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ON THE INTEGRAL EQUATIONS VIA HARTLEY SUPERCONVOLUTIONS

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ABSTRACT. The new superconvolutions for the Hartley integral transforms are formulated and their properties are studied. Furthermore, we also formulate the Young type theorem for these superconvolutions in the function space $L_s^{\alpha,\beta,\gamma}(\mathbb{R})$ and get its norm estimations. We apply them to solve integral equations and system of integral equations of Toeplitz plus Hankel type.

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1. INTRODUCTION

The Hartley integral transforms are closely related to the Fourier integral transform [2, 9]

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-iyx} dy, x \in \mathbb{R},$$

but they transform real-valued functions into real-valued functions [4]

$$\left(H_{\begin{Bmatrix}1\\2\end{Bmatrix}} f \right)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \operatorname{cas}(\pm xy)\, dy, y \in \mathbb{R},$$

where $\operatorname{cas}(u) = \cos u + \sin u$. The Hartley transforms are involutive

$$H_{\begin{Bmatrix}1\\2\end{Bmatrix}} \left(H_{\begin{Bmatrix}1\\2\end{Bmatrix}} f \right)(y) = f(x),$$

and unitary

$$\left\| H_{\begin{Bmatrix}1\\2\end{Bmatrix}} f \right\|_2 = \|f\|_2.$$

The convolution for the Hartley transform [5, 6]

(1.1)
$$\left(f \underset{H_1}{*} g \right)(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \left[g(x+u) + g(x-u) + g(u-x) - g(-x-u) \right] du,$$

satisfies the factorization property

$$H_1 \left(f \underset{H_1}{*} g \right)(y) = (H_1 f)(y) (H_1 g)(y). \tag{1.2}$$

The generalized convolution for the Hartley integral transform $\left(f \underset{H_1,H_1,H_2}{*} g \right)$ [2], is defined by

$$\begin{aligned} &\left(f \underset{H_1 H_1,H_2}{*} g \right)(x) \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[f(x+t) + f(x-y) + f(-x+t) + f(-x-t) \Big] g(t) dt, x \in \mathbb{R}, \end{aligned} \tag{1.3}$$

satisfies the following factorization property [2]

$$H_1 \left[\left(f \underset{H_1,H_1,H_2}{*} g \right) \right](y) = (H_1 f)(y)(H_2 g)(y). \tag{1.4}$$

The generalized convolution for the Hartley integral transform $\left(f \underset{H_2,H_1,H_1}{*} g \right)$ [2], is defined by

$$\begin{aligned} &\left(f \underset{H_2,H_1,H_1}{*} g \right)(x) \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[f(x+t) - f(x-y) + f(-x+t) + f(-x-t) \Big] g(t) dt, x \in \mathbb{R}, \end{aligned} \tag{1.5}$$

satisfies the following factorization property [2]

$$H_2\left[\left(f \underset{H_2,H_1,H1}{*} g\right)\right](y) = (H_1 f)(y)(H_1 g)(y). \tag{1.6}$$

In [10] the Hartley transform was used to solve in closed form some special cases of the Toeplitz plus Hankel integral equation [4, 13]

$$f(x) + \int_0^\infty f(u)\left[k_1(x+u) + k_2(x-u)\right] du = g(x), x > 0, \tag{1.7}$$

where g, k_1, k_2 are given, and f is an unknown function. This equation has many useful applications [8, 10, 11, 13]. However, it can be solved in closed form only in some particular cases of the Hankel kernel k_1 and the Toeplitz kernel k_2 [8, 10, 11, 13]. The solution of the integral Equation(1.7) in closed form in the general case is still open. The Hartley superconvolution was introduced in 2018 by N. M. Khoa, N. X. Thao and V. K. Tuan [7].

In this paper, we introduce and study some superconvolutions related to the Hartley integral transform. We apply these superconvolutions so solve some integral equations similar to the Toeplitz plus Hankel equation, but on the whole real line and system of two Toeplitz plus Hankel integral equations.

2. SOME NEW HARTLEY SUPERCONVOLUTIONS.

Definition 2.1. The Hartley superconvolutions of two functions f, g are defined by

$$\left(f \underset{1}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\left[g(x+u) + g(-x-u)\right] du, x \in \mathbb{R}, \tag{2.1}$$

$$\left(f \underset{2}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\left[g(-x+u) - g(x-u)\right] du, x \in \mathbb{R}. \tag{2.2}$$

Theorem 2.1. *Assume that $f, g \in L_1(\mathbb{R})$. Then the superconvolutions $\left(f \underset{k}{*} g\right)$, $k = 1, 2$ belong to $L_1(\mathbb{R})$ and satisfy following additive factorization indentities*

$$a)\ H_1\left(f \underset{1}{*} g\right)(y) = (H_2 f)(y)(H_2 g)(y) + (H_2 f)(y)(H_1 g)(y), \forall y \in \mathbb{R}, \tag{2.3}$$

$$H_2\left(f \underset{1}{*} g\right)(y) = (H_1 f)(y)(H_1 g)(y) + (H_1 f)(y)(H_2 g)(y), \forall y \in \mathbb{R}. \tag{2.4}$$

$$b)\ H_1\left(\underset{2}{*} g\right)(y) = (H_2 f)(y)(H_2 g)(y) - (H_2 f)(y)(H_1 g)(y), \forall y \in \mathbb{R}, \tag{2.5}$$

$$H_2\left(f \underset{2}{*} g\right)(y) = (H_1 f)(y)(H_1 g)(y) - (H_1 f)(y)(H_2 g)(y), \forall y \in \mathbb{R}. \tag{2.6}$$

Remark: The additive factorization formulas (2.3), (2.4), (2.5) and (2.6) make (2.1) and (2.2) different from classical convolutions, generalized convolutions, when only produtcs are involved in factorization indentities. It justifies why we call (2.1) and (2.2) superconvolution.

Proof. a) Now we prove that $\left(f \underset{1}{*} g\right)(x) \in L_1(\mathbb{R})$ for $f, g \in L_1(\mathbb{R})$. We have

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\left(f \underset{1}{*} g\right)(x)\right| dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \left|f(u)\left[g(x+u) + g(-x-u)\right]\right| du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(u)|\,|g(x+u)|\, du \\ &+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(u)|\,|g(-x-u)|\, du.\end{aligned}$$

With the substitutions $x+u=v$ and $-x-u=v$, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}|f(u)|\,|g(x+u)|\,du=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)|\,|g(v)|\,dudv,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}|f(u)|\,|g(-x-u)|\,du=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)|\,|g(v)|\,dudv$$

Hence

$$\int_{-\infty}^{\infty}\left|\left(f\underset{1}{*}g\right)(x)\right|dx\le 2.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(u)|\,du.\int_{-\infty}^{\infty}|g(v)|\,dv<\infty.$$

So $\left(f\underset{1}{*}g\right)(x)$ is well defined, belonging to space $L_1(\mathbb{R})$.

Now, we prove the factorization property (2.3). We have

$$\begin{aligned}
&(H_2f)(y)\,(H_2g)(y)+(H_2f)(y)\,(H_1g)(y)\\
&\qquad=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(-uy)\,f(u)\,du.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(-vy)\,g(v)\,dv\\
&\qquad+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(-uy)\,f(u)\,du.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(vy)\,g(v)\,dv\\
&\qquad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{cas}(-uy)\,f(u)\,du.\int_{-\infty}^{\infty}[\operatorname{cas}(-vy)+\operatorname{cas}(vy)]\,g(v)\,dv
\end{aligned}$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{cas}(-uv)\cos(vy)\,f(u)\,g(v)\,dudv. \tag{2.7}$$

On the other hand, one has

$$\begin{aligned}
\operatorname{cas}(-uy).\cos(vy)&=[\cos(uy)-\sin(uy)].\cos(vy)\\
&=\frac{1}{2}[\cos(u+v)y+\cos(u-v)y]-\frac{1}{2}[\sin(u+v)y+\sin(u-v)y]
\end{aligned}$$

$$=\frac{1}{2}\left\{\operatorname{cas}[-(u+v)y]+\operatorname{cas}[-(u-v)y]\right\}. \tag{2.8}$$

From (2.7) and (2.8), it follows that

$$\begin{aligned}
&(H_2f)(y)\,(H_2g)(y)+(H_2f)(y)\,(H_1g)(y)\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\operatorname{cas}[-(u+v)y]+\operatorname{cas}[-(u-v)y]\right\}f(u)\,g(v)\,dudv\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\operatorname{cas}(yx).g[-(u+x)]\,dx\right\}f(u)\,du\\
&\quad+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\operatorname{cas}(yx).g(u+x)\,dx\right\}f(u)\,du\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u).[g(x+u)+g(-x-u)]\operatorname{cas}(yx)\,dudx\\
&\quad=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(f\underset{1}{*}g\right)(x)\operatorname{cas}(yx)\,dx\\
&\quad=H_1\left(f\underset{1}{*}g\right)(y).
\end{aligned}$$

Thus, the identity (2.3) holds true. The identity (2.4) can easily get in a similar way.

b) We prove that $\left(f \underset{2}{*} g\right)(x) \in L_1(\mathbb{R})$ for $f, g \in L_1(\mathbb{R})$. We have

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\left(f \underset{2}{*} g\right)(x)\right| dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty}\left|f(u)\left[g(-x+u)-g(x-u)\right]\right| du \\ &\leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty}|f(u)|\,|g(-x+u)|\, du \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty}|f(u)|\,|g(x-u)|\, du.\end{aligned}$$

With the substitutions $-x+u=v$ and $x-u=v$, we get

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty}|f(u)|\,|g(-x+u)|\, du &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)|\,|g(v)|\, dudv, \\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty}|f(u)|\,|g(x-u)|\, du &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)|\,|g(v)|\, dudv.\end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty}\left|\left(f \underset{2}{*} g\right)(x)\right| dx \leq 2.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(u)|\, du.\int_{-\infty}^{\infty}|g(v)|\, dv < \infty.$$

Hence, $\left(f \underset{2}{*} g\right)(x)$ is well defined, belonging to space $L_1(\mathbb{R})$.

Now, we prove the factorization property (2.5). We have

$$\begin{aligned}&(H_2 f)(y)(H_2 g)(y) - (H_2 f)(y)(H_1 g)(y) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(-uy) f(u)\, du.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(-vy) g(v)\, dv \\ &- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(-uy) f(u)\, du.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{cas}(vy) g(v)\, dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{cas}(-uy) f(u)\, du.\int_{-\infty}^{\infty}\left[\operatorname{cas}(-vy)-\operatorname{cas}(vy)\right] g(v)\, dv \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{cas}(-uy)\sin(vy) f(u) g(v)\, dudv.\end{aligned}$$

From $\operatorname{cas}(-uy).sin(vy) = \frac{1}{2}\left[\operatorname{cas}(u+v)y - \operatorname{cas}(u-v)y\right]$, we get

$$\begin{aligned}&(H_2 f)(y)(H_2 g)(y) - (H_2 f)(y)(H_1 g)(y) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{-\operatorname{cas}\left[(u+v)y\right] + \operatorname{cas}\left[(u-v)y\right]\right\} f(u) g(v)\, dudv.\end{aligned}$$

With the suitable change of variable, we obtain

$$\begin{aligned}&(H_2 f)(y)(H_2 g)(y) - (H_2 f)(y)(H_1 g)(y) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{cas}(yx) f(u)\left[g(-x+u)-g(x-u)\right] dudx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(f \underset{2}{*} g\right)(x).\operatorname{cas}(yx)\, dx \\ &= H_1\left(f \underset{2}{*} g\right)(y).\end{aligned}$$

Thus, the identity (2.5) holds true. The identity (2.6) can easily get in a similar way. The theorem is proved. ∎

3. YOUNG TYPE THEOREM

In this part, we study Hartley superconvolutions in the function space $L_s^{\alpha,\beta,\gamma}(\mathbb{R})$ and get its norm estimations.

Theorem 3.1. *(*Young type theorem*) Let* $f \in L_p(\mathbb{R}),\ g \in L_q(\mathbb{R}),\ h \in L_r(\mathbb{R})$*, such that* $p, q, r > 1$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$*. Then the following estimations hold*

$$a) \left|\int_{-\infty}^{\infty} \left(f \underset{1}{*} g\right)(x)\, h(x) dx\right| \leq \|f\|_{L_p(\mathbb{R})}.\|g\|_{L_q(\mathbb{R})}.\|h\|_{L_r(\mathbb{R})}, \tag{3.1}$$

$$b) \left|\int_{-\infty}^{\infty} \left(f \underset{2}{*} g\right)(x)\, h(x) dx\right| \leq \|f\|_{L_p(\mathbb{R})}.\|g\|_{L_q(\mathbb{R})}.\|h\|_{L_r(\mathbb{R})}. \tag{3.2}$$

Proof. Let $p_1,\ q_1,\ r_1$ be the conjugate exponentials of $p,\ q,\ r$: $\frac{1}{p} + \frac{1}{p_1} = 1$; $\frac{1}{q} + \frac{1}{q_1} = 1$; $\frac{1}{r} + \frac{1}{r_1} = 1$. Then, we have

$$\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} = 1.$$

Set

$$U(x,u) = |g(x+u) + g(-x-u)|^{\frac{q}{p_1}}\, |h(x)|^{\frac{r}{p_1}} \in L_{p_1}(\mathbb{R}^2);$$
$$V(x,u) = |h(x)|^{\frac{r}{q_1}}.|f(u)|^{\frac{p}{q_1}} \in L_{q_1}(\mathbb{R}^2);$$
$$W(x,u) = |f(u)|^{\frac{p}{r_1}}.\, |g(x+u) + g(-x-u)|^{\frac{q}{r_1}} \in L_{r_1}(\mathbb{R}^2).$$

Then, we get

$$(U.V.W)(x,u) = |f(u)|.|h(x)|.\, |g(x+u) + g(-x-u)|\, . \tag{3.3}$$

Using the definition of the norm on the space $L_{p_1}(\mathbb{R}^2)$ and with the help of Fubini theorem, we can write

$$\begin{aligned}\|U\|^{p_1}_{L_{p_1}(\mathbb{R}^2)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x+u) + g(-x-u)|^q.|h(x)|^r dxdu \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |g(x+u) + g(-x-u)|^q du\right).|h(x)|^r dx.\end{aligned}$$

Given that function $t^q\ (q > 1)$ is a convex function, changing variable results in

$$\begin{aligned}\int_{-\infty}^{\infty} |g(x+u) + g(x-u)|^q du &\leq 2^{q-1}\left(\int_{-\infty}^{\infty} |g(x+u)|^q du + \int_{-\infty}^{\infty} |g(-x-u)|^q du\right) \\ &= 2^q \int_{-\infty}^{\infty} |g(u)|^q du.\end{aligned}$$

From the above inequality, we have

$$\begin{aligned}\|U\|^{p_1}_{L_{p_1}(\mathbb{R}^2)} &\leq 2^q \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |g(u)|^q du\right).|h(x)|^r dx \\ &= 2^q \|g\|^q_{L_q(\mathbb{R})}.\|h\|^r_{L_r(\mathbb{R})}.\end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\|W\|^{r_1}_{L_{r_1}(\mathbb{R}^2)} \leq 2^q \|f\|^p_{L_p(\mathbb{R})}.\|g\|^q_{L_q(\mathbb{R})}. \tag{3.5}$$

Obviously,

$$\|V\|_{L_{q_1}(\mathbb{R}^2)} \leq \|f\|^p_{L_p(\mathbb{R})}.\|h\|^r_{L_r(\mathbb{R})}. \tag{3.6}$$

From (3.3) and (3.6), using Hölder inequality for three function, we get

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\left(f\underset{1}{*}g\right)(x)\,h(x)dx\right| &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)|.|g(x+u)+g(-x-u)|^{q}.h(x)dx\\ &= \int_{-\infty}^{\infty}U(x,u)V(x,u)W(x,u)dudx\\ &\leq \frac{1}{2}\|U\|_{L_{p_1}(\mathbb{R}^2)}.\|V\|_{L_{q_1}(\mathbb{R}^2)}.\|W\|_{L_{r_1}(\mathbb{R}^2)}\\ &\leq \|f\|_{L_p(\mathbb{R})}.\|g\|_{L_q(\mathbb{R})}.\|h\|_{L_r(\mathbb{R})}.\end{aligned}$$

The proof of part (a) is complete. The proof of part (b) is the same of part (a).

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4. APPLICATION TO SOLVING SOME INTEGRAL EQUATIONS OF TOEPLITZ PLUS HANKEL TYPE AND SYSTEM OF INTEGRAL EQUATIONS OF TWO TOEPLITZ PLUS HANKEL INTEGRAL EQUATIONS

In this section, we use the generalized convolution $\left(f\underset{3}{*}g\right)$ in [12]. Let functions $f,g\in L_1(\mathbb{R})$. Then the generalized convolution $\left(f\underset{3}{*}g\right)$, respectively, defined by the formula:

$$\left(f\underset{3}{*}g\right)(x)=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(t\right)\left[g\left(x+t\right)+g\left(x-t\right)+g\left(-x-t\right)-g\left(-x+t\right)\right]dt. \tag{4.1}$$

For this convolution, the following corresponding factorization equality holds

$$H_2\left(f\underset{3}{*}g\right)(y)=\left(H_1f\right)(y)\left(H_2g\right)(y)\,,\forall y\in\mathbb{R}. \tag{4.2}$$

The functions $f,g\in L_1(\mathbb{R})$, the generalized convolution $\left(f\underset{4}{*}g\right)$ in [10], repectively, defined by the formula:

$$\left(f\underset{4}{*}g\right)=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[f(x+y)-f(x-y)+f(-x+y)+f(-x-y)\right]g(y)dy. \tag{4.3}$$

For this convolution, the following factorization equality holds

$$H_1\left(f\underset{4}{*}g\right)(y)=\left(H_2f\right)(y)\left(H_2g\right)(y). \tag{4.4}$$

4.1. Consider following integral equation.

$$\begin{aligned}&f(-x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(u\right)\big[g\left(x+u\right)+g\left(-x-u\right)+g\left(-x+u\right)-g\left(x-u\right)\big]du\\ &=h\left(x\right),x\in\mathbb{R},\end{aligned} \tag{4.5}$$

where $g(x),h(x)$ are given functions in $L_1(\mathbb{R})$, and $f(x)$ is an unknown function in $L_1(\mathbb{R})$.

Theorem 4.1. *Assume that g,h in $L_1(\mathbb{R});1+2\left(H_1g\right)(y)\neq 0,\forall y\in\mathbb{R}$, then there exists a unique solution $f\in L_1(\mathbb{R})$ of the integral Equation* (4.5)*. Moreover, the solution is given by the formula*

$$f(x)=h(x)-\left(h\underset{H_1}{*}l\right)(x).$$

Proof. We write the equation (4.5) in the form

$$f(-x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(u\right)\left[g\left(x+u\right)+g\left(-x-u\right)\right]du \tag{4.6}$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(u\right)\left[g\left(-x+u\right)-g\left(x-u\right)\right]du=h\left(x\right). \tag{4.7}$$

Using Definition 2.1, Equation (4.6) become

$$f(-x)+\left(f\underset{1}{*}g\right)(x)+\left(f\underset{2}{*}g\right)(x)=h(-x),x\in\mathbb{R}. \tag{4.8}$$

Applying the Hartley transform H_2 to (4.8) and using Theorem 2.1, we obtain

$$\begin{aligned}&\left(H_2f\right)(-y)+\left(H_1f\right)(y)\left(H_1g\right)(y)+\left(H_1f\right)(y)\left(H_2g\right)(y)+\left(H_1f\right)(y)\left(H_1\right)(y)\\&-\left(H_1f\right)(y)\left(H_2g\right)(y)=\left(H_2h\right)(-y).\end{aligned}$$

It is equivalent to

$$\left(H_1f\right)(y)+2\left(H_1f\right)(y)\left(H_1g\right)(y)=\left(H_1h\right)(y).$$

With the condition $1+2\left(H_1g\right)(y)\neq 0,\forall y\in\mathbb{R}$, we have

$$\left(H_1f\right)(y)=\left(H_1h\right)(y)\frac{1}{1+2\left(H_1g\right)(y)}=\left(H_1h\right)(y)\left[1-\frac{2\left(H_1g\right)(y)}{1+2\left(H_1g\right)(y)}\right].$$

According to the Wiener-Levy theorem [1] for the Hartley transform, if $1+2\left(H_1g\right)(y)\neq 0$ for all $y\in\mathbb{R}$, then there exists a function $l\in L_1(\mathbb{R})$ such that

$$\left(H_1l\right)(y)=\frac{2\left(H_1g\right)(y)}{1+2\left(H_1g\right)(y)}.$$

This together with (1.1) and (1.2) implies that

$$\left(H_1f\right)(y)=\left(H_1h\right)(y)-\left(H_1h\right)(y)\left(H_1l\right)(y)=\left(H_1h\right)(y)-H_1\left(h\underset{H_1}{*}l\right)(y).$$

Consequently,

$$f(x)=h(x)-\left(h\underset{H_1}{*}l\right)(x)\in L_1(\mathbb{R}).$$

The proof is complete. ∎

4.2. **Consider following integral equation.**

$$\begin{aligned}&f(-x)+\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left[g\left(x+t\right)+g\left(x-t\right)-g\left(-x+t\right)+g\left(-x-t\right)\right]dt\\&=h\left(-x\right),x\in\mathbb{R},\end{aligned} \tag{4.9}$$

where $g(x),h(x)$ are given functions in $L_1(\mathbb{R})$, and $f(x)$ is unknown function in $L_1(\mathbb{R})$.

Theorem 4.2. *Assume that $g,h\in L_1(\mathbb{R});1+\left(H_1g\right)(y)\neq 0,\forall y\in\mathbb{R}$, then there exists a unique solution $f\in L_1(\mathbb{R})$ of the integral Equation* (4.9)*. Moreover, the solution is given by the formula*

$$f(x)=h(x)-\left(l\underset{3}{*}h\right)(x).$$

Proof. The equation (4.9) can be written in the following form

$$f(-x) - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[g\left(x+t\right) - g\left(x-t\right) + g\left(-x+t\right) + g\left(-x-t\right)\right] dt +$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(t\right) \left[g\left(x+t\right) + g\left(-x-t\right)\right] dt = h\left(-x\right), x \in \mathbb{R}. \tag{4.10}$$

Using Definition 2.1 and (4.3) to Equation (4.10), we obtain

$$f(-x) - \left(f \underset{4}{*} g\right)(x) + \left(f \underset{1}{*} g\right)(x) = h(-x), x \in \mathbb{R}. \tag{4.11}$$

Applying the Hartley transform H_1 to (4.11) and using Theorem 2.1, we obtain

$$\left(H_1 f\right)(-y) - \left(H_2 f\right)(y)\left(H_2 g\right)(y) + \left(H_2 f\right)(y)\left(H_2 g\right)(y) + \left(H_2 f\right)(y)\left(H_1 g\right)(y)$$
$$= \left(H_1 h\right)(-y),$$

it is equivalent to

$$\left(H_2 f\right)(y)\big[1 + \left(H_1 g\right)(y)\big] = \left(H_2 h\right)(y).$$

With the condition $1 + \left(H_1 g\right)(y) \neq 0, \forall y \in \mathbb{R}$, we get

$$\left(H_2 f\right)(y) = \left(H_2 h\right)(y)\left[1 - \frac{\left(H_1 g\right)(y)}{1 + \left(H_1 g\right)(y)}\right].$$

According to the Wiener-Levy Theorem [1] for the Hartley transform, then there exists a function $l \in L_1(\mathbb{R})$ such that

$$\left(H_1 l\right)(y) = \frac{\left(H_1 g\right)(y)}{1 + \left(H_1 g\right)(y)}.$$

Which implies that

$$\left(H_2 f\right)(y) = \left(H_2 h\right)(y) - \left(H_2 h\right)(y). \left(H_1 l\right)(y). \tag{4.12}$$

Using (4.2), we can rewriten the dominator in (4.12) in the form

$$\left(H_2 f\right)(y) = \left(H_2 h\right)(y) - H_2\left(l \underset{3}{*} h\right)(y).$$

Consequently,

$$f(x) = h(x) - \left(l \underset{3}{*} h\right)(x) \in L_1(\mathbb{R}).$$

The theorem is proved. ∎

4.3. **Consider following integral equation.**

$$f(-x) + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[g\left(x+t\right) - g\left(x-t\right) + g\left(-x-t\right) + g\left(-x+t\right)\right] dt$$
$$= h\left(-x\right), x \in \mathbb{R}, \tag{4.13}$$

where $g(x), h(x)$ are given functions in $L_1(\mathbb{R})$, and $f(x)$ is unknown function in $L_1(\mathbb{R})$.

Theorem 4.3. *Assume that* $g, h \in L_1(\mathbb{R}); 1 + \left(H_1 g\right)(y) \neq 0, \forall y \in \mathbb{R}$*, then there exists a unique solution* $f \in L_1(\mathbb{R})$ *of the integral equation* (4.13)*. Moreover, the solution is given by the formula*

$$f(x) = h(x) - \left(h \underset{H_1}{*} l\right)(x).$$

Proof. We write the equation (4.13) in the form

$$f(-x)+\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left[g\left(x+t\right)+g\left(x-t\right)+g\left(-x-t\right)-g\left(-x+t\right)\right]dt$$
$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left[g\left(-x+t\right)-g\left(x-t\right)\right]dt=h\left(-x\right),x\in\mathbb{R}. \tag{4.14}$$

Using Definition 2.1 and (4.1), we get

$$f(-x)+\left(f\underset{3}{*}g\right)(x)+\left(f\underset{2}{*}g\right)(x)=h(-x),x\in\mathbb{R}. \tag{4.15}$$

Applying the Hartley transform H_2 to (4.15) and using Theorem 2.1, we obtain

$$\left(H_2f\right)(-y)+\left(H_1f\right)(y).\left(H_2g\right)(y)+\left(H_1f\right)(y)\left(H_1g\right)(y)-\left(H_1f\right)(y).\left(H_2g\right)(y)$$
$$=\left(H_2h\right)(-y),$$

equivalently,

$$\left(H_1f\right)(y)+\left(H_1f\right)(y).\left(H_1g\right)(y)=\left(H_1h\right)(y).$$

So, $\left(H_1f\right)(y)\left[1+\left(H_1g\right)(y)\right]=\left(H_1h\right)(y)$. With the condition $1+\left(H_1g\right)(y)\neq 0,\forall y\in\mathbb{R}$, we get

$$\left(H_1f\right)(y)=\left(H_1h\right)(y)\left[1-\frac{\left(H_1g\right)(y)}{1+\left(H_1g\right)(y)}\right]$$

According to the Wiener-Levy Theorem [1] for the Hartley transform, if $1+\left(H_1g\right)(y)\neq 0$ for all $y\in\mathbb{R}$, then there exists a function $l\in L_1(\mathbb{R})$ such that

$$\left(H_1l\right)(y)=\frac{\left(H_1g\right)(y)}{1+\left(H_1g\right)(y)}.$$

From that we obtain

$$\left(H_1f\right)(y)=\left(H_1h\right)(y)\left[1-\left(H_1l\right)(y)\right]=\left(H_1h\right)(y)+H_1\left(h\underset{H_1}{*}l\right)(y).$$

Consequently,

$$f(x)=h(x)-\left(h\underset{H_1}{*}l\right)(x)\in L_1(\mathbb{R}).$$

The theorem is proven. ∎

4.4. **Consider following system of integral equations.**

$$\begin{cases} f(-x)+\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}g(t)\big[\varphi(x+t)+\varphi(-x-t)\big]dt=h(-x) \\ \dfrac{1}{2\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\Big[f(x+t)-f(x-y)+f(-x+t)+f(-x-t)\Big]g(t)dt \\ +g(-x)=k(-x),x\in\mathbb{R}, \end{cases} \tag{4.16}$$

where φ,ψ,h,k are given functions in $L(\mathbb{R})$, f and g are unknown functions.

Theorem 4.4. *Assume that* $1 - H_1\left(\varphi \underset{H_1}{*} \psi\right)(y) - H_1\left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)(y) \neq 0, \forall y \in \mathbb{R}$, *then there exists a unique solution in* $L(\mathbb{R})$ *of* (4.16) *which is defined by*

$$\begin{aligned} f(x) &= h(x) - \left(k \underset{H_1}{*} \varphi\right)(x) - \left(k \underset{H_1,H_1,H_2}{*} \psi\right)(x) + \left(h \underset{H_1}{*} l\right)(x) \\ &\quad - \left[\left(k \underset{H_1}{*} \varphi\right) \underset{H_1}{*} l\right](x) - \left[\left(k \underset{H_1,H_1,H_2}{*} \psi\right) \underset{H_1}{*} l\right](x) \in L(\mathbb{R}), \\ g(x) &= k(x) + \left(k \underset{H_1}{*} l\right)(x) - \left(\psi \underset{H_1}{*} h\right)(x) - \left[\left(\psi \underset{H_1}{*} h\right) \underset{H_1}{*} l\right](x) \in L(\mathbb{R}), \end{aligned}$$

where

$$(H_1 l)(y) = \frac{H_1\left[\left(\varphi \underset{H_1}{*} \psi\right) - \left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)\right](y)}{1 - H_1\left[\left(\varphi \underset{H_1}{*} \psi\right) - \left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)\right](y)}.$$

Proof. Using (2.1) and (1.5) we can write the system (4.16) in the form

$$\begin{cases} f(x) + \left(\varphi \underset{1}{*} g\right)(x) = h(-x), \\ \left(f \underset{H_2,H_1,H_1}{*} \psi\right)(x) + g(-x) = k(-x), x \in \mathbb{R}. \end{cases} \tag{4.17}$$

Applying the Hartley transform H_2 to the system (4.17), and using Theorem 2.1 we obtain

$$\begin{cases} (H_2 f)(-y) + (H_1 g)(y)(H_1\varphi)(y) + (H_1 g)(H_2\psi) = (H_2 h)(-y) \\ (H_1 f)(y)(H_1\psi)(y) + (H_2 g)(-y) = (H_2 k)(-y). \end{cases}$$

It is equivalent to

$$\begin{cases} (H_1 f)(y) + (H_1 g)(y)\big[(H_1\varphi)(y) + (H_2\psi)\big] = (H_1 h)(y), \\ (H_1 f)(y)(H_1\psi)(y) + (H_1 g)(y) = (H_1 k)(y). \end{cases} \tag{4.18}$$

We calculate the determinants of the system (4.18) and using (1.4), we get

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & (H_1\varphi)(y) + (H_2\varphi)(y) \\ (H_1\psi)(y) & 1 \end{vmatrix} \\ &= 1 - H_1\left(\varphi \underset{H_1}{*} \psi\right)(y) - H_1\left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)(y) \\ \Delta_1 &= \begin{vmatrix} (H_1 h)(y) & (H_1\varphi)(y) + (H_2\psi)(y) \\ (H_1 k)(y) & 1 \end{vmatrix} \\ &= (H_1 h)(y) - H_1\left(k \underset{H_1}{*} \varphi\right)(y) - H_1\left(k \underset{H_1,H_1,H_2}{*} \psi\right)(y) \\ \Delta_2 &= \begin{vmatrix} 1 & (H_1 h)(y) \\ (H_1\psi)(y) & (H_1 k)(y) \end{vmatrix} = (H_1 k)(y) - H_1\left(\psi \underset{H_1}{*} h\right)(y). \end{aligned}$$

Since $1 - H_1\left(\varphi \underset{H_1}{*} \psi\right)(y) - H_1\left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)(y) \neq 0$, we have

$$\frac{1}{\Delta} = \left[1 + \frac{H_1\left[\left(\varphi \underset{H_1}{*} \psi\right) - \left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)\right](y)}{1 - H_1\left[\left(\varphi \underset{H_1}{*} \psi\right) - \left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)\right](y)}\right].$$

Furthermore, according to Wiener-Levy's Theorem [1] then there exists a function $l \in L(\mathbb{R})$ such that

$$(H_1 l)(y) = \frac{H_1\left[\left(\varphi \underset{H_1}{*} \psi\right) - \left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)\right](y)}{1 - H_1\left[\left(\varphi \underset{H_1}{*} \psi\right) - \left(\psi \underset{H_1,H_1,H_2}{*} \varphi\right)\right](y)}.$$

It follows that

$$\begin{aligned}(H_1 f)(y) &= \left[(H_1 h)(y) - H_1\left(k \underset{H_1}{*} \varphi\right)(y) - H_1\left(k \underset{H_1,H_1,H_2}{*} \psi\right)(y)\right]\left[1 + (H_1 l)(y)\right] \\ &= (H_1 h)(y) - H_1\left(k \underset{H_1}{*} \varphi\right)(y) - H_1\left(k \underset{H_1,H_1,H_2}{*} \psi\right)(y) \\ &\quad - H_1\left[\left(k \underset{H_1,H_1,H_2}{*} \psi\right) \underset{H_1}{*} l\right](y) + H_1\left(h \underset{H_1}{*} l\right)(y) - H_1\left[\left(k \underset{H_1}{*} \varphi\right) \underset{H_1}{*} l\right](y).\end{aligned}$$

Thus,

$$\begin{aligned}f(x) &= h(x) - \left(k \underset{H_1}{*} \varphi\right)(x) - \left(k \underset{H_1,H_1,H_2}{*} \psi\right)(x) + \left(h \underset{H_1}{*} l\right)(x) \\ &\quad - \left[\left(k \underset{H_1}{*} \varphi\right) \underset{H_1}{*} l\right](x) - \left[\left(k \underset{H_1,H_1,H_2}{*} \psi\right) \underset{H_1}{*} l\right](x) \in L(\mathbb{R}).\end{aligned}$$

Similarly, we get

$$\begin{aligned}(H_1 g)(y) &= \left[(H_1 k)(y) - H_1\left(\psi \underset{H_1}{*} h\right)(y)\right]\left[1 + (H_1 l)(y)\right] \\ &= (H_1 k)(y) + H_1\left(k \underset{H_1}{*} l\right)(y) - H_1\left(\psi \underset{H_1}{*} h\right)(y) - H_1\left[\left(\psi \underset{H_1}{*} h\right) \underset{H_1}{*} l\right](y).\end{aligned}$$

Hence

$$g(x) = k(x) + \left(k \underset{H_1}{*} l\right)(x) - \left(\psi \underset{H_1}{*} h\right)(x) - \left[\left(\psi \underset{H_1}{*} h\right) \underset{H_1}{*} l\right](x) \in L(\mathbb{R}).$$

The proof is completed. ∎

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