak{B}$.

Recalling that $\mathfrak{B}$ is said to as $*$-semi-simple if $\mathrm{Jr}_*(\mathfrak{B})=(0)$, and we obtain

$$\begin{aligned}\mathrm{Jr}_*(\mathfrak{B}/J) &= \cap\left\{\bar{M}:\mathrm{M}\text{ represents }^*-\text{ maximum ideal in }\mathfrak{B}\right\}\\ &= \cap\{M:\text{ represents }^*-\text{ maximum ideal in }\mathfrak{B}\}\\ &= \overline{\mathrm{Jr}}_*(\mathfrak{B})=\mathrm{Jr}_*(\mathfrak{B})/J\end{aligned}$$

An *-ideal J of $\mathfrak{B}$ is an ideal which is closed under involution; that is $J^*=\{a^*\in\mathfrak{B}:a\in J\}\subseteq J$.

Proposition 2.4. [11] *Let $\mathfrak{A}$ be an *-simple algebra, if $\mathfrak{A}$ is not simple. Then there exists a unitary simple subalgebra J of $\mathfrak{A}$ such that $A=J\oplus J^*$*

Lemma 2.5. *[*[4]*, Corollary 3.2.10]*
Let $\mathfrak{B}$ be a Banach algebra, $x\in\mathfrak{B}$, and suppose that $\rho(yx)=0$ for all $y\in\mathfrak{B}$. Then $x\in\mathrm{Jr}(\mathfrak{B})$.

Recall the concept of separating space of a linear operator, let $\mathfrak{B}$ and $\mathfrak{B}$ be two Banach algebras, and given $\psi:\mathfrak{B}\longrightarrow\mathfrak{B}$ a linear mapping. The separating space of ψ is determined by :

$$\mathfrak{S}(\psi)=\{\beta\in\mathfrak{B}:\text{there exists }(\alpha_m)_m\text{ in }\mathfrak{B}\text{ such that }\alpha_m\to 0\text{ and }\psi(\alpha_m)\to\beta\}$$

We know that $\mathfrak{S}(\psi)$ is a closed linear subspace of $\mathfrak{B}$. By the closed graph theorem, ψ is continuous if and only if $\mathfrak{S}(\psi)=\{0\}$ [2, 5.1.2]

Lemma 2.6. [5] *Let $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{W}$ be Banach spaces, T be a linear operator from $\mathfrak{U}$ into $\mathfrak{V}$, and S be a continuous operator from $\mathfrak{V}$ into $\mathfrak{W}$. Then :*

(1) *ST is continuous if and if $S\mathfrak{S}(T)=\{0\}$.*

(2) $\overline{S\mathfrak{S}(T)}=\mathfrak{S}(ST)$

Lemma 2.7. [4]
Let $\mathfrak{B}$ be a Banach algebra such that $xy=yx$. Then $\rho(x+y)\le\rho(x)+\rho(y)$ and $\rho(xy)\le\rho(x)\rho(y)$ for all $x,y\in\mathfrak{B}$

Lemma 2.8. *[Singer and Wermer* [7]*]*
Let $\mathfrak{B}$ be a commutative Banach algebra and let δ be a continuous derivation on $\mathfrak{B}$ then $d(x)\in Jr(\mathfrak{B})(x\in\mathfrak{B})$

3. MAIN RESULT

Proposition 3.1. *Assume that φ is a generalized derivation of a Banach algebra $\mathfrak{B}$. If $\mathfrak{B}$ is semi-simple and commutative, φ must be continuous.*

Proof. Given $y\in\mathfrak{S}(\varphi)$ there is a sequence (α_n) in $\mathfrak{B}$ such that
$\alpha_n\to 0$ in $\mathfrak{B}$ and $\varphi(\alpha_n)\to y$ in $\mathfrak{B}$.
As $\rho(\tau(\alpha_n))\leqslant\rho(\alpha_n)$ and $\rho(\delta(\alpha_n))\le\epsilon\rho(\alpha_n)$ for some $\epsilon\ge 0$
we have $\rho(\varphi(\alpha_n))=\rho(\delta(\alpha_n)+(\tau(\alpha_n))\le\rho(\delta(\alpha_n))+\rho(\tau(\alpha_n))$, and $\rho(\alpha_n)\to 0$
Then $\rho(\varphi(\alpha_n))\leqslant(\epsilon+1)\rho(\alpha_n)\to 0$.
However, in contrast.. $\rho(\varphi(\alpha_n))\to\rho(y)$. Therefore $\rho(y)=0$ because $\mathfrak{S}(\varphi)$ is an ideal in $\mathfrak{B}$.
Thus for every z in $\mathfrak{B}$, $yz\in\mathfrak{S}(\varphi)$ we conclude that $\rho(yz)=0$
since we conclude $y\in Jr(\mathfrak{B})$.Therefore, $\mathfrak{S}(\varphi)\subseteq\mathrm{Jr}\,\mathfrak{B}$.
Since $\mathfrak{B}$ is semi-simple, $\mathfrak{S}(\varphi)=0$ consequently φ is continuous.
∎

Theorem 3.2. *All generalized derivations on a Banach *-algebra $\mathfrak{B}$ which is *-simple are continuous.*

Proof. Let $\mathfrak{B}$ be an algebra $*$ simple, there exists simple unital subalgebra J of $\mathfrak{B}$ such that : $A = J \oplus J^*$ (Proposition 2.4); following algebraic isomorphism: $J \approx \mathfrak{B} \ / \ J^*$, therefore, J represents the maximal ideal in $\mathfrak{B}$. From where J (resp; J^*) is closed in $\mathfrak{B}$. Consequently, the algebra $\mathfrak{B}/J$ (resp; $\mathfrak{B}/J^*$) is a simple Banach $*$-algebra. Since J represents an ideal of $\mathfrak{B}$, then so is $\varphi(J) + J$; therefore $\varphi(J) + J/J$ represents an ideal of $\mathfrak{B}/J$. Since $\mathfrak{B}/J$ constitutes simple algebra, so

$\varphi(J) + J/J = \{\bar{0}\}$ or $\varphi(J) + J/J = \mathfrak{B}/J$. Since J represents a

maximal ideal in $\mathfrak{B}$, since $\varphi(J) + J = J$, so $\varphi(J) \subseteq J$. Think about the function $\widetilde{\varphi}$ on $\mathfrak{B}/J$, which is given by:

$\widetilde{\varphi}(a + J) = \varphi(a) + J$.

We show that $\widetilde{\varphi}$ represents a generalized derivation determined by a derivation $\widetilde{\delta}$ on $\mathfrak{B}/J$. Note that it is easy to show $\widetilde{\varphi}$ is linear operator. Moreover, for $a, b \in \mathfrak{B}$, $\widetilde{\varphi}(a + J)(b + J)) = \widetilde{\varphi}(ab+J)) = \varphi(ab)+J = \varphi(a)b+a\delta(b)+J$. But then, $\widetilde{\varphi}(a+J)(b+J)+(a+J)\widetilde{\delta}(b+J) = (a+J)(\delta(b)+J)+(\delta(a)+J)(b+J) = (\varphi(a)+J)(b+J)+(a+J)(\delta(b)+J) = a\delta(b)+\delta(a)b+J$. So $\widetilde{\varphi}$ is a generalized derivation on the simple Banach algebra $\mathfrak{B}/J$, then $\widetilde{\varphi}$ is continuous. Now,we show that φ is continuous,we consider the canonical surjection $\pi : \mathfrak{B} \to \mathfrak{B}/J; a \to a+J$ which is continuous. To show that φ is continuous, first we observe that $\pi o\varphi = \widetilde{\varphi} o\pi$ because for every a $\in \mathfrak{B}$, we have $\pi o\varphi(a) = \pi(\varphi(a)) = \varphi(a) + J$ and $\widetilde{\varphi} o\pi(a) = \widetilde{\varphi}(a + J) = \varphi(a) + J$. Since $\widetilde{\varphi} o\pi$ is continuous, then; we have $\mathfrak{S}(\widetilde{\varphi} \circ \pi) = \{\bar{0}\}$, And $\overline{\pi\mathfrak{S}(\varphi)} = \mathfrak{S}(\widetilde{\varphi} \circ \pi) = \{\bar{0}\}$ (Lemma 2.6) and this implied that $\mathfrak{S}(\varphi) \subset J$. Following the same steps, we show that $\mathfrak{S}(\varphi) \subset J^*$, then $\mathfrak{S}(\varphi) \subset J \cap J^* = \{0\}$. Therefore φ is continuous . ∎

Theorem 3.3. *Let $\mathfrak{B}$ be a *-prime Banach *-algebra . Then all generalized derivation φ on $\mathfrak{B}$ is continuous.*

Proof. Since $\mathfrak{B}$ is a *-prime algebra, there is a minimal prime nonzero P such that $\mathrm{P} \cap \mathrm{P}^* = \{0\}$ and $\mathrm{P} = \mathrm{Ann}(\mathrm{P}^*), \mathrm{P}^* = \mathrm{Ann}\ (\mathrm{P})$.
Let $\mathfrak{S}(\varphi)$ be the ideal separating in $\mathfrak{B}$. Suppose $\mathfrak{S}(\varphi) \not\subset \mathrm{P}$, then P is a closed ideal . On the other hand, if p is a nonzero element of P, then $\overline{p\mathfrak{S}(\varphi)} = \mathfrak{S}(\varphi)$. Therefore: $p\mathfrak{S}(\varphi) \subseteq P$, then $\overline{p\mathfrak{S}(\varphi)} = \mathfrak{S}(\varphi) \subseteq \bar{P} = P$. Which contradicts the assumption. Following $\mathfrak{S}(\varphi) \subseteq \mathrm{P}$. By the same reasoning, we show that $\mathfrak{S}(\varphi) \subseteq \mathrm{P}^*$. Which gives, $\mathfrak{S}(\varphi) \subseteq \mathrm{P} \cap \mathrm{P}^* = \{0\}$. Therefore, φ is continuous . ∎

REFERENCES

[1] B. AUPETIT, Caracterisation spectrale des algebres de Banach commutatives, *Pacific Math. J.*, **63** (1976), pp. 23-35.

[2] H. G. DALES, Banach algebras and automatic continuity, *Lond. Math. Soc.*, **24** (2000).

[3] P. G. DIXON, Automatic continuity of positive functionals on topological involution algebras, *Bull. Aust. Math. Soc.* **23** (1981).

[4] B. AUPETIT, *A Primer on Spectral Theory*, *Springer*, 1990.

[5] A. M. SINCLAIR, Automatic continuity of linear operators, *Lond. Math. Soc.*, **21** (1976).

[6] B. E. JOHNSON, The uniqueness of the (complete) norm topology, *Bull. Amer. Math. Soc.*, **73** (1967), pp. 537-539.

[7] I. M. SINGER and J. WERMER, Derivations on commutative normed algebras, *Math. Ann.*, **129** (1955), pp. 260-264.

[8] M. BRESAR and M. MATHIEU, Derivations mapping into the radical III, *J. Funct. Anal.*, **133** (1995), pp. 21-29.

[9] M. BRESAR, On the distance of the composition of two derivations to the generalized derivations, *Comm. Glasgow Math. J.*, **33** (1991), pp. 89-93.

[10] B. E JOHSON, *Continuity of Centralizers on Banach Algebras*, (1966).

[11] Y. TIDLI and L. OUKHTITE and A. TAJMOUATI, On automatic continuity of derivations for Banach algebras with involution, *Eur. J. Math. Compu. Scien.*, **4** (2017).