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SOME INEQUALITIES FOR THE GENERALIZED RIESZ POTENTIAL ON THE GENERALIZED MORREY SPACES OVER HYPERGROUPS

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ABSTRACT. We present in this paper some inequalities for the generalized Riesz potential on the generalized Morrey spaces over commutative hypergroups. The results can be found by employing the maximal operator.

Key words and phrases: Commutative hypergroup; Generalized; Maximal operator; Morrey spaces; Riesz potential.

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1. INTRODUCTION

Riesz potential, also known as fractional integral operator, I_α (for $0 < \alpha < n$), is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

When $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 < p < q < \infty$, [8] and [21] proved that I_α satisfies the strong type Hardy-Littlewood-Sobolev inequality

$$\|I_\alpha f\|_{L^q} \le C\|f\|_{L^p}.$$

For this strong type inequality, people often say that I_α is bounded from the Lebesgue spaces (over the Euclidean spaces) L^p to L^q. The strong type Hardy-Littlewood-Sobolev inequality has been extended in some other spaces (see for examples [3], [6], [9], [15], [17], [18], [19], [20]) and for the generalized Riesz potential (see for example [11], [13], [14], [15], [16]). The strong type inequality for the generalized Riesz potential

$$T_\rho f(x) := \int_{\mathbb{R}^n} \frac{\rho\left(|x-y|\right)}{|x-y|^n} f(y)\, dy$$

on the generalized Morrey spaces over Eucidean spaces has been established by [2] and [5]. Later on, for a non negative almost increasing function, $h : (0,\infty) \to (0,\infty)$, [7] proved that the generalized Riesz potential on hypergroup version

$$R_h f(x) := \int_K T^x \left(\frac{h\left(\rho(e,y)\right)}{\rho(e,y)^n} \right) f(y^\sim)\, d\mu(y) = \int_K \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n}\, d\mu(y)$$

also satisfied the strong type inequalities in Lebesgue spaces and Orlicz spaces over commutative hypergroups. Here, e denotes the identity of the comutative hypergroup, and T^x (for $x \in K$) denotes the generalized translation operators, in which

$$T^x f(y) := f\left(\delta_x * \delta_y\right) = \int_K f\, d\left(\delta_x * \delta_y\right).$$

A hypergroup $(K,*)$ is a locally compact Hausdorff space K equipped with a bilinear, associative, and weakly continuous convolution $*$ on $M^b(K)$ (i.e. the set of bounded Radon measure on K) satisfying the following properties:

(1) For all $x, y \in K$, the convolution $\delta_x * \delta_y$ of the point measures is a probability measure with compact support;
(2) The mapping $(x,y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ of $K \times K$ into the space of nonempty compact support subsets of K is continuous with respect to the Michael topology;
(3) There is an identity $e \in K$ such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$;
(4) There is a continuous involution $*$ (i.e. a homeomorphism $x \mapsto x^\sim$ of K onto itself with the property $(x^\sim)^\sim = x$ for all $x \in K$) such that $\delta_{x^\sim} * \delta_{y^\sim} = (\delta_x * \delta_y)^\sim$;
(5) For $x, y \in K$, we have $e \in \operatorname{supp}(\delta_x * \delta_y)$ if only if $x = y^\sim$.

We could find the definition of hypergroups in [1] and [10]. If $\delta_x * \delta_y = \delta_y * \delta_x$ for every x, y in K, then the hypergroup $(K,*)$ is a commutative hypergroup. For the simplicity, in the rest of the paper, we will write the comutative hypergroups $(K,*)$ as K only. According [4], every commutative hypergroup posseses a Haar measure (denoted by μ), that is μ satisfies

$$\int_K \int_K f\left(\delta_x * \delta_y\right)\, d\mu(y) = \int_K f(y)\, d\mu(y)$$

for every $x \in K$ and every $f \in C_c(K)$.

For the generalized Riesz potential R_h, in this paper, we will extend the inequalities established by [2] and [5] into the generalized Morrey spaces over hypergroups. Here, the generalized Morrey space $\mathcal{M}^{p,\phi}(K)$ for $1 < p < \infty$ consists of function in $L^p_{\text{loc}}(K)$ in which

$$\|f\|_{\mathcal{M}^{p,\phi}(K)} := \sup_{B=B(e,r)} \frac{1}{\phi\left(\mu\left(B\left(e,2r\right)\right)\right)} \left(\frac{\int_{B(e,r)} T^x|f(y^\sim)|^p \, d\mu(y)}{\mu\left(B\left(e,2r\right)\right)}\right)^{\frac{1}{p}}$$

is finite. We assume throughout this paper that $\phi : (0,\infty) \to (0,\infty)$ is an almost decreasing function.

2. MAIN RESULTS

In this section, we employ the condition of upper Ahlfors n-reguler by identity, that is

$$\mu(B(e,r)) \leq Cr^n \tag{2.1}$$

for some constant $C > 1$. This cosnstant is independent of $r > 0$. Furthermore, based on the result of [12] and the result on the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{\mu B(e,2r)} \int_{B(e,r)} T^x|f(y^\sim)| d\lambda(y)$$

given by [7], we find that the maximal operator is a bounded operator on the generalized Morrey spaces over commutative hypergroups $\mathcal{M}^{p,\phi}(K), 1 < p < \infty$, that is

$$\|Mf\|_{\mathcal{M}^{p,\phi}(K)} \leq C\|f\|_{\mathcal{M}^{p,\phi}(K)}, \tag{2.2}$$

if

$$\int_r^\infty \frac{\Phi(t)^p}{t} \leq C_1 \Phi(r)^p$$

for $C_1, t, r > 0$ and $g : t \to t^n, \Phi(t) = \phi(g(t))$. This enables us to find the following theorem.

Theorem 2.1. *Suppose that $\psi : (0,\infty) \to (0,\infty)$ is an almost decreasing function and $g : t \to t^n, \Phi(t) = \phi(g(t)), \Psi(t) = \psi(g(t))$ for positive t. If $1 < p < \infty$, $\int_r^\infty \frac{\Phi(t)^p}{t} \leq C_1 \Phi(r)^p$ and*

$$\Phi(r) \int_0^r \frac{h(t)}{t} \, dt + \int_r^\infty \frac{h(t)\Phi(t)}{t} \, dt \leq C_2 \Psi(r)$$

for $C_1, C_2 > 0$, then the inequality

$$\|R_h f\|_{\mathcal{M}^{p,\psi}(K)} \leq C\|f\|_{\mathcal{M}^{p,\phi}(K)},$$

holds.

Proof. We firstly write the generalized Riesz potential R_h into

$$\begin{aligned} R_h f(x) &= \int_{B(e,r)} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n} \, d\mu(y) + \int_{K \setminus B(e,r)} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n} \, d\mu(y) \\ &= A_1(x) + A_2(x). \end{aligned}$$

Then, we will estimate $|A_1(x)|$ and $|A_2(x)|$. The estimate for $|A_1(x)|$ is given by

$$\begin{aligned}|A_1(x)| &\le \sum_{j=-\infty}^{-1} \int_{2^j r\le \rho(e,y)<2^{j+1}r} \frac{h\left(\rho(e,y)\right) T^x f(y^{\sim})}{\rho(e,y)^n}\, d\mu(y)\\ &\le C \sum_{j=-\infty}^{-1} \frac{h\left(2^j r\right)}{\left(2^j r\right)^n} \int_{B(e,2^{j+1}r)} T^x |f(y^{\sim})|\, d\mu(y)\\ &\le C \sum_{j=-\infty}^{-1} \frac{h\left(2^j r\right)}{\left(2^{j+2} r\right)^n} \int_{B(e,2^{j+1}r)} T^x |f(y^{\sim})|\, d\mu(y).\end{aligned}$$

The condition of upper Ahlfors n-reguler by identity

$$\mu\left(B\left(e, 2^{j+2}r\right)\right) \le C\left(2^{j+2}r\right)^n$$

gives us

$$\begin{aligned}|A_1(x)| &\le C \sum_{j=-\infty}^{-1} \frac{h\left(2^j r\right)}{\mu\left(B\left(e,2^{j+2}r\right)\right)} \int_{B(e,2^{j+1}r)} T^x |f(y^{\sim})|\, d\mu(y)\\ &\le CMf(x) \sum_{j=-\infty}^{-1} h\left(2^j r\right).\end{aligned}$$

If $2^j r < t < 2^{j+1} r$, then $h\left(2^j r\right) < h(t) < h\left(2^{j+1} r\right)$. As a result

$$h\left(2^j r\right) \le C \int_{2^j r}^{2^{j+1} r} \frac{h(t)}{t}\, dt.$$

The last inquality give us

$$\begin{aligned}|A_1(x)| &\le CMf(x) \sum_{j=-\infty}^{-1} \int_{2^j r}^{2^{j+1} r} \frac{h(t)}{t}\, dt\\ &\le CMf(x) \int_0^r \frac{h(t)}{t}\, dt\\ &\le CMf(x) \frac{\Psi(r)}{\Phi(r)}\\ &\le CMf(x) \frac{\psi((r)^n)}{\phi(r^n)}\end{aligned}$$

Now, we could state the estimate for $|A_2(x)|$, that is

$$\begin{aligned}
|A_2(x)| &\leq \sum_{j=0}^{\infty} \int_{2^j r\leq \rho(e,y)<2^{j+1}r} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n}\, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} \frac{h\left(2^{j+1}r\right)}{\left(2^j r\right)^n} \int_{B(e,2^{j+1}r)} T^x |f(y^\sim)|\, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} \frac{h\left(2^{j}r\right)}{\left(2^{j+2} r\right)^n} \int_{B(e,2^{j+1}r)} T^x |f(y^\sim)|\, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} \frac{h\left(2^{j+1}r\right)}{\mu\left(B\left(e,2^{j+2}r\right)\right)} \int_{B(e,2^{j+1}r)} T^x |f(y^\sim)|\, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} h\left(2^{j+1}r\right) \left[\frac{\int_{B(e,2^{j+1}r)} T^x|f(y^\sim)|^p d\mu(y)}{\mu\left(B\left(e,2^{j+2}r\right)\right)}\right]^{\frac{1}{p}} \left[\frac{\int_{B(e,2^{j+1}r)} d\mu(y)}{\mu\left(B\left(e,2^{j+2}r\right)\right)}\right]^{1-\frac{1}{p}}.
\end{aligned}$$

By applying the condition of upper Ahlfors n-reguler by identity and the almost decreasing assumption of ϕ, we find that

$$\begin{aligned}
|A_2(x)| &\leq C \sum_{j=0}^{\infty} h\left(2^{j+1}r\right) \phi\left(\mu\left(B\left(e,2^{j+2}r\right)\right)\right) \|f\|_{\mathcal{M}^{p,\phi}(K)} \\
&\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)} \sum_{j=0}^{\infty} h\left(2^{j+1}r\right) \phi\left(C_3\left(2^{j+2}r\right)^n\right) \\
&\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)} \sum_{j=0}^{\infty} h\left(2^{j+1}r\right) \Phi\left(C_4\left(2^{j+1}r\right)\right) \\
&\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)} \sum_{j=0}^{\infty} \int_{2^j r}^{2^j r} \frac{h(t)\Phi(t)}{t} \\
&\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)} \int_{r}^{\infty} \frac{h(t)\Phi(t)}{t} \\
&\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)} \Psi(r) \\
&\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)} \psi\left(r^n\right)
\end{aligned}$$

for $C_3, C_4 > 0$.

From the estimate of $|A_1(x)|$ and equation (2.2), we get

$$\begin{aligned}
\frac{1}{\psi(\mu(B(e,2r)))^p \mu(B(e,2r))} \int_{B(e,r)} |A_1(x)|^p\, d\mu(x) &\leq C\|Mf\|^p_{\mathcal{M}^{p,\phi}(K)} \\
&\leq C\|f\|^p_{\mathcal{M}^{p,\phi}(K)}.
\end{aligned}$$

Meanwhile, from the estimate of $|A_2(x)|$, we get

$$\frac{1}{\psi(\mu(B(e,2r)))^p \mu(B(e,2r))} \int_{B(e,r)} |A_2(x)|^p\, d\mu(x) \leq C\|f\|^p_{\mathcal{M}^{p,\phi}(K)}.$$

As $R_h f(x) = A_1(x) + A_2(x)$, we find that

$$\frac{1}{\psi(\mu(B(e,2r)))^p \mu(B(e,2r))} \int_{B(e,r)} |R_h f(x)|^p\, d\mu(x) \leq C\|f\|^p_{\mathcal{M}^{p,\phi}(K)}.$$

This leads us to

$$\|R_h f\|_{\mathcal{M}^{p,\psi}(K)} \leq C\|f\|_{\mathcal{M}^{p,\phi}(K)},$$

which is our desired strong type inequality. ∎

Another inequality regarding the generalized Riesz potential R_h is presented in the following theorem.

Theorem 2.2. *Suppose that* $g : t \to t^n, \Phi(t) = \phi(g(t)), \Psi(t) = \psi(g(t))$ *for* $t > 0$ *and* Φ *is surjective. If* $1 < p < q < \infty$, $\int_r^\infty \frac{\Phi(t)^p}{t} \leq C_1 \Phi(r)^p$, *and*

$$\Phi(r) \int_0^r \frac{h(t)\Phi(t)}{t}\, dt + \int_r^\infty \frac{h(t)}{t}\, dt \leq C_2 \Phi(r)^{\frac{p}{q}}$$

for $C_1, C_2 > 0$, *then*

$$|R_h f(x)| \leq C\left(Mf(x)\right)^{\frac{p}{q}} \|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\phi}(K)}. \tag{2.3}$$

Proof. For any $r > 0$ and $f \in \mathcal{M}^{p,\phi}(K)$, where $1 < p < \infty$, we decompose R_h into

$$\begin{aligned} R_h f(x) &= \int_{B(e,r)} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n}\, d\mu(y) + \int_{K\setminus B(e,r)} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n}\, d\mu(y) \\ &= A_1(x) + A_2(x). \end{aligned}$$

By applying equation (2.1), we find the estimate for $|A_1(x)|$, that is

$$\begin{aligned} |A_1(x)| &\leq \sum_{j=-\infty}^{-1} \int_{2^j r \leq \rho(e,y) < 2^{j+1} r} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n}\, d\mu(y) \\ &\leq C \sum_{j=-\infty}^{-1} \frac{h\left(2^j r\right)}{\mu\left(B\left(e, 2^{j+2} r\right)\right)} \int_{B(e,2^{j+1}r)} T^x |f(y^\sim)|\, d\mu(y) \\ &\leq CMf(x) \sum_{j=-\infty}^{-1} h\left(2^j r\right) \\ &\leq CMf(x) \sum_{j=-\infty}^{-1} \int_{2^j r}^{2^{j+1} r} \frac{h(t)}{t}\, dt \\ &\leq CMf(x) \int_0^r \frac{h(t)}{t}\, dt \\ &\leq CMf(x)\Phi(r)^{\frac{p}{q}-1}. \end{aligned}$$

Meanwhile, the estimate for $|A_2(x)|$ is provided by

$$\begin{aligned}
|I_2(x)| &\leq \sum_{j=0}^{\infty} \int_{2^j r \leq \rho(e,y) < 2^{j+1} r} \frac{h\left(\rho(e,y)\right) T^x f(y^\sim)}{\rho(e,y)^n} \, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} \frac{h\left(2^{j+1} r\right)}{\mu\left(B\left(e, 2^{j+2} r\right)\right)} \int_{B(e,2^{j+1}r)} T^x |f(y^\sim)| \, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} h\left(2^{j+1} r\right) \left[\frac{\int_{B(e,2^{j+1}r)} T^x |f(y^\sim)|^p d\mu(y)}{\mu\left(B\left(e, 2^{j+2} r\right)\right)}\right]^{\frac{1}{p}} \left[\frac{\int_{B(e,2^{j+1}r)} d\mu(y)}{\mu\left(B\left(e, 2^{j+2} r\right)\right)}\right]^{1-\frac{1}{p}} \\
&\leq C \|f\|_{\mathcal{M}^{p,\phi}(K)} \sum_{j=0}^{\infty} h\left(2^{j+1} r\right) \phi\left(C_2 \left(2^{j+2} r\right)^n\right) \\
&\leq C \|f\|_{\mathcal{M}^{p,\phi}(K)} \sum_{j=0}^{\infty} h\left(2^{j+1} r\right) \Phi\left(C_3 \left(2^{j+1} r\right)\right) \\
&\leq C \|f\|_{\mathcal{M}^{p,\phi}(K)} \sum_{j=0}^{\infty} \int_{2^j r}^{2^j r} \frac{h(t)\Phi(t)}{t} \\
&\leq C \|f\|_{\mathcal{M}^{p,\phi}(K)} \int_r^{\infty} \frac{h(t)\Phi(t)}{t} \\
&\leq C \|f\|_{\mathcal{M}^{p,\phi}(K)} \Phi(r)^{\frac{p}{q}}.
\end{aligned}$$

Having surjective Φ, we take

$$\Phi(r) = \frac{Mf(x)}{\|f\|_{\mathcal{M}^{p,\phi}(K)}}.$$

Therefore, the estimate $|A_1(x)|$ and $|A_2(x)|$ give us

$$\begin{aligned}
|R_h f(x)| &\leq C \left((\Phi(r))^{\frac{p}{q}-1} Mf(x) + \|f\|_{\mathcal{M}^{p,\phi}(K)} (\Phi(R))^{\frac{p}{q}} \right) \\
&\leq C \left(\left(\frac{Mf(x)}{\|f\|_{\mathcal{M}^{p,\phi}(K)}} \right)^{\frac{p}{q}-1} Mf(x) + \|f\|_{\mathcal{M}^{p,\phi}(K)} \left(\frac{Mf(x)}{\|f\|_{\mathcal{M}^{p,\phi}(K)}} \right)^{\frac{p}{q}} \right) \\
&= C \left(Mf(x)\right)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\phi}(K)}^{1-\frac{p}{q}}.
\end{aligned}$$

Hence, the desired inequality is proved. ∎

From Theorem 2.2, we can derive the following theorem.

Theorem 2.3. *Suppose that positive* t *we have* $g : t \to t^n, \Phi(t) = \phi(g(t)), \Psi(t) = \psi(g(t))$ *for* $t > 0$. *If* $1 < p < q < \infty$, Φ *is surjetive and satisfies*

$$\int_r^{\infty} \frac{\Phi(t)^p}{t} \leq C_1 \Phi(r)^p$$

and

$$\Phi(r) \int_0^r \frac{h(t)\Phi(t)}{t} \, dt + \int_r^{\infty} \frac{h(t)}{t} \, dt \leq C_1 \Phi(r)^{\frac{p}{q}}$$

for positive C_1 *and* $C_2 > 0$, *then we have*

$$\|R_h f\|_{\mathcal{M}^{q,\phi^{p/q}}(K)} \leq C \|f\|_{\mathcal{M}^{p,\phi}(K)}.$$

Proof. Equation (2.3) gives us

$$\begin{aligned}
&\frac{1}{\phi\left(\mu\left(B(e,2r)\right)\right)^{p/q}}\left[\frac{1}{\mu(B(e,r))}\int_{B(e,r)}|R_h f(x)|^q\,d\mu(x)\right]^{1/q}\\
&\quad\leq \frac{C}{\phi\left(\mu\left(B(e,2r)\right)\right)^{p/q}}\left[\frac{1}{\mu(B(e,r))}\int_{B(e,r)}(Mf(x))^p\,\|f\|^{q-p}_{\mathcal{M}_{p,\phi}(K)\ d\mu(x)}\right]^{1/q}\\
&\leq C\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\phi}(K)}\left[\frac{C}{\phi\left(\mu\left(B(e,2r)\right)\right)}\left[\frac{1}{\mu(B(e,r))}\int_{B(e,r)}(Mf(x))^p\ d\mu(x)\right]^{1/p}\right]^{p/q}\\
&\leq C\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\phi}(K)}\|Mf\|^{p/q}_{\mathcal{M}^{p,\phi}(K)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\frac{1}{\phi\left(\mu\left(B(e,2r)\right)\right)^{p/q}}\left[\frac{1}{\mu(B(e,r))}\int_{B(e,r)}|R_h f(x)|^q\,d\mu(x)\right]^{1/q}\\
&\qquad\leq C\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\phi}(K)}\|f\|^{p/q}_{\mathcal{M}^{p,\phi}(K)}\\
&\qquad\leq C\|f\|_{\mathcal{M}^{p,\phi}(K)}
\end{aligned}$$

as the maximal operator M is bounded on $\mathcal{M}^{p,\phi}(K)$. ∎

3. CONCLUDING REMARKS

If $\phi(t) = t^{\frac{1}{p}}\left(\frac{\lambda}{n}-1\right)$, the spaces $\mathcal{M}^{p,\phi}(K)$ is reduced to Morrey spaces over commutative hypergroups $L^{p,\lambda}(K)$. So, our results in Theorem 2.1, 2.2, and 2.3 work in Morrey spaces over commutative hypergroups by taking $\phi(t) = t^{\frac{1}{p}}\left(\frac{\lambda}{n}-1\right)$ and some other conditions.

REFERENCES

[1] W. R. BLOOM and H. HEYER, *Harmonic Analysis of Probability Measures on Hypergroups,* de Gruyter, 1995.

[2] ERIDANI, On the boundedness of generalized fractional integral on the generalized Morrey Sapces, *Tamkang J. Math.*, **33**(4) (2004), pp. 245–261 .

[3] J. GARCÍA-CUERVA and J. M. MARTELL, Two weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces, *Indiana Univ. Math. J.* **50** (2001), PP. 1241–1280.

[4] G. GIGANTE, Transference for hypergroups. *Collect. Math.* **52**(2) (2001), pp. 127–155.

[5] H. GUNAWAN, A note on the generalized fractional integral operators, *J. Indones. Math. Soc. (MIHMI)* **9**(1) (2003), pp. 39–43.

[6] D. V. GORBACHEV, V. I. IVANOV and S. Y. TIKHONOV, Riesz ootential and maximal function for Dunkl transform. *Potential Anal.* **55** (2021), pp. 513–538.

[7] M. G. HAJIBAYOV, Generalized potentials on commutative hypergroups, *Azerbaijan J. Math* **5**(2) (2015), pp. 37–46.

[8] G.H. HARDY and J.E. LITTLEWOOD, Some properties of fractional integrals. I, *Math. Zeit.* **27** (1927), 565–606.

[9] K. P. HO, Potential type operators on weighted Morrey spaces, *Arch. Math.* **118** (2022), pp. 159–168.

[10] R. I. JEWETT, Spaces with an abstract convolution of measure, *Adv. Math.* **18** (1975), 1–101.

[11] A. KUCUKASLAN, Equivalence of norms of the generalized fractional integral operator and the generalized fractional maximal operator on the generalized weighted Morrey spaces,*Ann. Funct. Anal.* **11** (2020), pp. 1007–1026 .

[12] E. NAKAI, Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces, *Math. Nachr.* **166** (1994), pp. 95–103.

[13] E. NAKAI, Generalized fractional integrals on generalized Morrey spaces, *Math. Nachr.* **287**(2-3) (2014), pp. 339–351.

[14] E. NAKAI and G. SADASUE, Characterizations of boundedness for generalized fractional integrals on martingale Morrey spaces, *Math. Inequal. Appl.* **20**(4) (2017), pp. 929–947.

[15] Y. SAWANO, S. SUGANO, and H. TANAKA, Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces, *Trans. Amer. Math. Soc.*, **263**(12) (2011), pp. 6481–6503, .

[16] Y. SAWANO, and T. SHIMOMURA, Generalized fractional integral operators on generalized Orlicz-Morrey spaces of the second kind over non-doubling metric measure spaces, *Georgian Mathematical Journal*, **25** (2) (2018), pp. 303–311,

[17] I. SIHWANINGRUM, H. GUNAWAN, and E. NAKAI, Maximal and fractional integral operators on generalized Morrey spaces over metric measure spaces *Math. Nachr.* **291** (2018), pp. 1400–1417.

[18] I. SIHWANINGRUM, H. P. SURYAWAN and H. GUNAWAN, Fractional integral operators and Olsen inequalities on non-homogeneous spaces, *J. Math. Anal. Appl.* **7** (2010), No. 1, Article 14, [`Online: https://ajmaa.org/cgi-bin/paper.pl?string=v7n1/V7I1P14.tex`].

[19] I. SIHWANINGRUM, S. MARYANI and H. GUNAWAN, Extension of Hardy-Littlewood-Sobolev inequalities for Riesz potentials on hypergroups, *Mediterr. J. Math.* **17** (2020), pp. 203.

[20] I. SIHWANINGRUM, A. WARDAYANI and Y. DASRIL, An Adams type inequality of fractional integral operator on hypergroups, *J. Phys.: Conf. Ser.* **1494** (2020), 012019 .

[21] S. L. Sobolev, On a theorem in functional analysis (Russian), *Mat. Sob.* **46** (1938), pp. 471–497, [English translation in *Amer. Math. Soc. Transl. ser. 2*, **34** (1963), pp. 39–68].