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THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY FROM THE INTEGRAL OF $1/t$

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ABSTRACT. In this short note, we improve the arithmetic-geometric mean inequality using by the integral of $1/t$.

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1. INTRODUCTION

We define the arithmetic mean A_n and the geometric mean G_n by

$$A_n = \sum_{i=1}^{n} p_i a_i \quad and \quad G_n = \prod_{i=1}^{n} a_i^{p_i}$$

for real positive numbers $a_1, \ldots, a_n$ and $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n = 1$. Here, the inequality $A_n \geq G_n$ holds, with equality if and only if $a_1 = a_2 = \cdots = a_n$, which is called the arithmetic-geometric mean inequality and many proofs of the inequality are now known. The short proof by Alzer [1] using integrals is very interesting. We prove the inequality using a function different from that used by Alzer as follows. We may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then there exists integers k and l with $1 \leq k, l \leq n-1$ such that $a_k \leq G_n \leq a_{k+1}$ and $a_l \leq A_n \leq a_{l+1}$. Here, the inequality

$$\frac{A_n}{G_n} - 1 = \sum_{i=1}^{k} p_i \int_{a_i}^{G_n} \left(\frac{1}{t} - \frac{a_i}{t^2} \right) dt + \sum_{i=k+1}^{n} p_i \int_{G_n}^{a_i} \left(\frac{a_i}{t^2} - \frac{1}{t} \right) dt \geq 0$$

holds, with equality holds if and only if $a_i = G_n$ for $i = 1, \cdots n$, that is, if and only if $a_1 = \cdots = a_n$. Also, the inequality

$$\ln \frac{A_n}{G_n} = \sum_{i=1}^{l} p_i \int_{a_i}^{A_n} \left(\frac{1}{t} - \frac{a_i}{t^2} \right) dt + \sum_{i=l+1}^{n} p_i \int_{A_n}^{a_i} \left(\frac{a_i}{t^2} - \frac{1}{t} \right) dt \geq 0$$

holds, with equality holds if and only if $a_i = A_n$ for $i = 1, \cdots n$, that is, if and only if $a_1 = \cdots = a_n$. In this short note, we may show the refinements of the arithmetic-geometric mean inequality using by the integral of $1/t$. From the mean value theorem of integrals, for any real positive numbers a and b with $a < b$, we have a unique real number c with $a < c < b$ such that

$$\int_a^b \frac{1}{t} dt = \frac{1}{c} (b - a) \ .$$

Therefore, we have

$$\int_a^b \frac{1}{t} dt = \left(\frac{1}{a} - \epsilon \right) (b - a) = \left(\frac{1}{b} + \delta \right) (b - a) \ ,$$

where $\epsilon = 1/a - 1/c$ and $\delta = 1/c - 1/b$. Our main results are as follows.

Theorem 1.1. *For any real positive numbers $a_1, \cdots, a_n$ and $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n = 1$, we set real positive numbers ϵ_i such that if $a_i \leq G_n$ then*

$$\left(\frac{1}{G_n} + \epsilon_i \right) (G_n - a_i) = \int_{a_i}^{G_n} \frac{1}{t} dt$$

and if $G_n \leq a_i$ then

$$\left(\frac{1}{G_n} - \epsilon_i \right) (a_i - G_n) = \int_{G_n}^{a_i} \frac{1}{t} dt \, .$$

The inequality

$$A_n - G_n \geq G_n \sum_{i=1}^{n} p_i \epsilon |G_n - a_i|$$

holds, with equality holds if and only if $a_1 = \cdots = a_n$, where $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$.

Theorem 1.2. *For any real positive numbers* $a_1, \cdots, a_n$ *and* $p_1, \cdots, p_n$ *with* $p_1 + \cdots + p_n = 1$, *we set real positive numbers* δ_i *such that if* $a_i \leq A_n$ *then*

$$\left(\frac{1}{A_n} + \delta_i\right)(A_n - a_i) = \int_{a_i}^{A_n} \frac{1}{t} dt$$

and if $A_n \leq a_i$ *then*

$$\left(\frac{1}{A_n} - \delta_i\right)(a_i - A_n) = \int_{A_n}^{a_i} \frac{1}{t} dt \,.$$

The inequality

$$A_n - \left(e^{\sum_{i=1}^{n} p_i \delta |A_n - a_i|}\right) G_n \geq 0$$

holds, with equality holds if and only if $a_1 = \cdots = a_n$, *where* $\delta = \min_{1 \leq i \leq n} \delta_i$.

2. PROOF OF MAIN THEOREMS

Proof of Theorem 1.1. We may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then there exists an integers k with $1 \leq k \leq n-1$ such that $a_k \leq G_n \leq a_{k+1}$. We have only one positive real number ϵ_i such that

$$\left(\frac{1}{G_n} + \epsilon_i\right)(G_n - a_i) = \int_{a_i}^{G_n} \frac{1}{t} dt$$

for $1 \leq i \leq k$ and

$$\left(\frac{1}{G_n} - \epsilon_i\right)(a_i - G_n) = \int_{G_n}^{a_i} \frac{1}{t} dt$$

for $k + 1 \leq i \leq n$. If we set $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$, then we have followings.

$$\sum_{i=1}^{k} p_i \int_{a_i}^{G_n} \frac{1}{t} dt - \sum_{i=k+1}^{n} p_i \int_{G_n}^{a_i} \frac{1}{t} dt = 0 \,,$$

$$\sum_{i=1}^{k} p_i \left(\frac{1}{G_n} + \epsilon_i\right)(G_n - a_i) - \sum_{i=k+1}^{n} p_i \left(\frac{1}{G_n} - \epsilon_i\right)(a_i - G_n) = 0 \,,$$

$$\sum_{i=1}^{k} p_i \left(\frac{1}{G_n} + \epsilon\right)(G_n - a_i) - \sum_{i=k+1}^{n} p_i \left(\frac{1}{G_n} - \epsilon\right)(a_i - G_n) \leq 0 \,,$$

$$\sum_{i=1}^{k} p_i \left\{\left(1 - \frac{a_i}{G_n}\right) + \epsilon\,(G_n - a_i)\right\} - \sum_{i=k+1}^{n} p_i \left\{\left(\frac{a_i}{G_n} - 1\right) - \epsilon\,(a_i - G_n)\right\} \leq 0 \,,$$

$$\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} \frac{p_i a_i}{G_n} + \sum_{i=1}^{n} p_i \epsilon |G_n - a_i| \leq 0 \,,$$

$$1 - \frac{A_n}{G_n} + \sum_{i=1}^{n} p_i \epsilon |G_n - a_i| \leq 0 \,,$$

$$A_n - G_n \geq G_n \sum_{i=1}^{n} p_i \epsilon |G_n - a_i| \,.$$

Moreover, the equality holds if and only if $a_i = G_n$ for $i = 1, \cdots n$, that is, if and only if $a_1 = \cdots = a_n$. ∎

Proof of Theorem 1.2. We may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then there exists an intger l with $1 \leq l \leq n-1$ such that $a_l \leq A_n \leq a_{l+1}$. We have only one positive real number ϵ_i such that

$$\left(\frac{1}{A_n} + \delta_i\right)(A_n - a_i) = \int_{a_i}^{A_n} \frac{1}{t} dt$$

for $1 \leq i \leq l$ and

$$\left(\frac{1}{A_n} - \delta_i\right)(a_i - A_n) = \int_{A_n}^{a_i} \frac{1}{t} dt$$

for $l+1 \leq i \leq n$. If we set $\delta = \min_{1 \leq i \leq n} \delta_i$, then we have followings.

$$\sum_{i=1}^{l} p_i \int_{a_i}^{A_n} \frac{1}{t} dt - \sum_{i=l+1}^{n} p_i \int_{A_n}^{a_i} \frac{1}{t} dt = \ln \frac{A_n}{G_n} \,,$$

$$\sum_{i=1}^{l} p_i \left(\frac{1}{A_n} + \delta_i\right)(A_n - a_i) - \sum_{i=l+1}^{n} p_i \left(\frac{1}{A_n} - \delta_i\right)(a_i - A_n) = \ln \frac{A_n}{G_n} \,,$$

$$\sum_{i=1}^{l} p_i \left(\frac{1}{A_n} + \delta\right)(A_n - a_i) - \sum_{i=l+1}^{n} p_i \left(\frac{1}{A_n} - \delta\right)(a_i - A_n) \leq \ln \frac{A_n}{G_n} \,,$$

$$\sum_{i=1}^{l} p_i \left\{\left(1 - \frac{a_i}{A_n}\right) + \delta\,(A_n - a_i)\right\} - \sum_{i=l+1}^{n} p_i \left\{\left(\frac{a_i}{A_n} - 1\right) - \delta\,(a_i - A_n)\right\} \leq \ln \frac{A_n}{G_n} \,,$$

$$\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} \frac{p_i a_i}{A_n} + \sum_{i=1}^{n} p_i \delta |A_n - a_i| \leq \ln \frac{A_n}{G_n} \,,$$

$$\sum_{i=1}^{n} p_i \delta |A_n - a_i| \leq \ln \frac{A_n}{G_n} \,,$$

$$A_n - \left(e^{\sum_{i=1}^{n} p_i \delta |A_n - a_i|}\right) G_n \geq 0 \,.$$

Moreover, the equality holds if and only if $a_i = A_n$ for $i = 1, \cdots n$, that is, if and only if $a_1 = \cdots = a_n$. ∎

REFERENCES

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