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A NOTE ON EVALUATION OF A NEW CLASS OF INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

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ABSTRACT. In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalshütz play a key role. Applications of the above mentioned summation theorems are well known for the series ${}_3F_2$. In our present investigation, we aim to evaluate twenty five new class of integrals involving generalized hypergeometric function in the form of a single integral of the form:

$$\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ c+\frac{1}{2} \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta.$$

for $i, j = 0, \pm1, \pm2$ with $\omega = \sqrt{-1}$.
The results are established with the help of the generalizations of the classical Watson's summation theorem obtained earlier by Lavoie et al.. Fifty interesting integrals in the form of two integrals (twenty five each) have also been given as special cases of our main findings.

Key words and phrases: Generalized hypergeometric function; Watsons theorem; Definite integral; Beta integral.

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1. INTRODUCTION

The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric function ${}_pF_q$, where $p, q \in \mathbb{N}_0$ defined by [1, 6]

$$ {}_pF_q\left[\begin{array}{c} a_1,\ \dots,\ a_p \\ b_1,\ \dots,\ b_q \end{array} ;\ z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!} \tag{1.1} $$

where $(a)_n$ is the well known Pochhammer symbol (or the raised factorial or the shifted factorial since $(1)_n = n!$) defined for any complex $a \in \mathbb{C}$ by

$$ \begin{aligned} (a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)},\ \left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right) \\ &= \begin{cases} a(a+1)\dots(a+n-1), & (n \in \mathbb{N}) \\ 1, & (n = 0) \end{cases} \end{aligned} \tag{1.2} $$

where Γ is the well-known Gamma function.

For a detailed study about hypergeometric and generalized hypergeometric functions, we refer the standard texts [1, 6].

In the theory of hypergeometric and generalized hypergeometric functions, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ play a key role.

Later, the above mentioned classical summation theorems have been generalized by Lavoie et al. [2, 3, 4].

However, in our present investigation, we are interested in the following classical Watson's summation theorem [1].

$$ {}_3F_2\left[\begin{array}{c} a, b, c \\ \frac{1}{2}(a+b+1), 2c \end{array} ; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.3} $$

provided $\Re(2c-a-b) > -1$,
and its following generalization due to Lavoie et al. [2]

$$ \begin{aligned} & {}_3F_2\left[\begin{array}{c} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{array} ;\ 1\right] \\ =& \mathcal{A}_{i,j}\, \frac{2^{a+b+i-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}(a+b+|i+j|-j-1)\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\,\Gamma(b)} \\ & \times\left\{\mathcal{B}_{i,j}\,\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1-(-1)^i\right)\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{1}{4}(-1)^j\,\left(1-(-1)^i\right)\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right. \\ & \left.+\mathcal{C}_{i,j}\,\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\left[\frac{j+1}{2}\right]+\frac{1}{4}(-1)^j\,\left(1-(-1)^i\right)\right)\,\Gamma\left(c-\frac{1}{2}b+\left[\frac{j+1}{2}\right]\right)}\right\} \\ =& \Omega \text{ (let)} \end{aligned} \tag{1.4} $$

for $i,\ j = 0,\ \pm1,\ \pm2$.
For $i = j = 0$, the result (1.4) reduces to classical Watson's summation theorem (1.3).
Here, $[x]$ denotes the greatest integer less than or equal to x and the modulus is denoted by $|x|$.
Also, the coefficients $\mathcal{A}_{i,j}$, $\mathcal{B}_{i,j}$ and $\mathcal{C}_{i,j}$ are given in the Table 3.1, Table 3.2 and Table 3.3, at

the end of the paper.
In addition to this, we shall also require the following well-known and useful integral due to MacRobert [5]

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}d\theta = e^{\frac{\omega\pi\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.5}$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ with $\omega = \sqrt{-1}$.
The aim of this paper is to evaluate twenty five integrals involving generalized hypergeometric function in the form of a single integral of the form

$$\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ c+\frac{1}{2} \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta$$

The results are derived with the help of generalized Watson's summation theorem on the sum of a ${}_3F_2$ given by (1.4). Fifty interesting integrals in the form of two integrals (twenty five each) have also been given as special cases of our main findings.

2. MAIN INTEGRALS

The twenty five integrals in the form of a single integral to be evaluated in this paper is given in the following theorem.

Theorem 2.1. *For* $\omega = \sqrt{-1}$, $\Re(c) > 0$ *and* $\Re(2c-a-b+i+2j+1) > 0$, *for* $i, j = 0, \pm 1, \pm 2$, *the following integral formula holds true.*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ c+\frac{1}{2} \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta \\ = e^{\omega\frac{\pi}{2}c}\frac{\Gamma(c)\,\Gamma(c)}{\Gamma(2c)}\Omega\end{aligned} \tag{2.1}$$

where Ω *is the same as given in* (1.4).

Proof. The proof of our theorem is quite straight forward. For this, we proceed as follows. Denoting the left hand side of (2.1) by I, we have

$$I = \int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ c+\frac{1}{2} \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta$$

Now expressing ${}_3F_2$ as a series, changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series in the interval $(0, \frac{\pi}{2})$, we have

$$I = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,(c+\frac{1}{2})_n\,(-1)^n(\omega)^n 2^{2n}}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(2c+j)_{n!}}\int_0^{\frac{\pi}{2}} e^{\omega\frac{\pi}{2}(c+n)}(\sin\theta)^{c+n-1}(\cos\theta)^{c+n-1}d\theta$$

Evaluating the integral using (1.5),
we have, after some simplification

$$I = e^{\omega\frac{\pi}{2}c}\frac{\Gamma(c)\,\Gamma(c)}{\Gamma(2c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,(c)_n}{\left(\frac{1}{2}(a+b+1)\right)_n\,(2c+j)_n\,n!}$$

Now summing up the series, we have

$$I = e^{\omega\frac{\pi}{2}c}\frac{\Gamma(c)\,\Gamma(c)}{\Gamma(2c)}{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};1\right]$$

We now observe that the ${}_3F_2$ appearing can be evaluated with the help of known result (1.4) and we easily arrive at the right hand side of (2.1).

This completes the proof of the theorem. ∎

3. SPECIAL CASES

In this section, we shall mention a large number of very interesting special cases of our main findings.

For this, we observe here that, if in (2.1), we let $b = -2n$ and replace a by $a + 2n$ or we let $b = -2n - 1$ and replace a by $a + 2n + 1$. In each case, one of the two terms appearing on the right-hand side of (2.1) will vanish and we get fifty interesting special cases(twenty five each) given below in the form of two corollaries.

Corollary 3.1. *For $\omega = \sqrt{-1}$ and $i, j = 0, \pm 1, \pm 2$, the following twenty five results hold true.*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ c+\frac{1}{2}\\ \frac{1}{2}(a+i+1),\ 2c+j\end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta\\ = D_{i,j}\, e^{\omega\frac{\pi}{2}c}\frac{\Gamma(c)\,\Gamma(c)}{\Gamma(2c)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{1}{2}j+\frac{1}{4}\left(1+(-1)^i\right)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n}\end{aligned} \tag{3.1}$$

where the coefficients $D_{i,j}$ are given in the Table 3.4.

Corollary 3.2. *For $\omega = \sqrt{-1}$ and $i, j = 0, \pm 1, \pm 2$, the following twenty five results hold true.*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ c+\frac{1}{2}\\ \frac{1}{2}(a+i+1),\ 2c+j\end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta\\ = E_{i,j}\, e^{\omega\frac{\pi}{2}c}\frac{\Gamma(c)\,\Gamma(c)}{\Gamma(2c)}\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{1}{2}j+\frac{1}{4}\left(1+(-1)^i\right)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n}\end{aligned} \tag{3.2}$$

where the coefficients $E_{i,j}$ are given in the Table 3.5.

In particular, in (3.1), if we take $i = j = 0$, we get the following interesting result.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ c+\frac{1}{2}\\ \frac{1}{2}(a+b+1),\ 2c\end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]dx\\ = e^{\omega\frac{\pi}{2}c}\frac{\Gamma(c)\,\Gamma(c)}{\Gamma(2c)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n}\end{aligned} \tag{3.3}$$

Similarly, in (3.2), if we take $i = j = 0$, we get the following elegant result.

$$\int_0^{\frac{\pi}{2}} e^{2\omega c\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ c+\frac{1}{2}\\ \frac{1}{2}(a+b+1),\ 2c\end{matrix};-4\omega e^{2\omega\theta}\sin\theta\cos\theta\right]d\theta = 0 \tag{3.4}$$

Similarly, we can obtain other results. We, however, prefer to omit the details.

i \ j	-2	-1	0	1	2
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1	1	$\frac{1}{2(c+1)}$
-1	$\frac{1}{(c-1)}$	1	2	2	$\frac{2}{(c+1)}$
-2	$\frac{1}{2(c-1)}$	1	1	2	$\frac{2}{(c+1)}$

Table 3.1: Table for $A_{i,j}$

$$A_{-2,-1} = 2(c-1)(a+b-1) - (a-b)^2 + 1$$
$$A_{2,2} = 2c(c+1)[(2c+1)(a+b-1) - a(a-1) - b(b-1)]$$
$$- (a-b-1)(a-b+1)[(c+1)(2c-a-b+1) + ab]$$
$$A_{-2,-2} = 2(c-1)(c-2)[(2c-1)(a+b-1) - a(a+1) - b(b+1) + 2]$$
$$- (a-b-1)(a-b+1)[(c-1)(2c-a-b-3) + ab]$$

i \ j	-2	-1	0	1	2
2	$c(a+b-1)-(a+1)(b+1)+2$	$a+b-1$	$a(2c-a)+b(2c-b)-2c+1$	$2c(a+b-1)-(a-b)^2+1$	$B_{2,2}$
1	$c-b-1$	1	1	$2c-a+b$	$2c(c+1)-(a-b)(c-b+1)$
0	$(c-a-1)(c-b-1)+(c-1)(c-2)$	1	1	1	$(c-a+1)(c-b+1)+c(c+1)$
-1	$2(c-1)(c-2)-(a-b)(c-b-1)$	$2c-a+b-2$	1	1	$c-b+1$
-2	$B_{-2,-2}$	$B_{-2,-1}$	$a(2c-a)+b(2c-b)-2c+1$	$a+b-1$	$c(a+b-1)-(a-1)(b-1)$

Table 3.2: Table for $B_{i,j}$

$$\mathcal{B}_{2,2} = 2c(c+1)\{(2c+1)(a+b-1)-a(a-1)-b(b-1)\} \\ -(a-b-1)(a-b+1)\{(c+1)(2c-a-b+1)+ab\};$$

$$\mathcal{B}_{-2,-2} = 2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\} \\ -(a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)+ab\};$$

$$\mathcal{B}_{-2,-1} = 2(c-1)(a+b-1)-(a-b)^2+1;$$

i \ j	-2	-1	0	1	2
2	-4	$-(4c-a-b-3)$	-8	$-[8c^2-2c(a+b-1)-(a-b)^2+1]$	$-4(2c+a-b+1)(2c-a+b+1)$
1	$-(c-a-1)$	-1	-1	$-(2c+a-b)$	$-[2c(c+1)+(a-b)(c-a+1)]$
0	4	1	0	-1	-4
-1	$2(c-1)(c-2)+(a-b)(c-a-1)$	$2c+a-b-2$	1	1	$c-a+1$
-2	$4(2c-a+b-3)(2c+a-b-3)$	$C_{-2,-1}$	8	$4c-a-b+1$	4

Table 3.3: Table for $C_{i,j}$

$$C_{-2,-1}=8c^2-2(c-1)(a+b+7)-(a-b)^2-7$$

i \ j	-2	-1	0	1	2
2	$\frac{(a+1)[(c-1)(a-1)+2n(a+2n)]}{(c-1)(a+4n-1)(a+4n+1)}$	$\frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-4n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-8n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$D_{2,2}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\frac{a[(c+1)(2c-a)-2n(2c+a+4n+2)]}{(c+1)(2c-a)(a+4n)}$
0	$1-\frac{2n(a+2n)}{(c-1)(2c-a-3)}$	1	1	1	$1-\frac{2n(a+2n)}{(c+1)(2c-a+1)}$
-1	$1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)}$	$1-\frac{4n}{(2c-a-2)}$	1	1	$1+\frac{2n}{(c+1)}$
-2	$D_{-2,-2}$	$1-\frac{8n(a+2n)}{(a-1)(2c-a-3)}$	$1-\frac{4n(a+2n)}{(a-1)(2c-a-1)}$	1	$1+\frac{2n(a+2n)}{(c+1)(a-1)}$

Table 3.4: Table for $D_{i,j}$

$$D_{2,2} = \frac{(a+1)[(a-1)(c+1)(2c-a+1)(2c-a-1) - 2an(6c+a+5)(2c-a+1) + 4n^2(5a^2+4a-5-4c(3c-a+4)) + 64n^3(a+n)]}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)}$$

$$D_{-2,-2} = 1 - \frac{2an(6c+a-7)(2c-a-3) - 4n^2[5a^2-4a-21-4c(3c-a-8)] - 64n^3(a+n)}{(c-1)(a-1)(2c-a-3)(2c-a-5)}$$

i \ j	-2	-1	0	1	2
2	$\frac{(a+1)(2c-a-3)}{(c-1)(a+4n+1)(a+4n+3)}$	$\frac{(a+1)(4c-a-3)}{(a+4n+1)(a+4n+3)(2c-1)}$	$\frac{2(a+1)}{(a+4n+1)(a+4n+3)}$	$E_{2,1}$	$\frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(c+1)(2c-a-1)(a+4n+1)(a+4n+3)}$
1	$\frac{(c-a-2n-2)}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(a+4n+2)(2c-1)}$	$\frac{1}{a+4n+2}$	$\frac{(2c+a+4n+2)}{(2c+1)(a+4n+2)}$	$\frac{(c+a+2)(2c-a)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)}$
0	$\frac{-1}{(c-1)}$	$\frac{-1}{(2c-1)}$	0	$\frac{1}{(2c+1)}$	$\frac{1}{(c+1)}$
-1	$E_{-1,-2}$	$\frac{-(2c+a+4n)}{a(2c-1)}$	$\frac{-1}{a}$	$\frac{-(2c-a)}{a(2c+1)}$	$\frac{-(c-a-2n)}{a(c+1)}$
-2	$\frac{-(2c+a+4n-1)(2c-a-4n-5)}{(a-1)(c-1)(2c-a-5)}$	$E_{-2,-1}$	$\frac{-2}{(a-1)}$	$\frac{-(4c-a+1)}{(a-1)(2c+1)}$	$\frac{-(2c-a+1)}{(a-1)(c+1)}$

Table 3.5: Table for $E_{i,j}$

$$E_{2,1} = \frac{(a+1)[(4c+a+3)(2c-a-1)-8n(a+2n+2)]}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)}\ ;$$
$$E_{-2,-1} = -\frac{[(4c+a-1)(2c-a-3)-8n(a+2n+2)]}{(a-1)(2c-1)(2c-a-3)}$$
$$E_{-1,-2} = -\frac{[(c+a)(2c-a-4)-2n(3a-2c+4n+6)]}{a(c-1)(2c-a-4)}$$

4. CONCLUSIONS

In this paper, we have evaluated twenty five interesting integrals involving generalized hypergeometric function in the form of a single integral. The results are established with the help of generalization of classical Watson's summation theorem obtained earlier by Lavoie et al. Fifty interesting integrals in the form of two integrals (twenty five each) have also been evaluated as special cases of our main findings. We conclude this paper by remarking that the interesting applications of the integrals obtained in this paper are under investigations and will be published soon.

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