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THE PROJECTIVE RICCATI EQUATIONS METHOD FOR SOLVING NONLINEAR SCHRÖDINGER EQUATION IN BI-ISOTROPIC FIBER

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ABSTRACT. Bi-isotropic materials, characterized by their chiral and non-reciprocal nature, present unique challenges and opportunities in scientific research, driving the development of cutting-edge applications. In this paper, we explore the influence of chirality using a newly developed framework that emphasizes the nonlinear effects arising from the magnetization vector under a strong electric field. Our research introduces a novel formulation of constitutive relations and delves into the analysis of solutions for the nonlinear Schrödinger equation, which governs pulse propagation in nonlinear bi-isotropic media. By employing the Projective Riccati Equation Method with variable dispersion and nonlinearity, we systematically derive families of solutions to the nonlinear Schrödinger equation in chiral and non-reciprocal optical fibers. This approach provides valuable insights into the propagation of light in two polarization modes right circularly polarized (RCP) and left circularly polarized (LCP) each associated with distinct wave vectors in nonlinear bi-isotropic environments. The study presents several new exact solutions of optical solitons within these media.

Key words and phrases: Nonlinear Bi-isotropic media; Schrödinger equation; Optical fiber; The projective Riccati Equations Method; chiratity; non-reciprocity.

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1. INTRODUCTION

The construction of the exact solutions of nonlinear partial differential equations (PDEs) is one of the most important and essential tasks in nonlinear science. In the past few decades, many authors had mainly studied solitary wave solutions of nonlinear PDEs by using various methods, such as the inverse scattering method (see references in [4]), Backlund transformation [17], Hirota bilinear method [4], the tanh method[4], various extended methods[7], generalized hyperbolic-function method[6] and generalized Riccati equation expansion method[5] and so on. In Ref[8], Conte and Musette presented an indirect method to seek some solitary wave solutions of nonlinear PDEs that can be expressed as a polynomial in two elementary functions which satisfy a project Riccati system [17]. By use of this method, some solitary wave solutions of many nonlinear PDEs have been obtained [JBDFTM]. Recently, Yan[6] and Chen-Li[17] further developed Conte and Musette's method by introducing a more general projective Riccati equations and obtained many exact travelling wave solutions of some nonlinear PDEs.
Constructing exact solutions to nonlinear partial differential equations (PDEs) is a crucial aspect of nonlinear science. Over the years, numerous methods have been developed to study solitary wave solutions, including the inverse scattering method, Bäcklund transformations, Hirota's bilinear method, the tanh method, and various extensions like the generalized hyperbolic function and Riccati equation methods. Among these, Conte and Musette introduced an approach to obtain solitary wave solutions using a projective Riccati system, which was further refined by Yan and Chen-Li to produce exact traveling wave solutions for various PDEs [4]. In this work, we expand on the projective Riccati equation method to find soliton-like solutions for nonlinear PDEs. Specifically, we apply the method to the nonlinear Schrödinger equation (NLSE) with varying coefficients in optical fibers [17] . NLSE solitons are of particular interest as they are considered fundamental for next-generation ultrahigh-speed optical communication systems. The paper is organized as follows: Section 2 outlines the extended projective Riccati equation method. Section 3 applies this method to the NLSE in optical fibers, yielding four families of exact soliton-like solutions. Section 4 concludes with a summary and discussion.

2. DESCRIPTION OF THE PROJECTIVE RICCATI EQUATIONS METHOD

Consider a nonlinear PDE in the following form

$$P(u, u_z, u_t, u_{zz}, u_{tt}) = 0 \tag{2.1}$$

where $u = u(t, z)$ is an unknown function,P is a polynomial in $u = u(t, z)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps of the generalized projective Riccati equations method .
Step 1 [17]. We use the following transformation

$$u(t, z) = u(\xi); \xi = z - vt \tag{2.2}$$

to reduce (2.1) to the following nonlinear ODE

$$K(u, u', u'', ...) = 0 \tag{2.3}$$

where v is velocity of the propagation,K is a polynomial of $u(\xi)$ and its derivatives $u'(\xi)$, $u''(\xi)$, . . . where $u' = \frac{du}{\partial\xi}$. We assume that the solution of Equation (2.3) has the form

$$u(t, z) = a_0 + \sum_{i=1}^{m} \sigma^{i-1}(\xi) \left[a_i\tau(\xi) + b_i\sigma(\xi)\right] \tag{2.4}$$

where $a_0(t,z)$, $a_i(t,z)$, b_i, $(i = 1,2,...,m,\ \xi(x,t)$ are all unknown functions of (t,z), $\sigma(\xi)$ and $\tau(\xi)$ satisfy (2.1) and (2.2). The parameter m can be found by balancing the highest order derivative term and the nonlinear terms in (2.5) (m is usually a positive integer). Then substitute (2.7) into (2.5) and return to determine balance constant m again [17].

Step 2.[17] The principle of the extended Riccati projective equations method is to take full advantage of the following projective equations:

$$\sigma'(\xi) = \varepsilon\sigma(\xi)\tau(\xi); \tau'(\xi) = R + \varepsilon\tau^2(\xi) - \mu\sigma(\xi), \varepsilon = \pm 1 \tag{2.5}$$

$$\tau^2(\xi) = -\varepsilon\left[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)\right], R \neq 0 \tag{2.6}$$

where R, μ are constants and $' = \frac{\partial}{\partial\xi}$. We know that (2.5) and (2.6) have the following solutions:
When: $\varepsilon = -1$

$$\begin{cases} \tau_1(\xi) = \frac{\sqrt{R}tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1} & ;\sigma_1(\xi) = \frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1} \\ \tau_2(\xi) = \frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1} & ;\sigma_2(\xi) = \frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1} \end{cases} \tag{2.7}$$

When: $\varepsilon = 1$

$$\begin{cases} \tau_3(\xi) = \frac{\sqrt{R}tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1} & ;\sigma_3(\xi) = \frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1} \\ \tau_4(\xi) = \frac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1} & ;\sigma_4(\xi) = \frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1} \end{cases} \tag{2.8}$$

Using these results, we establish the method of extended projective Riccati equations as follows: Substituting (2.4) along with (2.1) and (2.2) into (2.3), extracting the numerator of the resulting system, we can obtain a set of algebraic polynomials for $\tau^i(\xi)$, $\sigma^j(\xi)$, $(i = 0,1; j = 0,1,2,...)$, setting the coefficients of these terms $\tau^i(\xi)\sigma^j(\xi)$ to zero, we get a system of over determined PDEs with respect to unknown functions a_0, a_i, b_i, $(i = 1,2,..,m)$, ξ.
Solving the above system by use of symbolic computation system Maple, we would end up with the explicit expressions for μ, a_0, a_i, b_i, $(i = 1,2,..,m)$, and ξ or the constraints among them.

3. EXACT FAMILY SOLUTIONS OF NLSE IN THE BI-ISOTROPIC FIBER

3.1. **Introduction to Bi-isotropic fiber ([4]).** The nonlinear Schrödinger equation (NLSE) plays a key role across various wave physics domains. It can be derived as a first-order approximation from the asymptotic expansion of the Korteweg De Vries (KdV) equation for weakly nonlinear wave packets, providing a description of the wave packet envelope's evolution. In the context of bi-isotropic nonlinear media, an electromagnetic approach using newly formulated constitutive relations allows for the derivation of the NLSE for chiroptic fibers. This first-order approximation offers valuable insight into the interaction between electromagnetic waves and bi-isotropic media. Such an understanding opens the door to potential applications in fields like optics and microwaves, as demonstrated through our study on chiroptical fibers. In this section, we focus on modeling light pulse propagation in bi-isotropic fibers using the extended generalized Riccati equation to solve the NLSE.

Theorem 3.1. [17] *The bi-isotropic fiber operattin in the third optical fiber is given by the nonlinear Schrödinger equation*

$$\frac{\partial}{\partial z}A(\widetilde{t},z)+\left(i\frac{1}{2}\beta_2\frac{\partial^2}{\partial\widetilde{t}^2}\right)A(\widetilde{t},z)=i\delta A(\widetilde{t},z)-i\rho\left|A(\widetilde{t},z)\right|^2A(\widetilde{t},z) \tag{3.1}$$

where β_2 is the chromatic dispersion coefficients associated, α is the attenuation coefficient and ρ is the fiber nonlinearity related to coefficient.

Proof. This approach enables us to derive exact solutions for nonlinearities in bi-isotropic fibers, as governed by the constitutive equations for bi-anisotropic effects [4]. As delineated in our formalism expounded in [7], the constitutive equations governing the bi anisotropic nonlinear effects are delineated as follows:

$$\overrightarrow{D}=\overrightarrow{\varepsilon}\overrightarrow{E}+\overrightarrow{\xi}_{EH}\overrightarrow{H} \tag{3.2}$$

$$\overrightarrow{B}=\overrightarrow{\mu}\overrightarrow{H}+{}^{g}\xi_{EH}\overrightarrow{E} \tag{3.3}$$

The medium effects are contained in the dyadic: $\overrightarrow{\varepsilon}$, $\overrightarrow{\mu}$, $\overrightarrow{\xi}_{EH}$ and $\overrightarrow{{}^{g}\xi_{EH}}$ due to anisotropy. The bi-isotropic medium has a Kerr type nonlinearity characterized by:

$$\varepsilon_g=\varepsilon+\varepsilon_{Kerr}\left|E^2\right| \tag{3.4}$$

$$\xi_{EH}^2=\xi_{EH}^{\star}+\xi_{EH}^{Kerr}\left|E^2\right| \tag{3.5}$$

$\xi_{EH}^{\star}$ is the linear bi-isotropy coefficient, and the term $\xi_{EH}^{Kerr}\left|E^2\right|$ corrects the bi-isotropic coefficient with a quantity proportional to the field intensity. The linear bi-isotropy factors are written as follows:

$$\xi_{EH}=\gamma-jk \tag{3.6}$$

γ is the nonlinear non-reciprocity parameter, and k is the nonlinear chirality parameter. γ^{Kerr} is the nonlinear non-reciprocity parameter, and k^{Kerr} is the nonlinear chirality parameter. There exist three distinct cases for the biisotropic medium:
1. The chiral medium, which is reciprocal (purely imaginary),$k\neq 0$ and $\gamma=0$.
2. The Tellegen medium, which is non-reciprocal (purely real),$k=0$ and $\gamma\neq 0$.
3. The biisotropic medium, characterized by both chirality and non-reciprocity (complex numbers), with $k\neq 0$ and $\gamma\neq 0$.

In this study, our focus centers on the third case. A biisotropic fiber refers to an optical fiber featuring a chiral core enveloped by an optical cladding. The core of the biisotropic fiber possesses a slightly higher refractive index compared to the sheath.

This variation in refractive index induces total internal reflection of light within the chiral core, enabling the propagation of light with two distinct modes: a right circular polarized wave (RCP) and a left circular polarized wave (LCP), each exhibiting different wave vectors.

From Maxwell's equations, which serve as the cornerstone of electromagnetism and locally describe the evolution and properties of electric and magnetic fields, we specifically consider Maxwell's first equation, known as the Maxwell-Faraday equation [4].

This equation elucidates the phenomenon of electromagnetic induction first discovered by Faraday:

$$\overrightarrow{\nabla}\times\overrightarrow{E}=-\frac{\partial\overrightarrow{B}}{\partial t} \tag{3.7}$$

As for the second equation, which is the Maxwell-Ampere equation, and which stems from Ampere's theorem, it links the evolution of the electric field as a function of the magnetic field. It is given by:

$$\overrightarrow{\nabla} \times \overrightarrow{H} = \frac{\partial \overrightarrow{D}}{\partial t} \tag{3.8}$$

What allowed us to deduce the equation of propagation in a Kerr-biisotropic medium our result is also a generalization:

(3.9)

$$\overrightarrow{\nabla^2}.\overrightarrow{E} - (\mu\varepsilon - \mu_0\varepsilon_0 \left|\xi_{EH}^2\right|)\frac{d^2\overrightarrow{E}}{\partial t^2}\overrightarrow{\nabla^2}.\overrightarrow{E} - (\mu\varepsilon - \mu_0\varepsilon_0 \left|\xi_{EH}^2\right|)\frac{d^2\overrightarrow{E}}{\partial t^2} - \sqrt{\mu_0\varepsilon_0}\left(\xi_{EH}^\star - \xi_{EH}\right).\frac{\partial\overrightarrow{\nabla}\times\overrightarrow{E}}{\partial t} = \Big(\mu\varepsilon_{Kerr} -$$

σ is the absorption coefficient. The electric field in the bi-isotropic fiber can be represented by wave propagating in the z direction Fig.1 [17]:

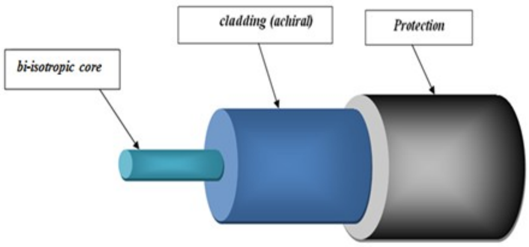

Fig.1.The bi-isotropic fiber

$$\overrightarrow{E}_{=}\left(\overrightarrow{e}_x \pm i\,\overrightarrow{e}_y\right)\Psi(r,t)e^{-i(kz-\omega_0 t)} \tag{3.10}$$

where the wave numbers k_+ (RCP) and k_- (LCP) can be written as:

$$k_+ = k\sqrt{\mu_0\varepsilon_0} + \sqrt{\mu\varepsilon - \gamma^2\mu_0\varepsilon_0} \tag{3.11}$$

$$k_- = -k\sqrt{\mu_0\varepsilon_0} + \sqrt{\mu\varepsilon - \gamma^2\mu_0\varepsilon_0} \tag{3.12}$$

The conditions of slowly variant envelope are given by:

$$\left|\frac{\partial^2}{\partial z^2}\Psi\right| \ll |2ik| \qquad \left|\frac{\partial}{\partial z}\Psi\right| \ll |i\omega_0\Psi| \tag{3.13}$$

$$\left|\frac{\partial^2}{\partial z^2}\left|\Psi\right|^2\Psi\right| \ll \left|i\omega_0\frac{\partial}{\partial t}\left|\Psi\right|^2\Psi\right| \ll \left|i\omega_0\left|\Psi\right|^2\Psi\right| \tag{3.14}$$

$$\frac{A(t,z)}{2k}\cdot\overrightarrow{e_z} = \overrightarrow{\Psi} \tag{3.15}$$

The phenomenon of dispersion is included in heuristic form through the relation

$$\Delta k = \frac{1}{v} \tag{3.16}$$

After algebraic manipulations, within the slowly varying amplitude approximation of Maxwell's equations, signals' propagating through bi-isotropic fiber is described using the following equation:

(3.17)
$$\left(\frac{\partial}{\partial}A(t,z)+\left(\beta_1\frac{\partial}{\partial t}+i\frac{1}{2}\beta_2\frac{\partial^2}{\partial t^2}-\frac{1}{6}\beta_3\frac{\partial^3}{\partial t^3}\right)A(t,z)\right)=i\delta A(t,z)+i\rho\left|A(t,z)\right|^2A(t,z)$$

When setting the variable $\widetilde{t}=t-\beta_1 z$, we obtain the nonlinear Schrödinger equation as follows:

(3.18)
$$\frac{\partial}{\partial z}A(\widetilde{t},z)+\left(i\frac{1}{2}\beta_2\frac{\partial^2}{\partial\widetilde{t}^2}-\frac{1}{6}\beta_3\frac{\partial^3}{\partial\widetilde{t}^3}\right)A(\widetilde{t},z)=\left(\frac{-\alpha}{2}+i\delta\right)A(\widetilde{t},z)-i\rho\left|A(\widetilde{t},z)\right|^2A(\widetilde{t},z)$$

In this step the bi-isotropic fiber is operattin in the third optical windows,where $\beta_3=0$ and neglecting absorption $\alpha=0$ the (3.18) becomes

$$\frac{\partial}{\partial z}A(\widetilde{t},z)+\left(i\frac{1}{2}\beta_2\frac{\partial^2}{\partial\widetilde{t}^2}\right)A(\widetilde{t},z)=i\delta A(\widetilde{t},z)-i\rho\left|A(\widetilde{t},z)\right|^2A(\widetilde{t},z) \tag{3.19}$$

we obtain our assertion [17]. ∎

3.2. **Application of the projective Riccati Equations Method in bi-isotropic fiber ([14].** Since $A(\widetilde{t},z)$ is a complex function, we assume that travelling wave transformation is in the form

$$A(\widetilde{t},z)=V(\widetilde{t},z)\exp(i\theta_(\widetilde{t},z)) \tag{3.20}$$

where $V_\pm(\widetilde{t},z)$ and $\theta_\pm(\widetilde{t},z)$ are the amplitude and phase functions respectively.
Substituting the wave transformation (3.17) into (3.19) and separating the real and imaginary parts, we have

$$-V\theta_z-\frac{1}{2}\beta_2(z)(V_{\widetilde{t}\widetilde{t}}-v\theta_{\widetilde{t}}^2)-\rho(z)V^3=0 \tag{3.21}$$

$$V_z\beta_2(z)(2\theta_z V_z+V\theta_{zz})-V=0 \tag{3.22}$$

where $\theta_z=\dfrac{d\theta}{dz},\theta_{zz}=\dfrac{d^2\theta}{dz^2}$ and $V_z=\dfrac{dV}{dz},V_{\widetilde{t}}=\dfrac{dV}{d\widetilde{t}}$ Considering the homogeneous balance in (3.21) and (3.22), we assume that (3.23) have the following solutions form

$$V(\widetilde{t},z)=a_0(z)+a_1(z)\tau(\xi)+b_1(z)\sigma(\xi);\xi=\widetilde{t}p(z)+q(z) \tag{3.23}$$

$$\theta(t,z)=tR(z)+S(z) \tag{3.24}$$

where $a_0(z)$, $a_1(z)$, $b_1(z)$, $p(z)$, $q(z)$, $R(z)$, $S(z)$. are functions of z to be determined,$\tau(\xi)$, $\sigma(\xi)$ satisfy (2.1) and (2.2).
Substituting (3.1), (3.2),(3.12) into (3.18), collecting coefficients of monomials of $\tau(\xi)$, $\sigma(\xi)$ and t of the resulting system's numerator, then setting each coefficients to zero, we obtain the following over-determined PDEs system with respect to differentiable functions : $a_0(z)$, $a_1(z)$, $b_1(z)$, $p(z)$, $q(z)$, $R(z)$, $S(z)$ with $\varepsilon=-1$,we have

$$R(\frac{\partial a_1}{\partial z}+a_1)=0 \tag{3.25}$$

$$-Rb_1\frac{\partial p}{\partial z}=0 \tag{3.26}$$

$$2b_1(-3\delta a_1^2+\delta b_1^2+\beta_2p^2\mu^2-\beta p^2+3\delta a_1^2\mu^2)=0 \tag{3.27}$$

$$2a_1(3b_1^2\delta R + \delta a_1^2\mu^2 - \delta a_1^2 + \beta_2 p^2\mu^2 - \beta_2 p^2) = 0 \tag{3.28}$$

$$-6\delta a_0 a_1^2 + 6\delta a_0 a_1^2\mu^2 + 6\delta a_0 b_1^2 R - 12\delta a_1^2 b_1\mu R - 3\beta p^2 b_1\mu R = 0 \tag{3.29}$$

$$R(\frac{\partial a_0}{\partial z} - a_0) = 0 \tag{3.30}$$

$$-b_1 R(\frac{\partial q}{\partial z} + \beta R p) = 0 \tag{3.31}$$

$$-2\frac{\partial R}{\partial z} a_0 R = 0 \tag{3.32}$$

$$-a_0 R(-2\delta a_0 - 3\delta a_1^2 R + 2\frac{\partial S}{\partial z} + R^2\beta) = 0 \tag{3.33}$$

$$-a_1 R(-4\delta a_1^2\mu - \mu p^2\beta + 12\delta a_0 b_1) = 0 \tag{3.34}$$

$$-a_1\frac{\partial p}{\partial z}(\mu - 1)(\mu + 1)) = 0 \tag{3.35}$$

$$a_1\mu\frac{\partial p}{\partial z} R = 0 \tag{3.36}$$

$$-a_1(\mu - 1)(\mu + 1)\left(\frac{\partial q}{\partial z} + \beta R p\right) = 0 \tag{3.37}$$

$$R\left(a_1\mu\frac{\partial q}{\partial z} + \frac{\partial b_1}{\partial z} + \beta R p a_1\mu - b_1\alpha\right) = 0 \tag{3.38}$$

$$-R\left(-6R\delta a_1^2 b_1 - R\beta p^2 b_1 + \beta R^2 b_1 + 12\mu a^2 a_0\delta - 6\delta a_0^2 b_1 + 2\frac{\partial S}{\partial z} b_1\right) = 0 \tag{3.39}$$

$$-2a_1\frac{\partial R}{\partial z} R = 0 \tag{3.40}$$

$$-2\frac{\partial R}{\partial z} b_1 R = 0 \tag{3.41}$$

Where a_0, b_0 denote $a_0(z)$ and $b_0(z)$, solving the algebraic system with the help of Mathematica, we get the following cases.

Case 1

$$\mu = a_0, vR(z) = C_1, b_1 = b_1, p = C_3, \delta = \delta$$
$$\beta = \frac{4\delta b_1^2 R}{C_3^2}, a_1 = \pm\sqrt{-R}b_1, \alpha = \frac{\partial b_1}{\partial z}\cdot\frac{1}{b_1}$$
$$q = \frac{-4C_1 R\int \delta b_1^2 R dz + C_6 C_3}{C_3}, S(z) = \frac{-R^2\int \delta b_1^2 dz - 2R\int \delta b_1^2 dz + C_5 C_3^2}{C_3^2}$$

The familly exact solutions of (3.19)

$$A(\widetilde{t}, z)_{11} = cRexp\left[\int \alpha(z)dz)\right]\left[\mp i\tanh\left(\sqrt{R}\xi + \sec\sqrt{R}\xi\right)\right] exp\left[i(C_1 t + S(z))\right] \tag{3.42}$$

$$A(\widetilde{t}, z)_{12} = cRexp\left[\int \alpha(z)dz)\right]\left[\mp i\coth\left(\sqrt{R}\xi + \csc\sqrt{R}\xi\right)\right] exp\left[i(C_1 t + S(z))\right] \tag{3.43}$$

Case2

$$\mu = a_0 = b_1 = 0, R(z) = C_1, a_1 = a_1, S(z) = \frac{1}{2}.\frac{-R^2 \int \delta a_1^2 dz C_2^2 R + \int \delta a_1^2 dz C_1^2 + 2C_3 C_2^2}{C_2^2}$$
$$\delta = \delta, q = \frac{C_1 \int \delta a_1^2 dz + C_4 C_2}{C_2}, \beta = -\frac{\delta a_1^2}{C_2^2}, p(z) = C_2, \alpha = \frac{\partial a_1}{\partial z}.\frac{1}{a_1}$$

The familly exact solutions of (3.19)

(3.44)
$$A(\widetilde{t}, z)_{21} = c\sqrt{R} exp\left[\int \alpha(z)dz)\right] \tanh\left[\sqrt{R}\left(C_2 t - C_1 C_2\right) \int \beta(z)dz\right].exp\left(i\left[C_1 t + S(z)\right)\right])$$

(3.45)
$$A(\widetilde{t}, z)_{22} = c\sqrt{R} exp\left[\int \alpha(z)dz)\right] \coth\left[\sqrt{R}\left(C_2 t - C_1 C_2\right) \int \beta(z)dz\right].exp\left(i\left[C_1 t + S(z)\right)\right])$$

Case3

$$\mu = \mp 1, a_0 = b_1 = 0, \delta = \delta, S(z) = \frac{\int \delta a_1^2 dz C_2 R + 2\int \delta a^2 dz C_1^2 + C_3 C_2^2}{C_2^2}$$
$$R(z) = C_1, \beta = -\frac{4a_1^2 \delta}{C_2^2}, a_1 = a_1, p = C_2, q = \frac{4C_1 \int \delta a_1^2 dz + C_2 C_4}{C_2}, \alpha = \frac{\partial a_1}{\partial z}.\frac{1}{a_1}$$

The familly exact solutions of (3.19)

$$A(\widetilde{t}, z)_{31} = \frac{c\sqrt{R}\tanh\left[\sqrt{R}\left(C_2 t - C_1 C_2 \int \beta(z)dz + C_4\right)\right]}{\pm\csc\left[\sqrt{R}\left(C_2 t - C_1 C_2 \int \beta(z)dz + C_4\right) + 1\right]} exp\left[i\left(C_1 t + S(z)\right)\right] \tag{3.46}$$

$$A(\widetilde{t}, z)_{32} = \frac{c\sqrt{R}\coth\left[\sqrt{R}\left(C_2 t - C_1 C_2 \int \beta(z)dz + C_4\right)\right]}{\pm\csc\left[\sqrt{R}\left(C_2 t - C_1 C_2 \int \beta(z)dz + C_4\right) + 1\right]} exp\left[i\left(C_1 t + S(z)\right)\right] \tag{3.47}$$

4. CONCLUSION

This study introduces a novel formulation of constitutive relations linked to magnetic effects, aiming to provide a deeper understanding of the physical nature of bi-isotropic effects and to extend existing macroscopic models. We derived the nonlinear Schrödinger equation for a bi-isotropic medium, incorporating a nonlinear magnetization term. Utilizing the extended Projective Riccati Equations Method [17], we effectively identified a family of solutions for the nonlinear Schrödinger equation in bi-isotropic optical fibers Theorem. (3.1). This powerful technique, based on perturbation expansion in terms of a dimensionless parameter, is applicable to both weak and strong nonlinearities. Additionally, the method accommodates variations in dispersion and nonlinearity, making it versatile for modeling diverse optical fibers [4]. Overall, this method offers valuable insights into the dynamics of nonlinear optical systems and holds significant potential for various applications in the field of nonlinear optics [5].

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