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DYADIC RIESZ WAVELETS ON LOCAL FIELDS OF POSITIVE CHARACTERISTICS

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ABSTRACT. In this research paper, we introduce a novel theory for the construction of a Riesz wavelet basis in the space $L^2(K)$, where K is a local field with positive characteristics. Our approach is two fold: firstly, we derive some essential characterizations of the scaling function associated with the structure of a Riesz MRA on a local field, and secondly, we review existing methods for constructing wavelet frames in $L^2(K)$. We also present a well elaborated example for a better comprehension of our theory. Due to mathematical convenience, we limit ourselves to the case of dyadic dilations only.

Key words and phrases: Local fields; Frame multiresolution analysis; Wavelet frames; Refinable function; $\mathfrak{p}$-substantial.

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1. INTRODUCTION

The emergence of wavelets and multiresolution analysis stemmed from the demand for improved signal processing methods that are both more efficient and precise. Conventional Fourier analysis, which breaks down signals into sinusoidal elements, encounters difficulties when it comes to capturing localized characteristics and managing non-stationary signals. In contrast, wavelets present a more adaptable and localized means of representing signals. In the traditional multiresolution analysis framework, a collection of scaling functions and wavelet functions establishes an orthonormal foundation for the signal space. Over the past decade, there has been substantial research into wavelet bases in both one and multiple dimensions. Y. Meyer [20], Mallat [18], C. Chui [8], I. Daubechies [10], and several other researchers made significant contributions to both the theoretical and practical aspects of orthonormal wavelet bases. The prevalent approach for constructing orthonormal wavelet bases stems from the concept of multiresolution analysis (MRA), which involves a nested sequence of approximation subspaces given as

$$\{0\} \to \cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots \to L^2(\mathbb{R})$$

generated by a scaling function $\phi \in V_0$, the concept can be understood as a family of transformations involving both dilations and translations,

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k),\ j,k \in \mathbb{Z}$$

which forms an orthonormal basis for V_j.

Extending the principles of multiresolution analysis (MRA) and wavelets to a local field K with positive characteristics entails tailoring the framework to align with the unique characteristics of this field. Incorporating a prime element of the field becomes crucial in shaping the foundations of MRA for such contexts, facilitating the analysis of signals within the framework of locally compact Abelian groups. In the following years, Dahlke [9] generalized the concept of MRA for arbitrary locally compact Abelian (LCA) groups. Then on, many authors have contributed to the field of construction of wavelet bases on a variety of groups [29, 15, 16]. Local fields of positive characteristic exhibit unique algebraic properties that differentiate them from the classical real and complex number fields. Benedetto [4, 5] established a Wavelet theory for local field. H. Jiang, D Li [13] defined MRA on a local field of positive characteristic and constructed the corresponding orthonormal wavelets. A recent advancement by [1, 2, 3, 26, 27] extended the concept of multiresolution analysis (MRA) to local fields with positive characteristics, diverging from the traditional Euclidean space framework. In this extension, the translation set operating on the scaling function to generate the subspace V_0 expands beyond a group structure, encompassing both $\mathcal{L}$ and translations of $\mathcal{L}$, where $\mathcal{L} = \{u(n) : n \in \mathbb{N}\}$ represents distinct coset representations of the unit disc $\mathfrak{D}$ within K^+.

Riesz wavelets are an essential class of wavelets that extend the concept of wavelet bases in functional analysis, particularly in the context of Riesz bases. Here, we wish to generalize this notion and thus construct a wavelet Riesz basis through MRA on a local field. The primary motivation for studying Reisz bases is that these bases are found to be very handy for studying the sampling of bandlimited signals (functions). It is well-known that, up to some transformation, Riesz bases are equivalent to the interpolation property. This makes them a robust tool in compress sensing and application to signal processing. Several works in the literature [12, 14, 22, 30] have addressed MRA and related structures for Riesz bases. These studies have extensively investigated the methods for constructing Riesz wavelet bases through MRA. Recently motivated by advancements in the theory of MRA, we have [17, 24, 23] undertaken the construction of Riesz MRA on locally compact Abelian groups.

The structure of this article is organized as follows: Section 2 presents the preliminary information and notation related to local fields and Riesz bases on local fields. In Section 3, we develop a Riesz MRA for local fields of positive characteristic. Section 4 introduces a comprehensive approach for constructing Riesz wavelets from the Riesz MRA. Finally, Section 5 provides a conclusion of our work.

2. PRELIMINARIES AND NOTATIONS

2.1. **A Background about Local Fields.** [3, 4, 5, 13, 28, 25, 27, 26] In this section, we establish the notations for local fields that will be utilized throughout the paper. A local field, denoted by K, is characterized as both an algebraic field and a topological space possessing the following properties: locally compact, complete, totally disconnected and non discrete. The additive and multiplicative groups associated with K are denoted by K^+ and K^*, respectively. A Haar measure, denoted by dx, can be chosen for K^+. Notably, if $\alpha \neq 0$ ($\alpha \in K$), then $d(\alpha x)$ also serves as a Haar measure, with $d(\alpha x) = |\alpha| dx$, where $|\alpha|$ is termed the absolute value or valuation of α, and $|0| = 0$.

The absolute value possesses the following properties:

(1) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
(2) $|xy| = |x| \cdot |y|$;
(3) $|x + y| \leq \max(|x|, |y|)$.

The last property, known as the ultrametric inequality, is significant.

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is known as the ring of integers in K, representing the unique maximal compact subring within K. Additionally, we define $\mathfrak{P} = \{x \in K : |x| < 1\}$ as the prime ideal in K. This prime ideal stands as the unique maximal ideal in $\mathfrak{D}$ and possesses both principal and prime properties.

Given the total disconnectedness of K, the absolute values $|x|$ as x traverses K constitute a discrete set, often represented as $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there exists an

element in $\mathfrak{P}$ with maximal absolute value. Let $\mathfrak{p}$ denote a fixed element with the maximum absolute value in $\mathfrak{P}$, referred to as a prime element of K. Notably, as an ideal in $\mathfrak{D}$, $\mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.
It is demonstrable that $\mathfrak{D}$ is both compact and open, thus implying $\mathfrak{P}$ also possesses these properties. Consequently, the residue space $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. For a detailed proof of this assertion, we refer to [21].

Remark 2.1. Since, in this paper, we focus on dyadic wavelet frames for mathematical convenience, therefore we take $p = q = 2$ in all the subsequent sections.

Now, for each $k \in \mathbb{Z}$, we define the fractional ideals $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$. Each $\mathfrak{P}^k$ is both compact and open and serves as a subgroup of K^+ [21]. Thus, if $x \in \mathfrak{P}^k$, where $k \in \mathbb{Z}$, x can be uniquely expressed as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$, where c_ℓ are coset representatives of the quotient group $\mathfrak{D}/\mathfrak{P}$.

2.1.1. ***Fourier analysis on local Fields***. Consider a measurable subset E of K, and denote the measure $|E|$ as the integral of the characteristic function $\chi_E(x)$ with respect to the normalized Haar measure dx on K, where $|\mathfrak{D}| = 1$.
It is evident that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$, with q defined as p^c, following straightforward observations. For a more in-depth exploration of these concepts, refer to [21].
In the realm of local fields, a significant feature emerges in the form of a nontrivial, unitary, continuous character denoted by Υ on K^+. It's established that K^+ exhibits self-duality (refer to [21]).
Now, let's focus on a specific character Υ on K^+ that is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$. Such a character can be constructed by initially selecting any nontrivial character and subsequently adjusting its scale. This construction is particularly relevant for a local field characterized by positive characteristics.
For $y \in K$, we define $\Upsilon_y(x) = \Upsilon(yx)$ for $x \in K$.
In the context of local fields, the Fourier transform of a function $f \in L^1(K)$ is denoted as $\widehat{f}(\omega)$ and defined by the integral:

$$\widehat{f}(\omega) = \int_K f(x)\overline{\Upsilon_\omega(x)}\, dx$$

It's worth noting that this expression can be alternatively written as:

$$\widehat{f}(\omega) = \int_K f(x)\Upsilon(-\omega x)\, dx$$

This definition is reminiscent of the standard Fourier analysis on the real line, emphasizing the adaptation to the locally compact, non-Archimedean nature of the local field K.
To define the Fourier transform of a function in $L^2(K)$, we introduce the characteristic functions Φ_k for $k \in \mathbb{Z}$. These functions are defined as the characteristic functions of the sets $\mathfrak{P}^k$.

Definition 2.1. For $f \in L^2(K)$, let $f_k = f\Phi_{-k}$, and define

$$\widehat{f}(\omega) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\Upsilon_\omega(x)}\, dx,$$

where the limit is taken in $L^2(K)$.

We can express the following theorem, as presented in Theorem 2.3 in [21]:

Theorem 2.1. *The Fourier transform is unitary on $L^2(K)$.*

2.1.2. *Operators on $L^2(K)$.* We know that for the operators of translation, modulation and dilation are an important part of wavelet theory. Thus, we wish to explicitly define these operators on the space $L^2(K)$.

(i). The translation operator: As a translating set for $L^2(K)$, we take the coset representatives of the quotient group $K^+/\mathfrak{D}$. More precisely, if we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\{u(n) : n \in \mathbb{N}_0\}$ to be the set of distinct coset representatives of $K^+/\mathfrak{D}$, then the translation operator $T : L^2(K) \to L^2(K)$ is given by

$$(T_n f)(x) = f(x - u(n)),\ n \in \mathbb{N}_0.$$

(ii). The modulation operator: Let $y \in K$ be give. Corresponding to this y, we define the modulation operator on $L^2(K)$ as

$$(E_y f)(x) = (\Upsilon_y f)(x) = \Upsilon(xy)\, f(x),\ x \in K.$$

(iii). The dilation operator: The dilation operator on the space $L^2(K)$ is defined as

$$(Df)(x) = q^{1/2}\, f(\mathfrak{p}^{-1}(x)).$$

If we take $\{u(n)\}_{n=0}^{\infty}$ to be a comprehensive collection of unique coset representatives of $\mathfrak{D}$ within K^+, then $\{\Upsilon_{u(n)}\}_{n=0}^{\infty}$ emerges as a set of distinct characters on $\mathfrak{D}$. In [21], it is shown that this set is exhaustive, leading to the following proposition.

Lemma 2.2. [28] *Let $\{u(n)\}_{n=0}^{\infty}$ represent a complete set of distinct coset representatives of $\mathfrak{D}$ within K^+. Then, the set $\{\Upsilon_n\}_{n=0}^{\infty}$ denotes a comprehensive collection of distinct characters on $\mathfrak{D}$. It's worth noting that for ease of notation, we shall denote Υ_n as $\Upsilon_{u(n)}$ for every n in $\mathbb{N}_0$. Furthermore, this set $\{\Upsilon_n\}_{n=0}^{\infty}$ not only constitutes a complete list of characters on $\mathfrak{D}$ but also establishes a complete orthonormal system over $\mathfrak{D}$.*

Given such a list of characters $\{\Upsilon_{u(n)}\}_{n=0}^{\infty}$, we define the Fourier coefficients of a function $f \in L^1(\mathfrak{D})$ as follows:

$$\widehat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\overline{\Upsilon_{u(n)}(x)}\, dx$$

The series

$$\sum_{n=0}^{\infty} \widehat{f}(u(n))\Upsilon_{u(n)}(x)$$

is termed the Fourier series of f. Drawing from the standard L^2 theory for compact Abelian groups, we infer that the Fourier series of f converges to f in $L^2(\mathfrak{D})$, and moreover,

$$\int_{\mathfrak{D}} |f(x)|^2\, dx = \sum_{n=0}^{\infty} |\widehat{f}(u(n))|^2$$

holds. Additionally, if $f \in L^1(\mathfrak{D})$ and $\widehat{f}(u(n)) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ almost everywhere. Note that, here $L^p(\mathfrak{D}) = \big\{f \in L^p(K) : f = 0\ a.e.\ \text{on } K\backslash\mathfrak{D}\big\}$.
In the subsequent sections, we will frequently encounter functions which are defined on whole of the field K and their restriction to $\mathfrak{D}$ lies in the space $L^2(\mathfrak{D})$. Such functions also repeat their values, and we observe that this repetition is dependent of set pf integers $\mathfrak{D}$. For our convenience, we define such functions here.

Definition 2.2. A function $f : K \to \mathbb{C}$ is said to be K-integral periodic if it satisfies

$$f(x + u(n)) = f(x),\ \forall\, x \in K, n \in \mathbb{N}_0.$$

We conclude this section by analysing some quotient groups. The proof of all the statements given below in this section may be found in [3, 4, 5, 13, 28]. Recall that, we have the following relation:

$$\mathfrak{P}^{-1}/\mathfrak{P} \cong \mathfrak{D}/\mathfrak{P} \cong GF(q).$$

Without loss of generality, we designate $\{u(n) : 0 \le n \le q-1\}$ as a set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$. This choice allows us to assert that $\{u(n)\mathfrak{p} : 0 \le n \le q-1\}$ forms a set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Furthermore, it has been demonstrated in [13, 25, 26, 27] that $u(0) = 0$. Combining these observations with the fact that $p = q = 2$ (Remark 2.1), we obtain:

$$\mathfrak{P}^{-1}/\mathfrak{D} = \{\mathfrak{D}, u(1) + \mathfrak{D}\} \quad \text{and} \quad \mathfrak{D}/\mathfrak{P} = \{\mathfrak{P}, u(1)\mathfrak{p} + \mathfrak{P}\}.$$

2.2. **Riesz Bases and Related Properties.** Let's delve into the concept of Riesz bases within an arbitrary, separable Hilbert space $\mathcal{H}$, and briefly touch upon some of their fundamental properties. For a comprehensive exploration, readers are directed to [7].

Definition 2.3. Consider a separable Hilbert space $\mathcal{H}$ and a countable index set $\mathbb{I}$. A sequence of elements $\{f_\beta\}_{\beta\in\mathbb{I}}$ is termed a Riesz basis for $\mathcal{H}$ if there exist an orthonormal basis $\{e_\beta : \beta \in \mathbb{I}\}$ and a bounded bijective operator $U : \mathcal{H} \to \mathcal{H}$ such that $f_\beta = Ue_\beta$ for all $\beta \in \mathbb{I}$.

An important property satisfied by a Riesz basis $\{f_\beta : \beta \in \mathbb{I}\}$ is the existence of constants A and B (both greater than zero) satisfying:

$$A||f||^2 \le \sum_{\beta\in\mathbb{I}} |\langle f, f_\beta\rangle|^2 \le B||f||^2, \quad \forall\, f \in \mathcal{H}.$$

In this context, A and B are known as the *Riesz bounds*, with A representing the *lower bound* and B the *upper bound*. From here, we can also conclude that a Riesz basis is, in fact, a special case of frames.

In wavelet theory, we often focus on families composed of translations of a single function. Therefore, it's essential to explore the conditions under which a family such as $\{T_k\phi : k \in \mathbb{N}_0\}$, where $\phi \in L^2(K)$, forms a Riesz sequence. Before delving into this analysis, let's introduce another notation denoted by Φ, representing a complex-valued function on K, defined as:

$$\Phi(\xi) = \sum_{n\in\mathbb{N}_0} |\widehat{\phi}(\xi + u(n))|^2. \tag{2.1}$$

It's evident that the function Φ is K-integral periodic and its restriction to $\mathfrak{D}$ belongs to the space $L^1(\mathfrak{D})$. For further insights, we refer [25] and related references. With this setup, we're prepared to state a lemma providing bounds for the function Φ. This lemma, serving as a generalized version of a result by Ron and Shen in [22], elucidates that the Riesz basis properties of $\{T_k\phi : k \in \mathbb{N}_0\}$ can be comprehensively characterized in terms of the function Φ.

Lemma 2.3. *Let $\phi \in L^2(K)$ be given. Then $\{T_n\phi : n \in \mathbb{N}_0\}$ is a Riesz sequence with bounds A and B if and only if $A \leq \Phi(\xi) \leq B$ for all $\xi \in K$.*

3. RIESZ MULTIRESOLUTION ANALYSIS ON $L^2(K)$

This section begins with a formal definition of Riesz multiresolution analysis (Riesz MRA) on a local field K of positive characteristic. The initial definition of Riesz MRA, specifically for the case where $G = \mathbb{R}$, was provided by R. A. Zalik in his paper [30]. Recently, Raj Kumar, Satyapriya and F.A. Shah introduced this concept to an arbitrary locally compact group G [23]. The following definition can be regarded as an analogous version of these definitions.

Definition 3.1. A Riesz multiresolution analysis (Riesz MRA) for $L^2(K)$ comprises a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ along with a function $\phi \in V_0$, satisfying the following criteria:

(i) The subspaces are nested, meaning

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots .$$

(ii) The subspaces have a dense union and a trivial intersection, i.e.

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K) \text{ and } \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

(iii) The subspaces are related through

$$V_j = D^j V_0.$$

(iv) The subspaces are translation invariant, implying

$$f \in V_j \implies T_n f \in V_0, \forall\, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

(v) The sequence $\{T_k\phi : k \in \mathbb{N}_0\}$ forms a Riesz basis for V_0.

In the context of a Riesz MRA, a function ϕ that generates the MRA is referred to as the *scaling function* associated with the Riesz MRA. Additionally, the subspaces V_j are termed *approximation spaces* or *multiresolution subspaces*.

If we replace the term “Riesz basis" in (v) of Definition 3.1 by “orthonormal basis", then we get the definition of the classical MRA ([13]) and if a similar replacement is made by “frame", then we get the definition of a frame MRA ([25]). Furthermore, condition (v) implies that $\overline{span}\{T_k\phi : k \in \mathbb{N}_0\} = V_0$. Additionally, in the case where ϕ generates an FMRA, we have:

$$V_j = D^j(\overline{\text{span}}\{T_k\phi : k \in \mathbb{N}_0\}) = \overline{\text{span}}\{D^jT_k\phi : k \in \mathbb{N}_0\}, \quad j \in \mathbb{Z}. \tag{3.1}$$

The conditions delineating when the scaling function ϕ generates an Riesz MRA can be derived by adapting certain modifications from the classical MRA scenario. The construction process of a Frame MRA has been thoroughly studied by F.A. Shah in his paper [25]. As, we have mentioned it earlier that a Riesz basis a special kind of frame, so all the results of [25] can be directly applied to the case of a Riesz MRA. Subsequently, we can write

(i) With the already chosen function ϕ and the definition of the subspaces V_j in (3.1), we get that the triviality of the intersection of the subspaces V_j is a redundant property.

(ii) The subspaces V_j are nested if there exist a K-integral periodic function m_0 in $L^\infty(\mathfrak{D})$ satisfying

$$\widehat{\phi}(\mathfrak{p}^{-1}(\xi)) = m_0(\xi)\widehat{\phi}(\xi). \tag{3.2}$$

The equation (3.2) is commonly referred to as the *refinement equation*, and any function ϕ satisfying such an equation is termed *refinable*. The K-integral periodic function m_0, which appears in (3.2), is known as the *two scale symbol* or the *refinement mask*.

This function m_0 and the function Φ, defined in (2.1), exhibit a notable relationship given via

$$\Phi(\mathfrak{p}^{-1}(\xi)) = |m_0(\xi)|^2\Phi(\xi) + |m_0(\xi + \mathfrak{p}u(1)|^2\Phi(\xi + \mathfrak{p}u(1)). \tag{3.3}$$

However, we find it necessary to mention here that the two scale symbol m_0 associated with a Frame MRA has multiple choices, but in case of a Riesz MRA, the two scale symbol m_0 is always unique. This can be derived using (3.3) along with the fact that the function Φ is never equal to zero whenever $\{T_n\phi : n \in \mathbb{N}_0\}$ is a Riesz sequence.

In the previous related works, the union property of the subspaces has been proved via the methods of [2]. In this article, we give an alternate method to prove this property. We will use the *substantiality* of the prime element $\mathfrak{p}$ to prove our claim. A similar process for the case of locally compact Abelian groups has been adopted in [15, 23]. To proceed further, we need to equip ourselves with some standard notations. Note that, we have written these notations analogous to their definition in [15].

Definition 3.2. Let K be a local field of positive characteristic having prime element $\mathfrak{p}$.

(i) Let $f, g \in L^1(K)$. Then their convolution is defined as

$$f * g(x) = \int_K f(y)g(x-y)dy, \ \forall \, x \in K.$$

(ii) A family $\mathcal{F}$ is called a zero divisor in $L^2(K)$ if there exists a non zero $g \in L^2(K)$ such that

$$f * g = 0, \ \forall \, f \in \mathcal{F}.$$

(iii) A function $f \in L^2(K)$ is called $\mathfrak{p}$-substantial if the family $\{D^j f : j \in \mathbb{Z}\}$ is not a zero divisor in $L^2(K)$.

Next, we give some important results which we will be requiring in proving the density property. We skip their trivial proofs.

Lemma 3.1. *Let K be a local field with positive characteristic having $\mathfrak{p}$ as its prime element. Then we have the following:*

(i) *The map $f \mapsto \widetilde{f}$, where*

$$\widetilde{f}(x) = \overline{f(-x)}, \ \forall \, x \in K$$

is a norm preserving conjugate linear bijection on $L^2(K)$.

(ii) *For any $f, g \in L^2(K)$,*

$$f * g(x) = \langle T_{-x} f, \widetilde{g} \rangle, \ \forall \, x \in K.$$

(iii) *If $W = \overline{\bigcup_{j \in \mathbb{Z}} V_j}$, then W is a translation invariant subspace of $L^2(K)$. Thus, we can write*

$$W = \overline{span}\{T_x D^j \phi : x \in K, \ j \in \mathbb{Z}\}.$$

Proof. Let $B : L^2(K) \to L^2(K)$ denote the map $f \mapsto \widetilde{f}$, where $\widetilde{f}(x) = \overline{f(-x)}$ for all $x \in K$. The bijectivity of this map is trivial exercise to prove and hence we skip it here. To see the conjugate linearity, observe that for any constants $a \in \mathbb{C}$ and $f, g \in L^2(K)$, we have

$$B(af+g)(x) = \overline{(af+g)(-x)} = \overline{af(-x) + g(-x)} = \overline{af(-x)} + \overline{g(-x)}, \ x \in K$$

This implies that

$$B(af+g)(x) = \overline{a}\, B(f)(x) + B(g)(x), \ \forall \, x \in K.$$

This shows that the map B is conjugate linear. Further, also note that

$$||Bf||^2 = \int_K \left| \widetilde{f}(x) \right|^2 dx = \int_K \left| \overline{f(-x)} \right|^2 dx.$$

Using the fact that $|f(x)| = |\overline{f(x)}|$ for all $x \in K$ along with the variable transformation $x \mapsto -x$, we obtain

$$||Bf||^2 = \int_K |f(x)|^2 \, dx = ||f||^2.$$

This proves (i). To see the proof of (ii), note that

$$f * g(x) = \int_K f(y)g(x-y)dy, \; \forall\, x \in K.$$

A variable transform $x - y \mapsto -t$ yields

$$f * g(x) = \int_K f(t+x)g(-t)dt = \int_K (T_{-x}f)(t)\overline{\widetilde{g}(t)}dt = \langle T_{-x}f, \widetilde{g}\rangle\,, \; \forall\, x \in K.$$

This proves (ii). We will prove (iii) in three steps.

Step 1: Here, we will show that the subspace $\bigcup\limits_{j\in\mathbb{Z}} V_j$ is translation invariant under translations from the set $L = \{\mathfrak{p}^j u(n) : j \in \mathbb{Z}\}$. First note that, for any $j \in \mathbb{Z}$, we have

$$(D^j T_n \phi)(x) = (T_{\mathfrak{p}^j(u(n))} D^j \phi)(x), \; \forall\, x \in K,$$

and thus, we can write $V_j = \overline{\text{span}}\left\{T_{\mathfrak{p}^j(u(n))} D^j\phi : n \in \mathbb{N}_0\right\}$. Now, let $f \in \bigcup\limits_{j\in\mathbb{Z}} V_j$ and $\lambda \in L$. This means that, there exists a $k_1 \in \mathbb{Z}$ such that $f \in V_{k_1}$, and a $k_2 \in \mathbb{Z}$ such that $\lambda = \mathfrak{p}^{k_2}(u(n_1))$ for some $n_1 \in \mathbb{N}_0$. Let $k = \max\{k_1, k_2\}$. Since, ϕ is refinable, therefore, the subspaces V_j are nested. This means that $f \in V_k$, and thus, for some scalar sequence $\{c_n\}_{n\in\mathbb{N}_0}$, we have

$$f = \sum_{n\in\mathbb{N}_0} c_n T_{\mathfrak{p}^k(u(n))} D^j \phi, \text{ i.e. } f(x) = q^{k/2} \sum_{n\in\mathbb{N}_0} c_n \phi(\mathfrak{p}^{-k}x - u(n)), \; \forall\, x \in G.$$

Now, consider the translate $T_\lambda f$ of f. For any $x \in G$, we have

$$(T_\lambda f)(x) = q^{k/2} \sum_{n\in\mathbb{N}_0} c_n \phi(\mathfrak{p}^{-k}(x - (\mathfrak{p}^k(u(n) + \mathfrak{p}^{k_2-k}(u(n_1)))))), \; \forall\, x \in G.$$

Some standard manipulation along with the properties of the sets $\mathfrak{P}^\ell$ give us

$$(T_\lambda f)(x) = q^{k/2} \sum_{n\in\mathbb{N}_0} c_n \phi(\mathfrak{p}^{-k}(x - (\mathfrak{p}^k(u(n))))), \; \forall\, x \in G.$$

This implies that $T_\lambda f = \sum\limits_{n\in\mathbb{N}_0} c_n T_{\mathfrak{p}^k(u(n))} D^k \phi$, which means that $T_\lambda f \in V_k$, and hence $T_\lambda f \in \bigcup\limits_{j\in\mathbb{Z}} V_j$. This proves our claim corresponding to Step 1.

Step 2: In this step, we will show that the set W is translation invariant under translations from K. To see this, first let $x \in K$ be arbitrary. Observe that the set L, defined in Step 1, is dense in K, therefore, there exists a net $\{\lambda_\beta\} \subset L$ such that $\lambda_\beta \to x$ in K. So, if $f \in \bigcup\limits_{j\in\mathbb{Z}} V_j$, then by Step 1, $T_{\lambda_\beta} f \in \bigcup\limits_{j\in\mathbb{Z}} V_j$ for all values of β. The continuity of the translations, now allows us to write $T_{\lambda_\beta} f \to T_x f$. This means that $T_x f \in W$. Now, let $g \in W$ be any element. Then there exits a sequence $\{g_\ell\}_{\ell\in\mathbb{Z}} \in \bigcup\limits_{j\in\mathbb{Z}} V_j$ such that $||g_n - g||_2 \to 0$ as $n \to \infty$. From here, it trivially follows that $||T_x g_n - T_x g||_2 \to 0$ as $n \to \infty$. Since, W, being a closed set, contains all its limit points, therefore $T_x g \in W$. This completes the proof for this step.

Step 3: In this step, we club together the observations made in previous two steps to obtain the claim made in (iii) of this theorem. Observe that, for any $k \in \mathbb{Z}$, by the definition of V_k, we have $V_k \subset \overline{\text{span}}\{T_x D^j \phi : x \in K,\ j \in \mathbb{Z}\}$. Now, this inclusion trivially implies that $W \subseteq \overline{\text{span}}\{T_x D^j \phi : x \in K,\ j \in \mathbb{Z}\}$. To see the converse, note that $\overline{\text{span}}\{T_x D^j \phi : x \in K,\ j \in \mathbb{Z}\}$ is the smallest closed translation invariant subspace of $L^2(K)$ containing the set $\{D^j \phi : j \in \mathbb{Z}\}$. Also, W is a subspace of $L^2(K)$ containing $\{D^j \phi : j \in \mathbb{Z}\}$, therefore, we must have $\overline{\text{span}}\{T_x D^j \phi : x \in K,\ j \in \mathbb{Z}\} \subseteq \overline{W}$. This allows us to conclude that $W = \overline{\text{span}}\{T_x D^j \phi : x \in K,\ j \in \mathbb{Z}\}$. ∎

Finally, we are now in a position to prove the density property of the approximation subspaces V_j. We give this property in terms of an equivalent condition of the scaling function ϕ being $\mathfrak{p}$-substantial.

Theorem 3.2. *Let ϕ be a refinable function in $L^2(K)$ and $\{V_j : j \in \mathbb{Z}\}$ be defined via* (3.1). *Then the following are equivalent:*

(i) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K)$.

(ii) *The scaling function ϕ is $\mathfrak{p}$-substantial.*

Proof. First assume that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K)$. Now, suppose that, for some $g \in L^2(K)$, we have

$$D^j\phi * g = 0;\ \ \forall\, j \in \mathbb{Z}.$$

Recall that, we denoted $\tilde{g}(x) = \overline{g(-x)}$ for all $x \in K$. Then using Lemma 3.1, we get that

$$\langle T_{-x} D^j \phi, \tilde{g}\rangle = 0,\ \forall\, x \in K \text{ and } \forall\, j \in \mathbb{Z}.$$

This means that $\tilde{g}$ is orthogonal to the set $\overline{span}\,\{T_x D^j\phi : x \in K, j \in \mathbb{Z}\}$, i.e.

$$\tilde{g} \perp \overline{span}\left\{T_x D^j \phi : x \in K, j \in \mathbb{Z}\right\}$$

Now, by our assumption, and Lemma 3.1-(iii), we get that $\tilde{g}$ must be zero, and hence $g = 0$. This implies that ϕ is $\mathfrak{p}$-substantial. The converse also follows once we trace back the steps. To avoid the repetition, we skip the proof corresponding to the converse part. ∎

We conclude the observations of this section in the following theorem, wherein, we just list all the conditions which need to be imposed on scaling function ϕ to get a Riesz MRA for $L^2(K)$.

Theorem 3.3. *A function $\phi \in L^2(K)$ generates a Riesz Multiresolution Analysis (Riesz MRA) if the following conditions are met:*

(i) *The subspaces $\{V_j : j \in \mathbb{Z}\}$ are defined as in Equation* (3.1).

(ii) *There exists a K-integral periodic function $m_0 \in L^\infty(\mathfrak{D})$ such that*

$$\widehat{\phi}(\mathfrak{p}^{-1}\xi) = m_0(\xi)\widehat{\phi}(\xi). \tag{3.2}$$

(iii) *The sequence $\{T_k\phi : k \in \mathbb{N}_0\}$ forms a Riesz sequence.*

(iv) *The function* ϕ *is* $\mathfrak{p}$*-substantial.*

However, more often than not, scaling function is given to us in terms of its Fourier transform, so in place of (iv) of the above theorem, we write

(iv) **(alternate)** $|\widehat{\phi}| > 0$ on a neighbourhood U of $0 \in K$.

3.1. **Example.** Consider a function $\phi \in L^2(K)$ given by

$$\phi(x) = \chi_{\mathfrak{P}}(x),\ x \in K,$$

and define the subspaces V_j by (3.1). With these choice, it is quite evident that the subspaces are V_j are translation invariant under translations from $\{u(n) : n \in \mathbb{N}\}$. Moreover, these subspaces also follow the dilation relation $V_j = D^j V_0$ amongst them. To verify that the subspaces V_j form a Riesz MRA for $L^2(K)$, we need to verify the remaining properties. Note that

$$\widehat{\phi}(\xi) = \int_K \phi(x)\overline{\Upsilon_\xi(x)}dx.$$

Recall that Υ is a character on K^+ which is trivial on $\mathfrak{D}$ but non trivial on $\mathfrak{P}^{-1}$. Now, the above equation changes to

$$\widehat{\phi}(\xi) = \int_{\mathfrak{P}} \overline{\Upsilon_\xi(x)}dx.$$

After some appropriate calculations and using definitions of ideals $\mathfrak{P}^k$, we obtain that

$$\widehat{\phi}(\xi) = q\ \chi_{\mathfrak{P}^{-1}}(\xi) = 2\ \chi_{\mathfrak{P}^{-1}}(\xi).$$

Clearly, the function $\widehat{\phi} \neq 0$ on a neighbourhood $U \subset \mathfrak{P}^{-1}$ of $0 \in K$. Hence by using a result of [25], we conclude that the subspaces V_j defined above indeed have a dense union in $L^2(K)$. Next, we observe that

$$\Phi(\xi) = \sum_{n\in\mathbb{N}_0} \left|\widehat{\phi}(\xi + u(n))\right|^2 = 4 \sum_{n\in\mathbb{N}_0} \left|\chi_{\mathfrak{P}^{-1}}(\xi + u(n))\right|^2.$$

We now use the representation of the quotient group $\mathfrak{P}^{-1}/\mathfrak{D}$ to obtain

$$\Phi(\xi) = 4 \sum_{n\in\mathbb{N}_0} \left|\chi_D(\xi + u(n))\right|^2 + 4 \sum_{n\in\mathbb{N}_0} \left|\chi_{u(1)+\mathfrak{D}}(\xi + u(n))\right|^2.$$

Now, if for some $n \in \mathbb{N}_0$, $\xi \in u(n) + \mathfrak{D}$, then $\xi + u(1) \in u(n) + u(1) + \mathfrak{D}$. This observation allows us to conclude that

$$4 \le \Phi(\xi) \le 8,\ \forall\, \xi \in K.$$

This means that the family $\{T_n\phi : n \in \mathbb{N}_0\}$ is a Riesz sequence and hence a Riesz basis for V_0 (Owing to (3.1)). Moreover, this also gives us that the triviality of the intersection of the subspaces V_j. Finally, it only remains to check for refinability of the function ϕ. To see this, note that

$$\widehat{\phi}(\mathfrak{p}^{-1}\xi) = 2\ \chi_{\mathfrak{P}^{-1}}(\mathfrak{p}^{-1}\xi) = 2\ \chi_{\mathfrak{D}}(\xi),\ \xi \in K.$$

If we define m_0 to be the K-integral periodic extension of the function $\chi_{\mathfrak{D}}$, then we get that the refinement for all $\xi \in K$. This means that the function ϕ is refinable with the two scale symbol m_0.

As we have seen, the function ϕ satisfies all the conditions to generate a Riesz MRA, therefore $\{V_j\}$ defined above is a Riesz MRA for $L^2(K)$.

4. Dyadic Riesz Wavelet Basis for $L^2(K)$

Our primary objective in this section is to construct a Riesz wavelet basis utilizing the provided Riesz MRA. Henceforth, we will consistently assume that ϕ generates a Riesz MRA, meaning that all the conditions outlined in Theorem 3.3 are satisfied. Additionally, we will assume that this Riesz MRA, generated by the function ϕ, follows dyadic dilations, i.e. $q = 2$.

To initiate this construction process, we begin by decomposing the space $L^2(K)$ into a more manageable structure, akin to what is done in classical MRA. Let W_j denote the orthogonal complement of the subspace V_j in V_{j+1}. This yields an orthogonal decomposition of the space $L^2(K)$, rendering it more amenable to analysis:

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Moreover, it's important to recognize that the construction of a Riesz wavelet basis for $L^2(K)$ hinges on the existence of functions in $L^2(K)$ whose translated families constitute a Riesz basis for W_0. This observation arises from the fact that the spaces W_j exhibit the same dilation property as V_j. The subsequent lemma succinctly encapsulates this insight, with its proof derivable from results outlined in [25].

Lemma 4.1. [25] *Assuming that $\phi \in L^2(K)$ generates a Riesz MRA, the following statements hold true:*

- (i) $W_j = D^j W_0$, *for all* $j \in \mathbb{Z}$.
- (ii) *If the functions* $\psi_1, \psi_2, \ldots, \psi_n \in W_0$ *form a Riesz for* W_0, *denoted by the family* $\{T_k\psi_i : k \in \mathbb{N}_0,\ 1 \le i \le n\}$, *then for every* $j \in \mathbb{Z}$, *the family* $\{D^jT_k\psi_i : k \in \mathbb{N}_0,\ 1 \le i \le n\}$ *constitutes a Riesz basis for* W_j. *Furthermore, the family* $\{D^jT_k\psi_i : k \in \mathbb{N}_0,\ 1 \le i \le n,\ j \in \mathbb{Z}\}$ *forms a Riesz basis for* $L^2(K)$. *It's notable that all these Riesz bases share the same Riesz bounds.*

These assertions shed light on the relationship between the multiresolution subspaces W_j and provide a systematic approach to constructing Riesz bases for $L^2(K)$ using similar bases for W_0. Notably, Lemma 4.1 indicates that our objective is simplified to the construction of functions $\psi_1, \psi_2, \ldots, \psi_n$ in $L^2(K)$ such that the family of translates $\{T_\lambda\psi_i : \lambda \in \Lambda, 1 \le i \le n\}$ forms a Riesz basis for W_0. Consequently, it becomes crucial for us to provide a characterization of the space W_0.

Lemma 4.2. [25] *Assume that $\phi \in L^2(K)$ generates a Riesz MRA of dydic dilation with two-scale symbol $m_0 \in L^\infty(\mathfrak{D})$. If, for any K-integral periodic function $F \in L^2(\mathfrak{D})$, we define $f \in V_1$ by*

$$\widehat{f}\big(\mathfrak{p}^{-1}(\xi)\big) = F(\xi)\widehat{\phi}(\xi), \tag{4.1}$$

then $f \in W_0$ if and only if

$$\Big(F\overline{m_0}\Phi\Big)(\xi) + \Big(F\overline{m_0}\Phi\Big)(\xi + \mathfrak{p}u(1)) = 0 \tag{4.2}$$

hold true for a.e. $\xi \in K$.

In previous studies on Multiresolution Analysis and Riesz Multiresolution Analysis, it has been demonstrated that whenever the underlying MRA structure generated by dyadic dilations, precisely one function is required to construct a Riesz basis for the space W_0. For an in-depth exploration of Riesz MRA with dyadic dilations in the case of locally compact Abelian groups, one may refer to [24, 17].

Remark 4.1. Drawing inspiration from previous research and assuming that the function ϕ generates an Riesz MRA with dyadic dilations, our goal here is to construct a function ψ such that the family

$$\big\{T_k\psi : k \in \mathbb{N}_0\big\} \tag{4.3}$$

forms a Riesz basis for W_0. We break down this process into two distinct steps:

- Firstly, we prove the existence of a functions $\psi \in W_0$ such that its translates generate W_0, i.e.,

  $$W_0 = \overline{\text{span}}\{T_k\psi : k \in \mathbb{N}_0\}.$$

  Additionally, we will provide an explicit expression for this function.
- Subsequently, we will establish that the family consisting of translates of function ψ, obtained in the previous step, indeed constitutes a Riesz basis for W_0.

The first task can be simplified considerably. We will provide an alternative characterization for the family $\big\{T_k\psi : k \in \mathbb{N}\big\}$ to generate the space W_0. In this alternative approach, we establish a sufficient condition that reduces our task to merely checking the solvability of a system of linear equations. These insights are encapsulated in the following theorem.

Theorem 4.3. *Assume that $\phi \in L^2(K)$ generates a Riesz MRA of dydic dilation and let for some K-integral periodic $m \in L^\infty(\mathfrak{D})$, the functions $\psi \in V_1$ be defined by:*

$$\widehat{\psi}(\mathfrak{p}^{-1}(\xi)) = m(\xi)\widehat{\phi}(\xi). \tag{4.4}$$

If there exist K-integral periodic functions G_0 and $G_1 \in L^\infty(\mathfrak{D})$ such that the equations

$$\left(\overline{m_0} m\Phi\right)(\xi + \mathfrak{p}u(0)) + \left(\overline{m_0} m\Phi\right)(\xi + \mathfrak{p}u(1)) = 0, \tag{4.5}$$

$$m_0(\xi)\, G_0(\xi) + m(\xi)\, G_1(\xi) = 1 \tag{4.6}$$

$$m_0(\xi + \mathfrak{p}u(1))\, G_0(\xi) + m(\xi + \mathfrak{p}u(1))\, G_1(\xi) = 0 \tag{4.7}$$

are satisfied for a.e. $\xi \in K$, then we have $W_0 = \overline{span}\{T_n\psi : n \in \mathbb{N}_0\}$.

Proof. Equation (4.5) along with Lemma 4.2 implies that $\psi \in W_0$. Furthermore, as W_0 is a closed and translation-invariant subspace of $L^2(K)$, we have the following implication:

$$\overline{\text{span}}\{T_n\psi : n \in \mathbb{N}_0\} \subseteq W_0. \tag{4.8}$$

For any $\xi \in K$ and for any $\ell \in \mathbb{N}_0$, an easy manipulation of the equations (4.6) and (4.7) yields

$$\frac{1}{2}\widehat{\phi}(\mathfrak{p}(\xi))\, \overline{\Upsilon_\ell(\mathfrak{p}(\xi))} = \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(u(n))+\ell}\, \Upsilon_n(\xi)\widehat{\phi}(\xi) + \sum_{n\in\mathbb{N}_0} g_{\mathfrak{p}(u(n))+\ell}\, \Upsilon_n(\xi)\widehat{\psi}(\xi);$$

Taking the inverse Fourier transform of the above equation and then writing it in operator form, we obtain

$$DT_\ell\phi = \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(u(n))+\ell}\, T_{-n}\phi + \sum_{n\in\mathbb{N}_0} g_{\mathfrak{p}(u(n))+\ell}\, T_{-n}\psi. \tag{4.9}$$

Since $\psi \in W_0$ and since $\phi \in V_0$ generates a Riesz MRA, therefore we get

$$\sum_{n\in\mathbb{N}_0} g_{\mathfrak{p}(u(n))+\ell}\, T_{-n}\psi \in W_0 \text{ and } \sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(u(n))+\ell}\, T_{-n}\phi \in V_0.$$

Now let $f \in W_0$ and let $\epsilon > 0$ be arbitrary. Since $\{DT_n\phi\}_{n\in\mathbb{N}_0}$ is a frame for V_1, therefore there exists a finite set $\mathbb{N}_\epsilon \subset \mathbb{N}_0$ and a finite sequence $\{b_\ell\}_{\ell\in\mathbb{N}_\epsilon}$ such that

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell DT_\ell\phi - f\right\|^2 < \epsilon.$$

A substitution from (4.9) now gives us

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell \left(\sum_{n\in\mathbb{N}_0} g^0_{\mathfrak{p}(u(n))+\ell}\, T_{-n}\phi + \sum_{n\in\mathbb{N}_0} g_{\mathfrak{p}(u(n))+\ell}\, T_{-n}\psi\right) - f\right\|^2 < \epsilon.$$

Now we use orthogonality of the two terms appearing on the right hand side of equation (4.9), to get

$$\left\|\sum_{\ell\in\mathbb{N}_\epsilon} b_\ell \sum_{n\in\mathbb{N}_0} g_{\mathfrak{p}(u(n))+\ell} T_{-n}\psi - f\right\|^2 < \epsilon;$$

and from this, we conclude that $f \in \overline{\text{span}}\{T_n\psi : n \in \mathbb{N}_0\}$. We now have the reverse inclusion in the expression (4.8) and thus we can write

$$W_0 = \overline{\text{span}}\{T_n\psi : n \in \mathbb{N}_0\}.$$

This completes the proof. ∎

Using the suficient condition given in the above lemma, we now proceed to prove our first aim, i.e. we find a function ψ generating the space W_0.

Theorem 4.4. *Let K be a local field and let $\phi \in L^2(K)$ generates an Riesz MRA of dydic dilation. Then there always exist a function $\psi \in W_0$ such that*

$$W_0 = \overline{span}\{T_n\psi : n \in \mathbb{N}_0\}.$$

Proof. Utilizing Lemma 4.3, it suffices to demonstrate that equations (4.5), (4.6), and (4.7) are satisfied almost everywhere on K. It's noteworthy that each term appearing in these equations is K-integral periodic, thereby reducing the requirement to establish their satisfaction almost everywhere on $\mathfrak{D}$.
To streamline our computations further, we partition the set $\mathfrak{D}$ into two disjoint parts as follows:

$$\mathfrak{D}_1 = \{\xi \in \mathfrak{D} : 0 \neq |m_0(\xi)| \geq |m_0\left((\xi + \mathfrak{p}u(1))\right)|\}$$
$$\mathfrak{D}_2 = \{\xi \in \mathfrak{D} : 0 \neq |m_0\left((\xi + \mathfrak{p}u(1))\right)| \geq |m_0(\xi)|\}$$

Observe that, if $\xi \in \mathfrak{D}_i$, then $\xi + \mathfrak{p}u(1) \in \mathfrak{D}_j$, where $1 \leq i \neq j \leq 2$.

Equation (4.5) now gives us

$$m(\xi) = -\frac{\left(\overline{m_0} m\Phi\right)(\xi + \mathfrak{p}u(1))}{\left(\overline{m_0}\Phi\right)(\xi)}.$$

From the equation above, we can make as choices as we want for the function m. To keep things simpler, we make the choice

$$m(\omega) = \begin{cases} -\frac{\left(\overline{m_0}\Phi\right)(\xi+\mathfrak{p}u(1))}{\left(\overline{m_0}\Phi\right)(\xi)}, & \omega \in \mathfrak{D}_1 \\ 1, & \omega \in \mathfrak{D}_2 \end{cases} \tag{4.10}$$

Corresponding to the above choice, we also observe that $|m(\omega)| \leq BA^{-1}$ for all $\xi \in \mathfrak{D}$. Hence, $m \in L^\infty(\mathfrak{D})$. We further observe that when we substitute this value of m in the equations (4.6) and (4.7), we get a kind of a system of linear equations in the variables G_0 and G_1. The functions G_0 and G_1 can now be chosen as

$$G_0(\omega) = \begin{cases} \frac{\overline{m_0(\xi)}\Phi(\xi)}{\Phi(\mathfrak{p}^{-1}\xi)}, & \omega \in \mathfrak{D}_1 \\ \frac{\overline{m_0(\xi)}\Phi(\xi)}{\Phi(\mathfrak{p}^{-1}\xi)}, & \omega \in \mathfrak{D}_2 \end{cases} \text{ and } G_1(\omega) = \begin{cases} -\frac{m_0(\xi+\mathfrak{p}u(1))\overline{m_0(\xi)}\Phi(\xi)}{\Phi(\mathfrak{p}^{-1}\xi)}, & \omega \in \mathfrak{D}_1 \\ \frac{\left(|H_0|^2\Phi\right)(\xi+\mathfrak{p}u(1))|}{\Phi(\mathfrak{p}^{-1}\xi)}, & \omega \in \mathfrak{D}_2 \end{cases}$$

Moreover, the functions G_0 and G_1 satisfy $||G_i|| \leq ||m_0||_\infty BA^{-1}$ where $1 \leq i, j \leq 2$. This implies that our task corresponding to finding functions satisfying (4.5)-(4.7) is complete. Thus for the above constructed function ψ, we conclude by Lemma 4.3 that $W_0 = \overline{\text{span}}\{T_n\psi : n \in \mathbb{N}_0\}$. ∎

In the proof of the above theorem, we mentioned that there are as many choices of m possible as we want. To justify our claim, we give another independent choice for the function m in the remark below:

Remark 4.2. If $\Delta(\xi)$ denotes the determinant

$$\Delta(\xi) = \begin{vmatrix} m_0(\xi) & m(\xi) \\ m_0(\xi + \mathfrak{p}u(1)) & m(\xi + \mathfrak{p}u(1)) \end{vmatrix},$$

then the functions $m,\ G_0$ and G_1 satisfying (4.5)-(4.7) can also be chosen as

$$\begin{aligned} m(\xi) &= (\overline{m_0}\Phi)(\xi + \mathfrak{p}u(1))\ \chi_{u(1)}(\xi) \\ G_0(\xi) &= \frac{\Phi(\xi)(m\Phi)(\xi + \mathfrak{p}u(1))}{\Delta(\xi)} \\ G_1(\xi) &= -\frac{\Phi(\xi)(m_0\Phi)(\xi + \mathfrak{p}u(1))}{\Delta(\xi)} \end{aligned}$$

This can be verified using suitable manipulations. We skip its proof here.

This completes our quest of a function ψ which generate the space W_0. We now show that the family of the type $\{T_n\psi : n \in \mathbb{N}_0\}$, constructed using the function obtained in above theorem, is indeed a Riesz basis for W_0.

Theorem 4.5. *Assume that $\phi \in L^2(K)$ generates a Riesz MRA of dyadic dilations and two scale symbol $m_0 \in L^\infty(\mathfrak{D})$. Further assume that the functions ψ is defined by* (4.4) *and the function m is assumed to be as it appears in equation* (4.10). *Then the family*

$$\{D^j T_n \psi : n \in \mathbb{N}_0\}$$

generates a Riesz basis for the space $L^2(K)$.

Proof. Analogous to the function Φ as defined in (2.1), we define the function Ψ by

$$\Psi(\xi) = \sum_{n \in \mathbb{N}_0} \left| \widehat{\psi}(\xi + u(n)) \right|^2.$$

It is easy to see that

$$\Psi(\mathfrak{p}^{-1}\xi) = |m(\xi)|^2 \Phi(\xi) + |F(\xi + \mathfrak{p}u(1))|^2 \Phi(\xi + \mathfrak{p}u(1)).$$

We now make use of Lemma 2.3 to show that the family $\{T_n\psi\}_{n \in \mathbb{N}_0}$ is a Riesz basis for W_0, i.e. we wish to find constants $C, D > 0$ such that

$$C \le \Psi(\xi) \le D,\ \ \forall\, \xi \in \mathfrak{D}.$$

But as it is more convenient for us to deal with the expression $\Psi(\mathfrak{p}^{-1}\xi)$, so we need to ensure that the bounds $C, D > 0$ which exist are such that

$$C \le \Psi(\mathfrak{p}^{-1}\xi) \le D,\ \ \forall\, \xi \in \mathfrak{p}\mathfrak{D}.$$

Analogous to the previous theorem, we divide the set $\mathfrak{p}\mathfrak{D}$ into two disjoint parts:

$$\mathfrak{p}\mathfrak{D}_1 = \{\xi \in \mathfrak{p}\mathfrak{D} : 0 \neq |m_0(\xi)| \geq |m_0(\xi + \mathfrak{p}u(1)|\}$$
$$\mathfrak{p}\mathfrak{D}_2 = \{\xi \in \mathfrak{p}\mathfrak{D} : 0 \neq |m_0(\xi + \mathfrak{p}u(1))| \geq |m_0(\xi)|\}.$$

Clearly, $\mathfrak{p}\mathfrak{D}_1 \subset \mathfrak{D}_1$ and $\mathfrak{p}\mathfrak{D}_2 \subset \mathfrak{D}_2$.
We nove move forward to finding suitable bounds for the function Ψ. First if we let $\xi \in \mathfrak{p}\mathfrak{D}_1$, then

$$\Psi(\mathfrak{p}^{-1}\xi) = \frac{|m_0(\xi + \mathfrak{p}u(1))|^2 \Phi(\xi + \mathfrak{p}u(1))^2}{|m_0(\xi)|^2 \Phi(\xi)} + \Phi(\xi).$$

It is then easy to see that

$$A \leq \Psi(\mathfrak{p}^{-1}\xi) \leq \frac{B}{A}(A + B).$$

The above inequality also holds when $\xi \in \mathfrak{p}\mathfrak{D}_2$. Thus, by virtue of Lemma 2.3, we can conclude that the family $\{T_n\psi : n \in \mathbb{N}_0\}$ generates a Riesz basis for W_0, and, further by Lemma 4.1, the family $\{D^j T_n\psi : n \in \mathbb{N}_0\}$ generates a Riesz basis for $L^2(K)$. ∎

5. CONCLUSION

This paper provides a comprehensive review of the theory behind constructing Riesz wavelet bases in the space $L^2(K)$, where K represents a local field with positive characteristics. A significant contribution of this work is the novel characterization of the scaling function within the Riesz multiresolution analysis (Riesz MRA) for such fields. Additionally, the paper introduces innovative approaches for constructing wavelet frames in $L^2(K)$, with particular attention to the property of density of the union, which is addressed in a completely new manner. By concentrating on dyadic dilations, the research successfully extends wavelet and MRA theory to the distinct properties of local fields, thus broadening its application beyond conventional Euclidean spaces. The paper also presents a detailed exposition of a novel method for constructing dyadic Riesz wavelet bases, further enriching the theoretical framework.
We emphasize that this paper deals majorly with the construction of Dyadic Riesz wavelet bases on local fields of positive characteristics, and to the best of our knowledge, the work done in this paper is a new work.

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