Aust. J. Math. Anal. Appl.
Vol. **21** (2024), No. 2, Art. 3, 24 pp.
AJMAA

MAXIMAL SINGULAR OPERATORS ON VARIABLE EXPONENT SEQUENCE SPACES AND THEIR CORRESPONDING ERGODIC VERSION

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Received 14 November, 2023; accepted 6 June, 2024; published 5 July, 2024.

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ABSTRACT. In this paper, we prove strong and weak type inequalities of singular operators on weighted $\ell^p_w(\mathbb{Z})$. Using these results, we prove strong type and weak type inequalities of the maximal singular operator of Calderón-Zygmund type on variable exponent sequence spaces $\ell^{p(\cdot)}(\mathbb{Z})$. Using the Calderón-Coifman-Weiss transference principle, we prove strong type, weak type inequalities of the maximal ergodic singular operator on $L^p_w(X, \mathcal{B}, \mu)$ spaces, where$(X, \mathcal{B}, \mu)$ is a probability space equipped with measure preserving transformation U.

Key words and phrases: Calderón-Zygmund decomposition; Maximal Singular Operator; Maximal Ergodic Singular Operator; Transference Method; Ergodic Rectangles; Ergodic Weights; Reverse Hölder inequality; Rubio de Francia extrapolation.

2010 *Mathematics Subject Classification.* Primary 37A46. Secondary 28D05.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Singular integral operators on the weighted $L^p_w(\mathbb{R})$ spaces are studied in [10]. In [10], the authors have proved that for $1 < p < \infty$ and if the non-negative function $w(x)$ satisfies A_p condition then the singular integral operators are bounded on $L^p_w(\mathbb{R})$. For $p = 1$, it has been proved that if $w(x)$ satisfies A_1 condition, then the singular integral operators satisfy weak type (1,1) inequality with respect to the weighted measure. The detailed proof of the same can also be seen in [6]. In [8], the authors have studied the singular operators on sequence spaces $\ell^p(\mathbb{Z})$ and their corresponding ergodic versions.

In this paper, we prove strong type, weak type inequalities of singular operators on weighted $\ell^p_w(\mathbb{Z})$ spaces. Using these results we prove strong type, weak type inequalities of maximal singular operator of Calderón-Zygmund type on variable sequence spaces $\ell^{p(\cdot)}(\mathbb{Z})$.

These results are achieved using Calderón-Zygmund decomposition for sequences, properties of A_p weights, reverse Hölder inequality and Rubio de Francia extrapolation. We also prove strong type, weak type inequalities of maximal ergodic singular operators on $L^p_w(X, \mathcal{B}, \mu)$ spaces, where $(X, \mathcal{B}, \mu)$ is a probability space equipped with an invertible measure preserving transformation U. We use Calderón-Coifman-Weiss transference principle to achieve these results.

In [4] the characterization of those positive functions w (known as ergodic A_p weights) for which the maximal ergodic singular operator associated with an invertible measure preserving transformation on a probability space is bounded on $L^p_w(X, \mathcal{B}, \mu)$ is given. In their proof the ergodic analogue of Calderón-Zygmund decomposition and the concept of ergodic rectangles are used. Using the same concept of ergodic rectangles, we prove that for $1 < p < \infty$, if the maximal ergodic Hilbert transform is bounded on $L^p_w(X, \mathcal{B}, \mu)$, then $w \in A_p(X)$. In [4], the authors have given direct proof of this result without using the corresponding results on weighted sequence spaces. In this paper we use the corresponding result on $\ell^p_w(\mathbb{Z})$ to prove this result.

2. DEFINITIONS AND NOTATION

Throughout this thesis, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{Z}_+$ denotes the set of all positive integers. For a given interval I in $\mathbb{Z}$ (we always mean finite interval of integers) , $|I|$ always denotes the cardinality of I. For each positive integer N, consider collection of disjoint intervals of cardinality 2^N,

$$\{I_{N,j}\}_{j\in\mathbb{Z}} = \left\{[(j-1)2^N + 1, \ldots, j2^N]\right\}_{j\in\mathbb{Z}}.$$

The set of intervals which are of the form $I_{N,j}$ where $N \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$ are called dyadic intervals. For fixed N, $I_{N,j}$ are disjoint.

Given a dyadic interval $I = \left\{[(j-1)2^N + 1, \ldots, j2^N]\right\}_{j\in\mathbb{Z}}$ and a positive integer m, we define

$$\begin{aligned}
2LI &= [(j-2)2^N + 1, \ldots, j2^N] \\
4LI &= [(j-4)2^N + 1, \ldots, j2^N] \\
2RI &= [(j-1)2^N + 1, \ldots, (j+1)2^N] \\
4RI &= [(j-1)2^N + 1, \ldots, (j+3)2^N] \\
3I &= 2LI \cup 2RI \\
5I &= 4LI \cup 4RI
\end{aligned}$$

For $k = 2, 3, 4, \ldots$ and $K \in \mathbb{Z}_+$, let $I(0, 2^k K)$ denotes the interval

$$[-2^{k-1}K, -2^{k-1}K + 1, \ldots, -1, 0, 1, 2, \ldots, 2^{k-1}K - 1, 2^{k-1}K].$$

For a given sequence $\{a(n) : n \in \mathbb{Z}\}$ and an interval I_j, $a(I_j) = \sum_{k\in I_j} a(k)$. For a sequence $\{p(n) : n \in \mathbb{Z}, p(n) \geq 1\}$, define $p_- = \inf\{p(n) : n \in \mathbb{Z}\}, p_+ = \sup\{p(n) : n \in \mathbb{Z}\}$. Throughout this paper, we assume $p_+ < \infty$ and $1 \leq p_- \leq p(n) < p_+ < \infty, n \in \mathbb{Z}$. We denote set of all such sequences $\{p(n) : n \in \mathbb{Z}\}$ by $\mathcal{S}$.

Maximal Operators. Let $\{a(n) : n \in \mathbb{Z}\}$ be a sequence. We define the following three types of Hardy-Littlewood maximal operators as follows:

Definition 2.1. If I_r is the interval $\{-r, -r+1, \dots, 0, 1, 2, \dots, r-1, r\}$, define centered Hardy-Littlewood maximal operator

$$M'a(m) = \sup_{r>0} \frac{1}{(2r+1)} \sum_{n\in I_r} |a(m-n)|$$

We define Hardy-Littlewood maximal operator as follows

$$Ma(m) = \sup_{m\in I} \frac{1}{|I|} \sum_{n\in I} |a(n)|$$

where the supremum is taken over all intervals containing m.

Definition 2.2. We define dyadic Hardy-Littlewood maximal operator as follows:

$$M_d a(m) = \sup_{m\in I} \frac{1}{|I|} \sum_{k\in I} |a(k)|$$

where supremum is taken over all dyadic intervals containing m.

Given a sequence $\{a(n) : n \in \mathbb{Z}\}$ and an interval I, let a_I denote average of $\{a(n) : n \in \mathbb{Z}\}$ on I. Let, $a_I = \frac{1}{|I|} \sum_{m\in I} a(m)$. Define the sharp maximal operator $M^{\#}$ as follows

$$M^{\#}a(m) = \sup_{m\in I} \frac{1}{|I|} \sum_{n\in I} |a(n) - a_I|$$

where the supremum is taken over all intervals I containing m. We say that sequence $\{a(n) : n \in \mathbb{Z}\}$ has bounded mean oscillation if the sequence $M^{\#}a$ is bounded. The space of sequences with this property is denoted by BMO($\mathbb{Z}$).

We define a norm in BMO($\mathbb{Z}$) by $\|a\|_\star = \left\|M^{\#}a\right\|_\infty$. The space BMO($\mathbb{Z}$) is studied in [9].

Norm in Variable Sequence Spaces.

Definition 2.3. Given a bounded sequence $\{p(n) : n \in \mathbb{Z}\}$ which takes values in $[1, \infty)$, define $\ell^{p(\cdot)}(\mathbb{Z})$ to be set of all sequences $\{a(n) : n \in \mathbb{Z}\}$ such that for some $\lambda > 0$,

$$\sum_{k\in\mathbb{Z}} (\frac{|a(k)|}{\lambda})^{p(k)} < \infty.$$

We define modular functional for variable sequences spaces associated with $p(\cdot)$ as

$$\rho_{p(\cdot)}(a) = \sum_{k\in\mathbb{Z}} |a(k)|^{p(k)}$$

Further for a given sequence $\{a(k) : k \in \mathbb{Z}\}$ in $\ell^{p(\cdot)}(\mathbb{Z})$, we define

$$\|a\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{a}{\lambda}) \leq 1\right\}$$

$\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$ is a norm in $\ell^{p(\cdot)}(\mathbb{Z})$ [7].

Weights.

Definition 2.4. For a fixed p, $1 < p < \infty$, we say that a non-negative sequence $\{w(n) : n \in \mathbb{Z}\}$ belongs to class A_p if there is a constant C such that, for all intervals I in $\mathbb{Z}$, we have

$$\left(\frac{1}{|I|}\sum_{k\in I} w(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} w(k)^{-\frac{1}{p-1}}\right)^{p-1} \le C.$$

Infimum of all such constants C is called A_p constant.
We say that $\{w(m) : m \in \mathbb{Z}\}$ belongs to class A_1 if there a constant C such that, for all intervals I in $\mathbb{Z}$,

$$\frac{1}{|I|}\sum_{k\in I} w(k) \le Cw(m)$$

for all $m \in I$. Infimum of all such constants C is called A_1 constant.
Let $1 \le p < \infty$ and $\{w(n) : n \in \mathbb{Z}\} \in A_p(\mathbb{Z})$. We say that a sequence $\{a(n) : n \in \mathbb{Z}\}$ is in $\ell^p_w(\mathbb{Z})$ if

$$\sum_{n\in\mathbb{Z}} |a(n)|^p w(n) < \infty.$$

We define norm in $\ell^p_w(\mathbb{Z})$ by

$$\|a\|_{\ell^p_w(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |a(k)|^p w(k)\right)^{\frac{1}{p}}.$$

For a subset A of $\mathbb{Z}$, $w(A)$ denotes $\sum_{k\in A} w(k)$.
For a given sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_w(\mathbb{Z})$, the weighted weak type (p,p) inequality for a non-negative weight sequence $\{w(n) : n \in \mathbb{Z}\}$ is as follows:

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \le \frac{C}{\lambda^p}\sum_{m\in\mathbb{Z}} |a(m)|^p w(m)$$

Definition 2.5. Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. Suppose $1 < p < \infty$ and $w : X \to \mathbb{R}$ be a non-negative integrable function. The function w is said to satisfy ergodic A_p condition,

$$esssup_{x\in X}\sup_{N\ge 1}\left(\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x)\right)\left(\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x)^{\frac{-1}{p-1}}\right)^{p-1} \le C.$$

The function w is said to satisfy ergodic A_1 condition,

$$esssup_{x\in X}\sup_{N\ge 1}\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x) \le Cw(U^m x)$$

for $m = -N, -N+1, \ldots, N$.

3. RELATIONS BETWEEN MAXIMAL OPERATORS

In the following lemmas, we give relations between maximal operators. For the proofs of the following lemmas, refer [1]. These relations will be used when we prove the weighted inequalities for maximal ergodic operators.

Lemma 3.1. *Given a sequence $\{a(m) : m \in \mathbb{Z}\}$, the following relation holds:*

$$M'a(m) \leq Ma(m) \leq 3M'a(m)$$

Lemma 3.2. *If $\boldsymbol{a} = \{a(k) : k \in \mathbb{Z}\}$ is a non-negative sequence with $\boldsymbol{a} \in \ell_1$, then*

$$|\{m \in \mathbb{Z} : M'a(m) > 4\lambda\}| \leq 3|\{m \in \mathbb{Z} : M_d a(m) > \lambda\}|$$

In the following lemma, we see that in the norm of BMO($\mathbb{Z}$) space, we can replace the average a_I of $\{a(n) : n \in \mathbb{Z}\}$ by a constant b. The proof is similar to the proof in continuous version [6].The second inequality follows from $||a| - |b|| \leq |a| - |b|$.

Lemma 3.3. *Consider a non-negative sequence $\boldsymbol{a} = \{a(k) : k \in \mathbb{Z}\}$ in BMO($\mathbb{Z}$). Then the following are valid.*

1. $\frac{1}{2}\|a\|_\star \leq \sup_{m \in I} \inf_{b \in \mathbb{Z}} \frac{1}{|I|}|a(m) - b| \leq \|a\|_\star$
2. $M^{\#}(|a|)(i) \leq M^{\#}a(i), i \in \mathbb{Z}$

4. WEIGHTED CLASSICAL RESULTS FOR MAXIMAL OPERATORS

Let $1 \leq p < \infty$. In this section, for a given sequence $\{a(n) : n \in \mathbb{Z}\}$ in $\ell^p_w(\mathbb{Z})$, we prove weighted weak type (p,p) inequality with respect to the weight sequence$\{w(n) : n \in \mathbb{Z}\} \in A_p$ which is stated in Theorem[4.2].

The proof of the following theorem is similar to the proof of corresponding result in continuous version [6]. We state here without proof.

Theorem 4.1. *Let $\{a(n) : n \in \mathbb{Z}\}$ be a non-negative sequence and $\{w(n) : n \in \mathbb{Z}\} \in A_p, 1 \leq p < \infty$ be a non-negative weight sequence. Let I be an interval such that $a(m) > 0$ for some $m \in I$. Then,*

(1)

$$w(I)\left(\frac{a(I)}{|I|}\right)^p \leq C\sum_{m \in I}|a(m)|^p w(m) \tag{4.1[A]}$$

(2) *Given a finite set $S \subset I$,*

$$w(I)\left(\frac{|S|}{|I|}\right)^p \leq Cw(S) \tag{4.1[B]}$$

$4.1[A]$ follows from Hölder's inequality and the A_p condition. $4.1[B]$ follows by taking $a = \chi_S$ in $4.1[A]$.

Theorem 4.2. *Assume $\{w(n) : n \in \mathbb{Z}\} \in A_p$. Given a non-negative sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_w(\mathbb{Z})$, for $1 \leq p < \infty$, the weighted weak(p,p) inequality holds:*

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \leq \frac{C}{\lambda^p}\sum_{m \in \mathbb{Z}}|a(m)|^p w(m)$$

For the proof of Theorem[4.2], refer [3].

Theorem 4.3. *If $w \in A_p, 1 < p < \infty$, then M is bounded on $\ell^p_w(\mathbb{Z})$.*

The proof follows from Theorem[4.2] and Marcinkiewicz interpolation theorem.

5. Properties of A_p Weights

We state the properties of A_p weights for sequences without proofs. The proofs are similar to the proofs of corresponding results in the continuous version.

Property 1. $A_p \subset A_q, 1 \leq p < q$.

Property 2. $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$

Let $w(m) \in A_p, m \in I$. Proof follows by noting $A_{p'}$ condition for $w^{1-p'}$. For the converse A_p condition can be verified.

Property 3. $w_0, w_1 \in A_1 \implies w_0 {w_1}^{1-p} \in A_p$.

Here, we state reverse Hölder inequality for weighted sequences. For continuous version of these proofs, refer to [6].

Property 4. *[Reverse Hölder Inequality] Let* $w \in A_p, 1 \leq p < \infty$. *Then , there exists constants* c *and* $\epsilon > 0$, *depending only on* p *and the* A_p *constants of* w, *such that for any interval* I,

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \leq \frac{C}{|I|}\sum_{m\in I} w(m)$$

Property 5. $A_p = \cup_{q<p} A_q, 1 < p < \infty$.

Property 6. *If* $w \in A_p, 1 \leq p < \infty$, *then there exists* $\epsilon > 0$ *such that* $w^{1+\epsilon} \in A_p$

Property 7. *If* $w \in A_p, 1 \leq p < \infty$, *then there exists* $\delta > 0$ *such that given a interval I and* $S \subset I$,

$$\frac{w(S)}{w(I)} \leq C\left(\frac{|S|}{|I|}\right)^{\delta}$$

6. Calderón-Zygmund decompostion for Sequences

For the proof of Theorem[6.1], we refer [2].

Theorem 6.1. *Let* $0 \leq \alpha < 1$. *Take a real number* p *such that* $1 \leq p < \frac{1}{\alpha}$ *(If* $\alpha = 0$ *then* $1 \leq p < \infty$*). Let* $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$. *Then*
there exists a sequence of disjoint intervals $\left\{I_j^t\right\}$ *such that*

$$(i) \quad t < \frac{1}{|I_j^t|^{1-\alpha}}\sum_{k\in I_j^t}|a(k)| \leq 2t, \forall j \in \mathbb{Z}.$$

$$(ii) \quad \forall n \notin \cup_j I_j^t, \quad |a(n| \leq t.$$

$$(iii) \quad \text{If} \quad t_1 > t_2, \text{ then each } I_j^{t_1} \text{is subinterval of some } I_m^{t_2} \quad \forall j,m \in \mathbb{Z}.$$

7. Weighted Good Lambda Estimate

Lemma 7.1. *Let* $\{a(n) : n \in \mathbb{Z}\}$ *be a non-negative sequence in* $\ell^p_w(\mathbb{Z})$. *Let* $w \in A_p, 1 \leq p_0 \leq p < \infty$. *If* $\{a(n) : n \in \mathbb{Z}\}$ *is such that* $M_d a \in \ell^{p_0}_w(\mathbb{Z})$, *then*

$$\sum_{m\in\mathbb{Z}}|M_d a(m)|^p w(m) \leq C\sum_{m\in\mathbb{Z}}|M^{\#}a(m)|^p w(m)$$

where M_d *is the dyadic maximal operator and* $M^{\#}$ *is the sharp maximal operator, whenever, the left hand side is finite.*

Proof. In order to prove Lemma[7.1], first we prove the good-λ inequality, which is as follows:
For some $\delta > 0$,

$$w(\{m \in \mathbb{Z} : M_d a(m) > 2\lambda, M^{\#}a(m) \le \gamma\lambda\}) \le C\gamma^{\delta} w(\{x \in \mathbb{Z} : M_d a(m) > \lambda\})$$

Since $\{m \in \mathbb{Z} : M_d a(m) > \lambda\}$ can be decomposed into disjoint dyadic cubes, it is enough to show that for each such interval I,

$$w(\{m \in I : M_d a(m) > 2\lambda, M^{\#}a(m) \le \gamma\lambda\}) \le C\gamma^{\delta} w(I)$$

The above inequality can be proved using the same argument as in Lemma[4.6] from [1] and property[7] . Now we prove Lemma[7.1]
Consider, for any positive integer N

$$I_N = \int_0^N p\lambda^{p-1} |w\{n \in \mathbb{Z} : M_d a(k) > \lambda\}| d\lambda$$

Since $a \in \ell^{p_0}$ implies $M_d a \in \ell^{p_0}$, I_N is finite,

$$\begin{aligned}
I_N &= \int_0^N p\lambda^{p-1} |w\{n \in \mathbb{Z} : M_d a(k) > \lambda\}| d\lambda \\
&= 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M_d a(k) > 2\lambda\}| d\lambda = \\
&\le 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M_d a(k) > 2\lambda, M^{\#}a(k) \le \gamma\lambda\}| d\lambda + \\
&2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M_d a(k) > 2\lambda, M^{\#}a(k) > \gamma\lambda\}| d\lambda \\
&\le 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} C\gamma^{\delta} w\{n \in \mathbb{Z} : M_d a(k) > \lambda\} d\lambda + \\
&2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M^{\#}a(k) > \gamma\lambda\}| d\lambda \\
&\le 2^p C\gamma^{\delta} \int_0^N p\lambda^{p-1} w\{n \in \mathbb{Z} : M_d a(k) > \lambda\} d\lambda \quad + \\
&2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M^{\#}a(k) > \gamma\lambda\}| d\lambda
\end{aligned}$$

It follows that

$$(1 - 2^p C\gamma^{\delta}) I_N \le 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M^{\#}a(k) > \gamma\lambda\}| d\lambda$$

Now take $(1 - 2^p C\gamma^{\delta}) = \frac{1}{2}$. Then,

$$\begin{aligned}
\frac{1}{2} I_N &\le 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M^{\#}a(k) > \gamma\lambda\}| d\lambda \\
&\le \frac{2^p}{\gamma^p} \int_0^{\frac{N}{2}} p\lambda^{p-1} |w\{n \in \mathbb{Z} : M^{\#}a(k) > \lambda\}| d\lambda
\end{aligned}$$

Now, take $N \to \infty$, we get

$$\sum_{m \in \mathbb{Z}} M_d a(m)^p w(m) \le C \sum_{m \in \mathbb{Z}} M^{\#}a(m)^p w(m)$$

∎

8. CALDERÓN-ZYGMUND SINGULAR OPERATOR

In this section, we study Calderón-Zygmund singular operator on weighted $\ell_w^{p(\cdot)}(\mathbb{Z})$ spaces. This operator on $\ell^p(\mathbb{Z})$ spaces is studied in [8].

Definition 8.1. A sequence $\{\phi(n)\}$ is said to be a singular kernel if there exist constants C_1 and $C_2 > 0$ such that

If $\phi = \{\phi(n)\}$ is a singular kernel and $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z}), 1 \leq p < \infty$, define
(S1): $\sum_{n=-N}^{N} \phi(n)$ converges as $N \to \infty$.
(S2): $\phi(0) = 0$ and $|\phi(n)| \leq \frac{C_1}{|n|}, n \neq 0$
(S3): $|\phi(n+1) - \phi(n)| \leq \frac{C_2}{n^2}, \quad n \neq 0.$

$$T_\phi a(n) = (\phi \star a)(n) = \sum_{k \in \mathbb{Z}} \phi(n-k)a(k)$$

Since S2 implies that $\phi \in \ell^r$ for all $1 < r \leq \infty$, the above convolution is defined.

The operator T_ϕ defined above is called discrete singular operator.

The maximal singular operator corresponding to this singular operator is defined as

$$T_\phi^\star a(n) = \sup_N |\sum_{k=-N}^{N} \phi(k)a(n-k)|$$

If ϕ is a singular kernel and we let K be the linear extension of ϕ to $\mathbb{R}$, then K is locally integrable and satisfies:

(K1)

$$\int_{\epsilon < |x| < \frac{1}{\epsilon}} K(x) \quad dx \quad \text{converges as } \epsilon \to 0$$

(K2)

$$|K(x)| \leq \frac{C}{|x|}$$

(K3)

$$|K(x) - K(x-y)| \leq \frac{C|y|}{x^2} \text{ for} |x| > 2|y|$$

The function $K(x)$ which satisfies $(K1), (K2), (K3)$ is known as Calderón-Zygmund singular kernel on $\mathbb{R}$. The principal value integral

$$T_K f(x) = \lim_{\epsilon > 0} \int_{|x-y| > \epsilon} K(x-y)f(y)dy$$

and the maximal singular integral operator

$$T_K^\star f(x) = \sup_{\epsilon > 0} |\int_{|x-y| > \epsilon} K(x-y)f(y)dy|$$

satisfy strong type (p,p) and weak type (1,1) inequalities [8].

The proof of following theorem can be found in [8]. The following theorem states that the discrete maximal singular operator and discrete singular operator are bounded on $\ell^p(\mathbb{Z}), 1 < p < \infty$ and they satisfy weak(1,1) inequality. For proof, we refer to [8].

Theorem 8.1 ([8]). *Let $\phi = \{\phi(n)\}$ be a singular kernel. Then there exists constant $C_p > 0$ such that*

(1) *If* $1 < p < \infty, \|T_\phi a\|_p \leq C_p \|a\|_p, \forall a \in \ell^p(\mathbb{Z})$.

(2) $|\{n : |T_\phi a(n)| > \lambda\}| \leq \frac{C_1}{\lambda} \|a\|_1, \forall a \in \ell^1(\mathbb{Z})$ *and* $\lambda > 0$.

Theorem 8.2 ([8]). *Let ϕ be a singular kernel and $1 \leq p < \infty$. Then there exists a constant $C_p > 0$ such that (i)*

$$\left\|T_\phi^\star a\right\|_p \leq C_p \|a\|_p \quad \forall a \in \ell^p(\mathbb{Z}), \quad \textit{if} \quad 1 < p < \infty$$

(ii)

$$|\left\{j \in \mathbb{Z} : T_\phi^\star a(j) > \lambda\right\}| \leq \frac{C_1}{\lambda} \|a\|_1 \quad \forall \lambda > 0 \quad \textit{and} \quad a \in \ell^1(\mathbb{Z})$$

Now, we prove the strong type and weak type inequalities for the discrete singular operator T_ϕ on $\ell_w^{p(\cdot)}(\mathbb{Z})$ spaces. For this we require the following lemmas.

Lemma 8.3. *Let ϕ be a singular kernel. Given an interval I which contains integers m,n, then for $r \notin 5I$,*

$$|\phi(m-r) - \phi(n-r)| \leq \frac{C|I|}{|n-r|^2}$$

Proof. If $m > n$, then

$$\begin{aligned}
&|\phi(n-r) - \phi(m-r)| \\
&\leq |\phi(n-r) - \phi(n-r+1) + \phi(n-r+1) - \phi(n-r+2) \cdots + \\
&+ \phi(n-r+m-n-1) - \phi(n-r+m-n)| \\
&\leq |\phi(n-r) - \phi(n-r+1)| + |\phi(n-r+1) - \phi(n-r+2)| \cdots + \\
&+ |\phi(n-r+m-n-1) - \phi(n-r+m-n)| \\
&\leq \frac{C}{|n-r|^2} + \frac{C}{|n-r+1|^2} + \cdots + \frac{C}{|m-r-1|^2} \\
&\leq C\frac{|n-m|}{|n-r|^2} \\
&\leq C\frac{|I|}{|n-r|^2}
\end{aligned}$$

By the same argument, if $n > m$, then

$$|\phi(m-r) - \phi(n-r)| \leq \frac{C|I|}{|m-r|^2}$$

Also

$$|m-r| = |(m-n) + (n-r)| \geq |n-r| - |m-n| \geq |n-r| - |I|$$

Since $r \in \mathbb{Z} \backslash 5I$, we have $|n-r| \geq 2|I|$. Hence for $r \in \mathbb{Z} \backslash 5I$. $|m-r| \geq |n-r| - \frac{|n-r|}{2} \geq \frac{|n-r|}{2}$ i.e $\frac{1}{|m-r|} \leq \frac{2}{|n-r|}$. Therefore, in this case also, $|\phi(m-r) - \phi(n-r)| \leq \frac{C|I|}{(n-r)^2}$. ∎

Lemma 8.4. *If T_ϕ is a singular operator, then for each $s > 1$, there exists a constant $C_s > 0$ such that*

$$M^{\#}(T_\phi a(m)) \leq C_s \left[M(|a|^s)(m)\right]^{\frac{1}{s}}$$

for each integer $m \in \mathbb{Z}$.

Proof. Fix $s > 1$. Given an integer m and an interval I which contains m, by Lemma[3.3] , it is enough to find a constant h such that

$$\frac{1}{|I|}\sum_{n\in I}|T_\phi a(n) - h| \leq CM(|a|^s)(m)^{\frac{1}{s}}$$

Decompose $a = a_1 + a_2$, where $a_1 = a\chi_{5I}, a_2 = a - a_1$. Now let $h = T_\phi a(m)$, then

$$\frac{1}{|I|}\sum_{n\in I}|T_\phi a(n) - h| \leq \frac{1}{|I|}\sum_{n\in I}|T_\phi a_1(n)| + \frac{1}{|I|}\sum_{n\in I}|T_\phi a_2(n) - T_\phi a_2(m)|$$

Since $s > 1$, T_ϕ is bounded on $\ell^s(\mathbb{Z})$. Therefore,

$$\begin{aligned}
\frac{1}{|I|}\sum_{n\in I}|T_\phi a_1(n)| &\leq \left(\frac{1}{|I|}\sum_{n\in I}|T_\phi a_1(n)|^s\right)^{\frac{1}{s}}\\
&\leq C\left(\frac{1}{|I|}\sum_{n\in\mathbb{Z}}|a_1(n)|^s\right)^{\frac{1}{s}}\\
&\leq C\left(\frac{5}{|5I|}\sum_{n\in 5I}|a(n)|^s\right)^{\frac{1}{s}}\\
&\leq 5^{\frac{1}{s}}C\Big[M(|a|^s)(m)\Big]^{\frac{1}{s}}
\end{aligned}$$

To deal with a_2, we require the estimate from Lemma[8.3].
Now, we estimate the second term as follows.

$$\begin{aligned}
&\frac{1}{|I|}\sum_{n\in I}|T_\phi a_2(n) - T_\phi a_2(m)|\\
&\leq \frac{1}{|I|}\sum_{n\in I}\Big|\sum_{r\in\mathbb{Z}\setminus 5I}\Big(\phi(n-r) - \phi(m-r)\Big)a(r)\Big|\\
&\leq \frac{1}{|I|}\sum_{n\in I}\sum_{r\in\mathbb{Z}\setminus 5I}|\phi(n-r) - \phi(m-r)||a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{r\in\mathbb{Z}\setminus 5I}\frac{|I|}{|n-r|^2}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\sum_{2^k|I|<|n-r|\leq 2^{k+1}|I|}\frac{|I|}{|n-r|^2}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{|I|}{2^{2k}|I|^2}\sum_{|n-r|\leq 2^{k+1}|I|}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{1}{2^{2k}|I|}\sum_{|n-r|\leq 2^{k+1}|I|}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{2}{2^k 2^{k+1}|I|}\sum_{|n-r|\leq 2^{k+1}|I|}|a(r)|
\end{aligned}$$

$$\leq 2CMa(m)\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{1}{2^k}$$
$$\leq CMa(m)\frac{1}{|I|}\sum_{n\in I}1$$
$$= CMa(m) \leq CM(|a|^s)(m)^{\frac{1}{s}}$$

The last inequality follows by using Hölder's inequality. ∎

Theorem 8.5. *If T_ϕ is a singular operator, then for any $w \in A_p, 1 < p < \infty$, T_ϕ is bounded on $\ell^p_w(\mathbb{Z})$.*

Proof. Let $w \in A_p$. Since $A_p = \cup_{q<p}A_q$, we can find s such that $p > s > 1$ and $w \in A_{\frac{p}{s}}$. Consider a sequence $\{a(n) : n \in \mathbb{Z}\}$ such that $a(n) = 0$ outside the interval $[-R, -R + 1, \ldots, R]$.

Therefore,

$$\sum_{m\in\mathbb{Z}}|T_\phi a(m)|^p w(m)$$
$$\leq \sum_{m\in\mathbb{Z}}\left[M_d\left[T_\phi a(m)\right]\right]^p w(m) \qquad Lemma[7.1]$$
$$\leq C\sum_{m\in\mathbb{Z}}\left[M^{\#}\left[T_\phi a(m)\right]\right]^p w(m) \qquad Theorem[8.4]$$
$$\leq C\sum_{m\in\mathbb{Z}}\left[M(|a(m)|^s)\right]^{\frac{p}{s}} w(m)$$
$$\leq C\sum_{m\in\mathbb{Z}}|a(m)|^p w(m)$$

In the second step, we use [Lemma[7.1](Weighted Good -Lambda estimate) provided

$$\sum_{m\in\mathbb{Z}}\left[M_d(T_\phi a(m))\right]^p w(m)$$

is finite. To show this it is enough to show that $T_\phi a \in \ell^p_w(\mathbb{Z})$.
We have to prove

$$\sum_{m\in\mathbb{Z}}\left(T_\phi a(m)\right)^p w(m) < \infty.$$

To show that this is finite, we split this sum as

$$\sum_{m\leq 2R}\left(T_\phi a(m)\right)^p w(m)$$

and

$$\sum_{m>2R}\left(T_\phi a(m)\right)^p w(m).$$

The former sum

$$\sum_{m\leq 2R}\left(T_\phi a(m)\right)^p w(m) < \infty$$

is trivial as shown below.

For$|m| \leq 2R$,

$$|T_\phi a(m)| \leq C \sum_{|n|\leq 2R, m\neq n} |a(n)| \frac{C}{|m-n|} \leq C \left\|a\right\|_\infty 4R < \infty. \tag{A4}$$

For $|m| > 2R$,

$$|T_\phi a(m)| = |\sum_{n\in\mathbb{Z}} a(n)\phi(m-n)| \leq C \sum_{|n|<R, m\neq n} \frac{|a(n)|}{|m-n|} \leq C \frac{\|a\|_\infty}{|m|}$$

Further, $I(0, 2R) \subset I(0, 2^{k+1}R)$ and $w(I(0, 2R))$ is a constant independent of m. Also, since $w \in A_p$, by Lemma[6], there exists $q < p$ such that $w \in A_q$. Then by Lemma[4.1]

$$w(I(0, 2^{k+1}R)) \leq Cw(I(0, 2R)) \left(\frac{|2^{k+1}R|}{|2R|} \right)^q \leq Cw(I(0, 2R))(2^k)^q \leq C(w, R)2^{kq}$$

So,

$$\begin{aligned}
\sum_{|m|>2R} |T_\phi a(m)|^p w(m) &\leq C \sum_{k=1}^{\infty} \sum_{2^k R<|m|\leq 2^{k+1}R} \frac{w(m)}{|m|^p} \\
&\leq C \sum_{k=1}^{\infty} (2^k R)^{-p} \sum_{|m|\leq 2^{k+1}R} w(m) \\
&\leq C \sum_{k=1}^{\infty} (2^k R)^{-p} C(w, R) 2^{kq} \\
&= C(w, R) \sum_{k=1}^{\infty} 2^{k(q-p)} = C(w, R) \sum_{k=1}^{\infty} \left(\frac{1}{2^{p-q}} \right)^k < \infty
\end{aligned}$$

Combining both results, $T_\phi a \in \ell^p_w(\mathbb{Z})$.

∎

Theorem 8.6. *Let T_ϕ be a Calderón-Zygmund operator and let $w \in A_1$. Then for any $\{a(n) : n \in \mathbb{Z}\} \in \ell^1_w(\mathbb{Z})$,*

$$w(\{m \in \mathbb{Z} : |T_\phi a(m)| > \lambda\}) \leq \frac{C}{\lambda} \sum_{m\in\mathbb{Z}} |a(m)| w(m)$$

Proof. Perform Calderón-Zygmund decomposition (Theorem[6.1]) of sequence $\{a(n) : n \in \mathbb{Z}\}$ at height λ and obtain disjoint dyadic intervals $\{I_j\}$ which satisfy

$$\lambda \leq \frac{1}{|I_j|} \sum_{m\in I_j} |a(m)| \leq 2\lambda$$

Decompose $a(m) = g(m) + b(m), m \in \mathbb{Z}$

$$g(m) = \begin{cases} a(m) & \text{if} \quad m \notin \Omega \\ \frac{1}{|I_j|} \sum_{k\in I_j} a(k) & \text{if} \quad m \in I_j \end{cases}$$

where $\Omega = \cup_j I_j$

$$b(m) = \sum_{j=1}^{\infty} b_j(m)$$

where

$$b_j(m) = \left(a(m) - \frac{1}{|I_j|} \sum_{k \in I_j} a(k) \right) \chi_{I_j}(m)$$

Write

$$w(\{m \in \mathbb{Z} : |T_\phi a(m)| > \lambda\})$$
$$\leq w(\left\{ m \in \mathbb{Z} : |T_\phi g(m)| > \frac{\lambda}{2} \right\}) + w(\left\{ m \in \mathbb{Z} : |Tb(m)| > \frac{\lambda}{2} \right\})$$

To estimate the first term, note that $w \in A_1$ implies $w \in A_2$.
Further since T_ϕ is bounded on $\ell^2_w(\mathbb{Z})$, it follows that

$$w(\left\{ m \in \mathbb{Z} : |T_\phi g(m)| > \frac{\lambda}{2} \right\})$$
$$\leq \frac{4}{\lambda^2} \sum_{m \in \mathbb{Z}} |T_\phi g(m)|^2 w(m)$$
$$\leq \frac{C}{\lambda^2} \sum_{m \in \mathbb{Z}} |g(m)|^2 w(m)$$
$$= \frac{C}{\lambda^2} \left(\sum_{m \in \Omega^c} |g(m)|^2 w(m) + \sum_{m \in \Omega} |g(m)|^2 w(m) \right)$$

Now,

$$\sum_{m \in \Omega^c} |g(m)|^2 w(m)$$
$$\leq \lambda \sum_{m \in \Omega^c} |g(m)| w(m) \leq \lambda \sum_{m \in \Omega^c} |a(m)| w(m)$$

Note $w \in A_1$ implies $\frac{w(I)}{|I|} \leq Cw(m) \quad \forall m \in I$. So on Ω,

$$\sum_{m \in \Omega} |g(m)|^2 w(m) \leq 4\lambda^2 \sum_{m \in \Omega} w(m)$$
$$= 4\lambda \sum_j \left(\left(\frac{1}{|I_j|} \sum_{k \in I_j} |a(k)| \right) \left(\sum_{m \in I_j} w(m) \right) \right.$$
$$= 4\lambda \sum_j \left(\left(\frac{1}{|I_j|} \sum_{k \in I_j} |a(k)| \right) \left(w(k)|I_j| \right) \right.$$
$$= 4\lambda \sum_j \left(\left(\sum_{k \in I_j} |a(k)| \right) \left(w(k) \right) \right)$$
$$\leq 4C\lambda \sum_j \left(\sum_{m \in I_j} |a(m)| w(m) \right)$$

$$\leq 4C\lambda \sum_{m\in\mathbb{Z}} |a(m)|w(m)$$

From above estimates we get

$$w(\left\{m \in \mathbb{Z} : |T_\phi g(m)| > \frac{\lambda}{2}\right\}) \leq \frac{C}{\lambda}|a(m)|w(m)$$

Consider,

$$w(\left\{m \in \mathbb{Z} : Tb(m) > \frac{\lambda}{2}\right\}) \leq w(\cup_j 3I_j) + w(\left\{m \in \mathbb{Z} \setminus \cup_j 3I_j : |Tb(m)| > \frac{\lambda}{2}\right\})$$

For the second estimate, by Lemma[4.1]

$$w(\cup_j 3I_j) \leq C \sum_j w(I_j) \leq C \sum_j \frac{w(I_j)}{|I_j|}|I_j|$$
$$\leq C \sum_j \frac{w(I_j)}{|I_j|} \frac{C}{\lambda} \left(\sum_{k\in I_j} |a(k)| \right)$$
$$\leq \frac{C}{\lambda} \sum_j \left(\sum_{k\in I_j} |a(k)| \frac{w(I_j)}{|I_j|} \right)$$
$$\leq \frac{C}{\lambda} \left(\sum_{k\in I_j} |a(k)|w(k) \right)$$
$$\leq \frac{C}{\lambda} \sum_{k\in\Omega} |a(k)|w(k)$$
$$\leq \frac{C}{\lambda} \sum_{k\in\mathbb{Z}} |a(k)|w(k)$$

Now let c_j be center of I_j. Then, since b_j has zero average on I_j.

$$w(\left\{m \in \mathbb{Z} \setminus \cup_j 3I_j : |T_\phi b(m)| > \frac{\lambda}{2}\right\})$$
$$\leq \frac{C}{\lambda} \sum_{m\in\mathbb{Z}\setminus\cup_j 3I_j} |T_\phi b(m)|w(m)$$
$$= \frac{C}{\lambda} \sum_{m\in\mathbb{Z}\setminus\cup_j 3I_j} |\sum_{n\in\mathbb{Z}} \phi(m-n) b_j(n)|w(m)$$
$$\leq \frac{C}{\lambda} \sum_{m\in\mathbb{Z}\setminus\cup_j 3I_j} |\sum_j \sum_{n\in I_j} \phi(m-n) b_j(n)|w(m)$$
$$\leq \frac{C}{\lambda} \sum_{m\in\mathbb{Z}\setminus\cup_j 3I_j} |\sum_j \sum_{n\in I_j} [\phi(m-n) - \phi(c_j - m)] b_j(n)|w(m)$$

If $m \in \mathbb{Z} \setminus \cup_j 3I_j$ and $n \in I_j$ then $|m-n| \geq |I_j| \quad \forall j$. So, it follows that $\forall j \in \mathbb{Z}$, from Lemma[8.3] $|\phi(m-n) - \phi(c_j - m)| \leq C\frac{|I_j|}{|m-n|^2}$.

It follows that,

$$w(\{m \in \mathbb{Z} \setminus \cup_j 3I_j : |Tb(m)| > \lambda\})$$

$$
\begin{aligned}
&\leq \frac{C}{\lambda} \sum_{m\in\mathbb{Z}\setminus 3I_j} \sum_{j} \sum_{n\in I_j} \bigg(\frac{C|I_j|}{|m-n|^2} w(m) \bigg) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} \sum_{m\in\mathbb{Z}\setminus 3I_j} \bigg(\frac{|I_j|}{|m-n|^2} w(m) \bigg) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} \sum_{s=0}^{\infty} \sum_{2^s|I_j|<|m-n|\leq 2^{s+1}|I_j|} \bigg(\frac{|I_j|}{|m-n|^2} w(m) \bigg) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} \sum_{s=0}^{\infty} \frac{I_j}{2^{2s}|I_j|^2} \sum_{2^s|I_j|<|m-n|\leq 2^{s+1}|I_j|} w(m) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} Mw(n) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} w(n) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} \Big(|a_j(n)| + |g_j(n)| \Big) \chi_{I_j}(n) w(n) \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} |a_j(n)| w(n) + \frac{C}{\lambda} \sum_{j} \sum_{n\in I_j} |g_j(n)| w(n) \\
&\leq \frac{C}{\lambda} \sum_{n\in\mathbb{Z}} |a(n)| w(n) + \frac{C}{\lambda} \sum_{n\in\mathbb{Z}} |g(n)| w(n) \\
&\leq \frac{C}{\lambda} \sum_{m\in\mathbb{Z}} |a(n)| w(n)
\end{aligned}
$$

Combining both estimates for $T_\phi g, T_\phi b$, we get desired result. ∎

Now, we prove the weak and strong type inequalities for the maximal singular operator $T_\phi^\star$ operator on $l_w^p(\mathbb{Z})$ spaces. Here, we use transference method to transfer the corresponding results on $\mathbb{R}$.

The following lemma, whose proof is obvious, is used in the proof of Theorem[8.8]

Lemma 8.7. *Suppose $\{w(n) : n \in \mathbb{Z}\}$ is a sequence in $A_p(\mathbb{Z}), 1 \leq p < \infty$. Put*

$$
w'(x) = \begin{cases} w(j) & if \quad x \in [j - \frac{1}{4}, j + \frac{1}{4}], \quad j \in \mathbb{Z} \\ 0 & \textit{otherwise} \end{cases}
$$

If $w \in A_p(\mathbb{Z})$, then $w' \in A_p(\mathbb{R}), 1 \leq p < \infty$.
If $w \in A_1(\mathbb{Z})$, then $w' \in A_1(\mathbb{R})$.

Theorem 8.8. *If T_ϕ is a singular kernel operator, then for $1 < p < \infty$, $T_\phi^\star$ is bounded on $\ell_w^p(\mathbb{Z})$ if $w \in A_p$ and $T_\phi^\star$ is weak (1,1) with respect to w if $w \in A_1$.*

Proof. Let $K(x)$ be the linear extension of ϕ. Also for a given sequence $\{a(n) : n \in \mathbb{Z}\}$, we define a function $f(x) = \sum_{m\in\mathbb{Z}} a(m)\chi_{I_m}(x)$ where $I_m = (m - \frac{1}{4}, m + \frac{1}{4})$.

The following inequality which gives the relation between the maximal singular operator on $\mathbb{Z}$ and the maximal singular integral operator on $\mathbb{R}$ is proved in [8].

(A5) $$T_\phi^\star a(m) \leq C(T_K^\star f(x) + Sf(x)), \quad x \in I_m$$

where

$$\begin{aligned} Sf(x) &= \int_{|x-y|>\frac{1}{2}} \frac{|f(y)|}{(x-y)^2} dy \\ &= \sum_{k=0}^{\infty} \int_{2^k \geq |x-y| > 2^{k-1}} \frac{|f(y)|}{(x-y)^2} dy \\ &\leq \sum_{k=0}^{\infty} \frac{4}{2^{2k}} \int_{|x-y|\leq 2^k} |f(y)| dy \\ &\leq CMf(x) \end{aligned}$$

Now,

$$\begin{aligned} \|f\|^p_{L^p_{w'(\mathbb{R})}} &= \int_{\mathbb{R}} |f(x)|^p w'(x) dx \\ &= \sum_{m\in\mathbb{Z}} \int_{I_m} |a(m)|^p w(m) dx = \sum_{m\in\mathbb{Z}} \frac{1}{2} |a(m)|^p w(m) = \frac{1}{2} \|a\|_{\ell^p_w(\mathbb{Z})} \end{aligned}$$

and

$$\|Sf\|_{L^p_{w'(\mathbb{R})}} = \left(\int_{\mathbb{R}} | \int_{|x-y|>\frac{1}{2}} \frac{|f(y)|}{(x-y)^2} dy|^p w'(x) dx \right)^{\frac{1}{p}}$$

Therefore, using Lemma[8.7]

$$\begin{aligned} \left\|T^\star_\phi a\right\|_{l^p_w(\mathbb{Z})} &= \left(\sum_{m\in\mathbb{Z}} |T^\star_\phi a(m)|^p w(m) \right) \\ &\leq \sum_{m\in\mathbb{Z}} 2 \int_{I_m} |T^\star_\phi a(m)|^p w(m) dx \\ &\leq \left(2C \sum_{m\in\mathbb{Z}} \int_{I_m} \left[T^\star_k f(x) + Sf(x) \right]^p w'(x) dx \right)^{\frac{1}{p}} \\ &\leq \left(2C \int_{\mathbb{R}} \left[T^\star_k f(x) + Sf(x) \right]^p w'(x) dx \right)^{\frac{1}{p}} \\ &\leq \left(2C \int_{\mathbb{R}} \left[T^\star_k f(x) + Mf(x) \right]^p w'(x) dx \right)^{\frac{1}{p}} \\ &\leq 2C \left(\|T^\star_K f\|_{L^p_{w'(\mathbb{R})}} + \|Mf\|_{L^p_{w'(\mathbb{R})}} \right) \\ &\leq C \|f\|_{L^p_{w'(\mathbb{R})}} \\ &= C \|a\|_{\ell^p_{w(\mathbb{Z})}} \end{aligned}$$

where we used $T^\star_K$ is of strong type (p,p) on $L^p_w(\mathbb{R})$ and $Sf(x)$ is also of strong type (p,p) on $L^p_w(\mathbb{R})$. Refer [8]. It follows that $T^\star_\phi$ is strong type (p,p) on $\ell^p_w(\mathbb{Z})$.

Now, we shall prove the weak type (1,1) inequality.

From [$A5$], we have

$$\left\{ m \in \mathbb{Z} : T^\star_\phi a(m) > \lambda \right\}$$

$$\subseteq \left\{x \in I_m : T_K^\star f(x) > \frac{\lambda}{2C}\right\} \cup \left\{x \in I_m : Sf(x) > \frac{\lambda}{2C}\right\}$$
$$\subseteq \left\{x \in I_m : T_K^\star f(x) > \frac{\lambda}{2C}\right\} \cup \left\{x \in I_m : Mf(x) > \frac{\lambda}{2C}\right\}$$

Therefore, the weighted analogue would give for each $x \in I_m$,

$$|w(\{m \in \mathbb{Z} : T_\phi^\star a(m) > \lambda\})|$$
$$\leq |w(\left\{x \in I_m : T_K^\star f(x) > \frac{\lambda}{2C}\right\})| + |w(\left\{m \in \mathbb{Z} : Mf(x) > \frac{\lambda}{2C}\right\})|$$

Hence, $T_K^\star$ is of weak type (1,1) and M is also of weak type (1,1) on $L_w^1(\mathbb{Z})$. Refer [8]. This gives $T_\phi^\star$ is weak type (1,1) on $\ell_w^1(\mathbb{Z})$. ∎

Now, we want to prove that if $T_\phi^\star$ is bounded on $\ell_w^p(\mathbb{Z}), 1 < p < \infty$, then $w \in A_p(\mathbb{Z})$ when $T_\phi^\star$ is maximal Hilbert transform $H^\star$ whose kernel is given by

$$\phi(k) = \begin{cases} \frac{1}{k} & \text{if} \quad k \neq 0 \\ 0 & k = 0 \end{cases}$$

The methodology used in our proof is given in [5]. Observe that for $1 < p < \infty$, if $H^\star$ is bounded on $\ell_w^p(\mathbb{Z})$ then H is bounded on $\ell_w^p(\mathbb{Z})$,

Theorem 8.9. *If for $1 < p < \infty$ and any positive sequence $\{w(n) : n \in \mathbb{Z}\}$*

$$\sum_{m\in\mathbb{Z}} |Ha(m)|^p w(m) \leq C \sum_{m\in\mathbb{Z}} |a(m)|^p w(m) \quad \forall \{a(n) : n \in \mathbb{Z}\}$$

then w satisfies the discrete A_p condition which is as follows

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \leq C$$

for any interval I in $\mathbb{Z}$.

Proof. Let $I_1 = [m, m+1, \ldots n]$ be any interval in $\mathbb{Z}$. Consider doubling interval I_1 and relabel it as

$$I_0 = [m, m+1, \ldots n, n+1, n+2, \ldots 2n - m + 1]$$

so that $I_0 = I_1 \cup I_2$, where

$$I_2 = [n+1, n+2 \ldots 2n - m + 1]$$

Take a non-negative sequence $\{a(n) : n \in \mathbb{Z}\}$ supported in I_1. Observe that

$$|Ha(m)| = |\sum_{n\in I_1} \frac{a(n)}{m-n}| = \sum_{n\in I_1} \frac{a(n)}{|m-n|}$$

So, for $m \in I_2$ we get

$$|Ha(m)| \geq \frac{1}{2}\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)\chi_{I_2}(m) \quad \forall n \in I_1$$

Now, using boundedness of H on $\ell_w^p(\mathbb{Z})$ i.e,

$$\sum_{m\in\mathbb{Z}} |Ha(m)|^p w(m) \leq C \sum_{m\in\mathbb{Z}} |a(m)|^p w(m)$$

Since support of $\{a(n) : n \in \mathbb{Z}\}$ is in I_1, we have,

$$\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)^p \left(\sum_{m\in I_2} w(m)\right) \le \sum_{m\in\mathbb{Z}}\left(\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)^p \chi_{I_2}(m)w(m)\right)$$

$$\le \sum_{m\in\mathbb{Z}} \chi_{I_2}(m)|Ha(m)|^p w(m) \le C\sum_{m\in\mathbb{Z}}|a(m)|^p w(m) = C\sum_{m\in I_1}|a(m)|^p w(m)$$

It follows that

$$\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)^p \left(\sum_{m\in I_2} w(m)\right) \le C\sum_{m\in I_1}|a(m)|^p w(m) \tag{A7}$$

Take $a(n) = 1 \quad \forall n \in \mathbb{Z}$ in $[A7]$ and by interchanging I_1 and I_2, we have the following two inequalities.

$$\sum_{m\in I_2} w(m) \le C\sum_{m\in I_1} w(m) \tag{A8}$$

$$\sum_{m\in I_1} w(m) \le C\sum_{m\in I_2} w(m) \tag{A9}$$

Likewise, take $a(n) = w(n)^{\frac{-1}{p-1}} \quad \forall n \in \mathbb{Z}$ in $[A7]$ to get

$$\left(\sum_{m\in I_2} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m\in I_1} w(m)^{\frac{-1}{p-1}}\right)^p \le C\sum_{m\in I_1} w(m)^{\frac{-p}{p-1}} w(m)$$

So,

$$\left(\sum_{m\in I_2} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m\in I_1} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \le C$$

Therefore,

$$\left(\frac{1}{|I_1|}\sum_{m\in I_1} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m\in I_1} w(m)^{\frac{-1}{p-1}}\right)^{p-1}$$

$$\le \left(\frac{C}{|I_1|}\sum_{m\in I_2} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m\in I_1} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \le C$$

It follows that $w \in A_p(\mathbb{Z})$. ∎

9. Maximal Singular Operator on Variable Sequence Spaces $\ell^{p(\cdot)}(\mathbb{Z})$

In this section, we prove weak type, and strong type inequalities for the maximal singular operator on $\ell^{p(\cdot)}(\mathbb{Z})$ spaces, $1 \le p < \infty$, using Rubio de Francia extrapolation method given in [7].

Lemma 9.1. *Given $p(\cdot)$ such that M is bounded on $\ell^{p(\cdot)}(\mathbb{Z})$, for each $h \in \ell^{p(\cdot)}(\mathbb{Z})$, define*

$$Rh(m) = \sum_{k=0}^{\infty} \frac{M^k h(m)}{2^k \|M\|^k_{B(\ell^{p(\cdot)}(\mathbb{Z}))}}$$

where for $k \ge 1$, $M^k = M \circ \ldots M$ where $\circ$ denotes composition operator acting k times and $M^0 = |I|$, I being identity operator. Then
(a) For all $m \in \mathbb{Z}, |h(m)| \le Rh(m)$

(b) R is bounded on $\ell^{p(\cdot)}(\mathbb{Z})$ *and* $\|Rh\|_{\ell^{p(\cdot)}} \leq 2\|h\|_{p(\cdot)}$
(c) $Rh \in A_1$ *and* $[Rh]_{A_1} \leq 2\|M\|_{B(\ell^{p(\cdot)}(\mathbb{Z}))}$*, where* $[]_{A_1}$ *denotes constant of* A_1 *weight.*

Proof of (a) is obvious.The proof of (b) and (c) are same as in the case of $\mathbb{R}$. For the corresponding results on $\mathbb{R}$, refer [7].

Lemma 9.2. *Given, a sequence* $\boldsymbol{a} = \{a(n)\}$*, and* $p(\cdot) \in \mathcal{S}$ *then for all* $s, \frac{1}{p_-} \leq s < \infty$*,*

$$\||a|^s\|_{p(\cdot)} = \|a\|_{sp(\cdot)}^s$$

This follows at once from the definition of $\ell^{p(\cdot)}(\mathbb{Z})$ norm. For details refer [7].

Theorem 9.3. *Given a sequence* $\{a(n) : n \in \mathbb{Z}\}$*, suppose* $p(\cdot) \in \mathcal{S}$ *such that* $p_- > 1$*. Let* $T_\phi^\star$ *be a maximal singular operator.*
Then,

$$\left\|T_\phi^\star a\right\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$$

If $p_- = 1$*, then for all* $t > 0$

$$\left\|t\chi_{\{n:|T_\phi^\star a(n)|>t\}}\right\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$$

Proof. We will prove strong type inequality when $p_- > 1$.

Take p_0 such that $1 < p_0 \leq p_- \leq p_+ < \infty$. Here we use $Rh \in A_1(\mathbb{Z})$ and hence $Rh \in A_p(\mathbb{Z}), 1 < p < \infty$ and the boundedness of $T_\phi^\star$ on $\ell^p_{Rh}(\mathbb{Z})$.

Therefore by Lemma[9.2]

$$\begin{aligned}
\left\|(T_\phi^\star a)\right\|_{\ell^{p(\cdot)}(\mathbb{Z})}^{p_0} &= \left\|(T_\phi^\star a)^{p_0}\right\|_{\ell^{\frac{p(\cdot)}{p_0}}(\mathbb{Z})} \\
&= \sup_{h\in\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z}),\|h\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |T_\phi^\star a(k)|^{p_0}|h(k)| \\
&\leq \sup_{h\in\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z}),\|h\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |T_\phi^\star a(k)|^{p_0}Rh(k) \\
&\leq \sup_{h\in\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z}),\|h\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |a(k)|^{p_0}Rh(k) \\
&\leq C \sup_{h\in\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z}),\|h\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})}=1} \left\||a|^{p_0}\right\|_{\ell^{\frac{p(\cdot)}{p_0}}(\mathbb{Z})} \|Rh\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})} \\
&= C \sup_{h\in\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z}),\|h\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})}=1} \||a|\|_{\ell^{p(\cdot)}(\mathbb{Z})}^{p_0} \|Rh\|_{\ell^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{Z})} \\
&\leq 2C\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}^{p_0}
\end{aligned}$$

Now we are going to prove type weak type (1,1) inequality stated in the theorem.

Let $A = \left\{m \in \mathbb{Z} : |T_\phi^\star a(m)| > t\right\}$. Then,

$$\left\|(t\chi_{\{m\in\mathbb{Z}:|T_\phi^\star a(m)|>t\}})\right\|_{p(\cdot)}$$

$$\begin{aligned}
&\leq \sup_{h\in\ell^{p(\cdot)'}(\mathbb{Z}),\|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |t\chi_{\left\{m\in\mathbb{Z}:|T_\phi^\star a(m)|>t\right\}}(k)||h(k)| \\
&= \sup_{h\in\ell^{p(\cdot)'}(\mathbb{Z}),\|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} t\chi_A(k)Rh(k) \\
&= \sup_{h\in\ell^{p(\cdot)'}(\mathbb{Z}),\|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} tRh(A) \\
&= \sup_{h\in\ell^{p(\cdot)'}(\mathbb{Z}),\|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} t\frac{C}{t}\sum_{k\in\mathbb{Z}}|a(k)|Rh(k) \\
&\leq \sup_{h\in\ell^{p(\cdot)'}(\mathbb{Z}),\|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} C\sum_{k\in\mathbb{Z}}|a(k)|Rh(k) \\
&= C\sum_{k\in\mathbb{Z}}|a(k)|Rh(k) \\
&\leq C\,\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}\,\|Rh\|_{\ell^{p(\cdot)'}(\mathbb{Z})} \\
&\leq 2C\,\|a\|_{p(\cdot)}
\end{aligned}$$

∎

10. Maximal Ergodic Singular Operator

Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. We define the truncated maximal ergodic singular operator and maximal ergodic singular operator as follows:

$$\tilde{T}^\star_{\phi,N} f(x) = \sup_{1\leq n\leq N} |\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)|$$

$$\tilde{T}^\star_{\phi} f(x) = \sup_{n} |\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)|$$

Now, we prove the strong type, weak type inequalities for the maximal ergodic singular operator on weighted $L^p_w(X, \mathcal{B}, \mu)$ spaces.

Theorem 10.1. *Let $(X, \boldsymbol{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. If w is an ergodic A_p weight, $1 < p < \infty$, then the maximal ergodic singular operator*
satisfies

(1)
$$\left\|\tilde{T}^\star_{\phi} f\right\|_{L^p_w(X)} \leq C_p \|f\|_{L^p_w(X)} \quad \textit{if} \quad 1 < p < \infty$$
where C_p is independent of N.

(2) *If $w \in A_1$, then*
$$\int_{\left\{x\in X:|\tilde{T}^\star_\phi f(x)|>\lambda\right\}} w(x)d\mu(x) \leq \frac{C}{\lambda}\int_X |f(x)|w(x)d\mu(x)$$
where C_1 is independent of N.

Proof. Fix $N > 0$ and take a function $f \in L^p_w(X)$.

$$\tilde{T}^{\star}_{\phi,N} f(x) = \sup_{1\leq n\leq N} |\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)|$$

It is enough to prove that $\tilde{T}^{\star}_{\phi,N}$ satisfies (1) and (2) with constants not depending on N. Let $\lambda > 0$ and put

$$E_\lambda = \left\{x \in X : |\tilde{T}^{\star}_{\phi,N} f(x)| > \lambda\right\}$$

For x lying outside a μ null set and a positive integer L, define sequences

$$a_x(k) = \begin{cases} f(U^{-k}x) & if \quad |k| \leq L+N \\ 0 & otherwise \end{cases}$$

$$w_x(k) = \begin{cases} w(U^{-k}x) & if \quad |k| \leq L+N \\ 0 & otherwise \end{cases}$$

Therefore,

$$\begin{aligned}
&w(\left\{x \in X : |\tilde{T}^{\star}_{\phi,N} f(x)| > \lambda\right\}) = \int_{E_\lambda} w(x)d\mu(x) = \frac{1}{\lambda^p}\int_{E_\lambda} \lambda^p w(x)d\mu(x) \\
&\leq \frac{1}{\lambda^p}\int_{E_\lambda} |\tilde{T}^{\star}_{\phi,N} f(x)|^p w(x)d\mu(x) \\
&\leq \frac{1}{\lambda^p}\int_{X} |\tilde{T}^{\star}_{\phi,N} f(x)|^p w(x)d\mu(x) \\
&= \frac{1}{\lambda^p}\frac{1}{2L+1}\sum_{m=-L}^{L}\int_X |\tilde{T}^{\star}_{\phi,N} f(U^{-m}x)|^p w(U^{-m}x)d\mu(x) \\
&\leq \frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-L}^{L} |\tilde{T}^{\star}_{\phi,N} f(U^{-m}x)|^p w(U^{-m}x)d\mu(x) \\
&\leq \frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-L}^{L} |T^{\star}_{\phi,N} a_x(m)|^p w_x(m)d\mu(x) \\
&\leq \frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-\infty}^{\infty} |T^{\star}_{\phi,N} a_x(m)|^p w_x(m)d\mu(x) \\
&\leq C\frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-\infty}^{\infty} |a_x(m)|^p w_x(m)d\mu(x) \\
&= C\frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-(L+N)}^{(L+N)} |a_x(m)|^p w_x(m)d\mu(x) \\
&= C\frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-(L+N)}^{(L+N)} |f(U^{-m}x)|^p w(U^{-m}x)d\mu(x) \\
&\leq \frac{C}{\lambda^p}\frac{1}{2L+1}\sum_{m=-(L+N)}^{(L+N)}\int_X |f(U^{-m}x)|^p w(U^{-m}x)d\mu(x)
\end{aligned}$$

$$= \frac{C}{\lambda^p}\frac{1}{2L+1}\sum_{m=-(L+N)}^{(L+N)}\int_X |f(x)|^p w(x)d\mu(x)$$
$$\leq \frac{C}{\lambda^p}\frac{1}{2L+1}(2(L+N)+1)\,\|f\|^p_{L^p_w(X)}$$
$$\leq \frac{C}{\lambda^p}\,\|f\|^p_{L^p_w(X)}$$

by choosing L appropriately. Conclusion (1) of the theorem now follows by using the Marcinkiewicz interpolation theorem. ∎

Now, we prove the converse of the above theorem when $\tilde{T}^\star_{\phi,N}$ with singular kernel as

$$\phi(k) = \begin{cases} \frac{1}{k} & \text{if} \quad k \neq 0 \\ 0 & k = 0 \end{cases}$$

The singular operator associated with this particular singular kernel is known as the maximal ergodic Hilbert transform and is denoted by $\tilde{H}^\star$. Here, we further assume that the associated measure preserving transformation is ergodic.

Definition 10.1 (Ergodic Rectangle)**.** Let E be a subset of X with positive measure and let $K \geq 1$ be such that $U^iE \cap U^jE = \phi \quad \text{if} \quad i \neq j$ and $-K \leq i,j \leq K$. Then the set $R = \cup_{i=-K}^{K} U^iE$ is called ergodic rectangle of length $2K+1$ with base E.

For the proof of following lemma, refer[4].

Lemma 10.2. *Let $(X, \boldsymbol{B}, \mu)$ be a probability space, U an ergodic invertible measure preserving transformation on X and K a positive integer.*

(1) *If $F \subseteq X$ is a set of positive measure then there exists a subset $E \subseteq F$ of positive measure such that E is base of an ergodic rectangle of length $2K+1$.*
(2) *There exists a countable family $\{E_j\}$ of bases of ergodic rectangles of length $2K+1$ such that $X = \cup_j E_j$.*

Theorem 10.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space, U an invertible ergodic measure preserving transformation on X. If $\tilde{H}^\star f$ is bounded on $L^p_w(X)$ for some $1 < p < \infty$, then $w \in A_P(X)$.*

Proof. For the given function w on X, for a.e $x \in X$ define the sequence $w_x(k) = w(U^{-k}x)$. We shall prove that

$$esssup_{x\in X}\left(\frac{1}{|I|}\sum_{k\in I}|w_x(k)|\right)\left(\frac{1}{|I|}\sum_{k\in I}|w_x(k)|^{p'-1}\right)^{p-1} \leq C$$

This will prove that $w \in A_p(X)$. In order to prove this, we shall prove that the maximal Hilbert transform $H^\star$ is bounded on $\ell^p_{w_x}(\mathbb{Z})$ and

$$\|H^\star a\|_{\ell^p_{w_x}(\mathbb{Z})} \leq C_p\,\|a\|_{\ell^p_{w_x}(\mathbb{Z})}$$

where C_p is independent of x. In order to prove the above inequality, take a sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_{w_x}(\mathbb{Z})$.

Let $R = \cup_{k=-2J}^{2J} U^kE$ be an ergodic rectangle of length $4J+1$ with base E. Let F be any measurable subset of E. Then F is also base of an ergodic rectangle of length $4J+1$. Let $R' = \cup_{k=-2J}^{2J} U^kF$. Define function f and w as follows.

$$f(U^{-k}x) = \begin{cases} a(k) \quad if \quad x \in F \quad \text{and} - J \leq k \leq J \\ 0 \quad otherwise \end{cases}$$

Then as shown in [3]

$$\|f\|^p_{L^p_w(X)} = \|a\|_{\ell^p_{w_x}(\mathbb{Z})} \mu(F)$$

It is easy to observe that for $-J \leq m \leq J$ and $x \in F$

$$\tilde{H}^\star_J f(U^{-m}x) = H^\star_J a(m)$$

Now,

$$\begin{aligned} C\|f\|^p_{L^p_w(X)} &\geq \int_X |\tilde{H}^\star_J f(x)|^p w(x) d\mu(x) \\ &= \int_{R'} |\tilde{H}^\star_J f(x)|^p w(x) d\mu(x) \\ &= \sum_{k=-J}^{J} \int_{U^k F} |\tilde{H}^\star_J f(x)|^p w(x) d\mu(x) \\ &= \sum_{k=-J}^{J} \int_F |\tilde{H}^\star_J f(U^{-k}x)|^p w(U^{-k}x) d\mu(x) \\ &= \sum_{k=-J}^{J} \int_F |H^\star_J a(k)|^p w_x(k) d\mu(x) \\ &= \int_F \sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) d\mu(x) \end{aligned}$$

So from the above estimates

$$\frac{1}{\mu(F)} \int_F \sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) d\mu(x) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}$$

Since F was an arbitrary subset of E, we get

$$\sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}$$

a.e $x \in E$. Since U is ergodic, X can be written as countable union of bases of ergodic rectangles of length $4J+1$. Therefore for a.e $x \in X$,

$$\sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}$$

Since C is independent of J, a.e $x \in X$,

$$\sum_{k\in\mathbb{Z}} |H^\star a(k)|^p w_x(k) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}$$

It follows that the sequence $\{w_x(n) : n \in \mathbb{Z}\}$ as defined by $w_x(k) = w(U^k x)$ belongs to $A_p(\mathbb{Z})$ a.e $x \in X$ and A_p weight constant for w_x is independent of x so that $w \in A_p(X)$. ∎

Remark 10.1. Using the boundedness of maximal ergodic singular operator and Rubio de Francia method, we can prove that the maximal ergodic singular operator is bounded on variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces. But Rubio de Francia method assumes maximal ergodic operator is bounded on the variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces. With this assumption we can prove the boundedness of maximal ergodic singular operator to variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces.

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