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SEMICOMMUTATIVE AND SEMIPRIME PROPERTIES IN BI-AMALGAMATED RINGS

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ABSTRACT. Let $\alpha : A \to B$ and $\beta : A \to C$ be two ring homomorphisms and I and I' be two ideals of B and C, respectively, such that $\alpha^{-1}(I) = \beta^{-1}(I')$. In this paper, we give a characterization for the bi-amalgamation of A with (B, C) along (I, I') with respect to (α, β) (denoted by $A \bowtie^{\alpha,\beta} (I, I')$) to be a SIT, semiprime, semicommutative and semiregular. We also give some characterization for these rings.

Key words and phrases: Idempotent; Tripotent; Semicommutative; Semiprime ring; Semiregular ring.

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1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity. For a ring R, we will use $J(R)$, $Id(R)$, $Tr(R)$ and $C(R)$, to denote the Jacobson radical, the set of idempotents, the set of all tripotents and the centre of R, respectively.

In 2016, Zhiling Ying et.all [22] investigated that rings for which every element is a sum of an idempotent and a tripotent that commute.

Let A and B be two commutative rings with unity, let I be an ideal of B and let $\alpha : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^{\alpha} I := \{(a, \alpha(a) + i) \mid a \in A, i \in I\}$$

called *the amalgamation of A with B along I with respect to α* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [11, 12]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [8, 9, 10]).

Let $\phi_1 : A \longrightarrow C$, $\phi_2 : A \longrightarrow C$ and $\alpha : A \longrightarrow B$ be ring homomorphisms. In the aforementioned papers [11, 12], the authors studied amalgamated algebras within the frame of pullback $\phi_1 \times \phi_2$ such that $\phi_1 = \phi_2 \circ \alpha$ [11, Proposition 4.2 and 4.4]. In this motivation, the authors created the new constructions, called bi-amalgamated algebras which arise as pullbacks $\phi_1 \times \phi_2$ such that the following diagram of ring homomorphims

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\phi_1} \\ C & \xrightarrow{\phi_2} & D \end{array}$$

is commutative with $\phi_1 \circ \pi_B(\phi_1 \times \phi_2) = \phi_1 \circ \alpha(A)$, where π_B denotes the canonical projection of $B \times C$ over B. Namely, let $\alpha : A \longrightarrow B$ and $\beta : A \longrightarrow C$ be two ring homomorphisms and let I and I' be two ideals of B and C, respectively, such that $\alpha^{-1}(I) = \beta^{-1}(I')$. The bi-amalgamation of A with (B, C) along (I, I') with respect to (α, β) is the subring of $B \times C$ given by

$$A \bowtie^{\alpha,g} (I, I') := \{(\alpha(a) + i, \beta(a) + i') | a \in A, (i, i') \in I \times I'\}.$$

A ring R is called semicommutative if for any $a, b \in R, ab = 0$ implies $aRb = 0$. R is semicommutative if and only if the right (left) annihilator over R is an ideal of R. Every commutative ring is semicommutative. Therefore, if A and B are commutative, then the ring $A \times B$ is commutative, and so is $A \bowtie^{\alpha} I$ as a subring of $A \times B$. A ring R is called nil-semicommutative [17], if $ab = 0$ implies $aRb = 0$ for every nilpotent elements $a, b \in R$. Every semicommutative ring is nil-semicommutative. Another version of semicommutativity is weakly semicommutativity. In [17, 7], weakly semicommutative rings were investigated. The ring R is called weakly semicommutative if for any $a, b \in R, ab = 0$ implies arb is nilpotent for any $r \in R$. Clearly, semicommutative rings are weakly semicommutative. There is no implication between nil-semicommutative rings and weakly semicommutative rings.

In [2, 14], authors have studied semicommutativity of amalgamated rings and SIT-rings of amalgamated algebra along an ideal. This motivates as we study many ring theoretical properties of the bi-amalgamation ring $A \bowtie^{\alpha,\beta} (I, I')$.

In this paper, we study many ring theoretical properties of the bi-amalgamation ring $A \bowtie^{\alpha,\beta} (I, I')$, in the case where the rings are not assumed to be commutative. We give characterizations for the bi-amalgamation ring $A \bowtie^{\alpha,\beta} (I, I')$ to be SIT-ring, semiregular, semicommutative, semiprime, nil-semicommutative, weakly semicommutative rings.

2. SIT-RING PROPERTY IN BI-AMALGAMATED RINGS

We start with a definition and examples of SIT-rings.

Definition 2.1. [22] A ring is said to be a (strong) SIT-ring if every element is a sum of an idempotent and a tripotent (that commute).

Proposition 2.1. [22] *The class of SIT-rings is closed under homomorphic images.*

Definition 2.2. Let $\alpha : A \longrightarrow B$ and $\beta : A \longrightarrow C$ be two ring homomorphisms and let I and I' be two ideals of B and C, respectively, such that $I_0 := \alpha^{-1}(I) = \beta^{-1}(I')$. The bi-amalgamation of A with (B, C) along (I, I') with respect to (α, β) is the subring of $(B \times C)$ given by $A \bowtie^{\alpha,\beta} (I, I') := \{(\alpha(a) + i, \beta(a) + i')|a \in A, (i, i') \in I \times I'\}$.

Following [15], the above definition was introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013.

Example 2.1. *Let* $A = \mathbb{Z}_2$ *and* $B = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ *and* $C = \begin{pmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{pmatrix}$ *be rings and* $I = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ *an ideal of* B *and* $I' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{pmatrix}$ *an ideal of* C.

Let $\alpha : A \longrightarrow B$ *defined by* $\alpha(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, *where* $a \in \mathbb{Z}_2$ *and* $\beta : A \longrightarrow C$ *defined by* $\beta(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$ *where* $a \in \mathbb{Z}_2$.

Then $\alpha(A) + I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ *and* $\beta(A) + I' = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

Hence, $\alpha(A) + I$ *and* $\beta(A) + I'$ *are SIT-rings.*

Also, $A \bowtie^{\alpha,g} (I, I') = \left\{ (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}), (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}), \right.$ $\left. (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}), (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \right\}$ *is a SIT-ring.*

Proposition 2.2. *If* $A \bowtie^{\alpha,\beta} (I, I')$ *is a SIT-ring then* $\alpha(A) + I$ *and* $\beta(A) + I'$ *are SIT-rings.*

Proof. Clearly, homomorphic image of a SIT-ring is a SIT-ring. Thus, in view of [15, Proposition 4.1], we have the following isomorphism of rings $\frac{A \bowtie^{\alpha,\beta} (I,I')}{0 \times I'} \cong \alpha(A) + I$ and $\frac{A \bowtie^{\alpha,\beta} (I,I')}{I \times 0} \cong \beta(A) + I'$. Hence, $\alpha(A) + I$ and $\beta(A) + I'$ are SIT-rings. ∎

Definition 2.3. A ring R is called uniquely SIT-ring if each element in R can be written uniquely as the sum of an idempotent and a tripotent.

Proposition 2.3. *Assume that* A *is a SIT-ring and* $\dfrac{\alpha(A) + I}{I}$ *and* $\dfrac{\beta(A) + I'}{I'}$ *are uniquely SIT-rings. Then* $A \bowtie^{\alpha,\beta} (I, I')$ *is a SIT-ring if and only if* $\alpha(A) + I$ *and* $\beta(A) + I'$ *are SIT-rings.*

Proof. If $A \bowtie^{\alpha,\beta} (I, I')$ is a SIT-ring, then so are $\alpha(A) + I$ and $\beta(A) + I'$. Conversely, assume that $\alpha(A) + I$ and $\beta(A) + I'$ are SIT-rings. Since A is a SIT-ring, we can write $a = e + t$, where $e \in Id(A)$ and $t \in Tr(A)$. On the other hand, since $\alpha(A) + I$ is a SIT-ring, $\alpha(a) + i = \alpha(x) + i_1 + \alpha(y) + i_2$ with $\alpha(x) + i_1$ and $\alpha(y) + i_2$ are respectively an idempotent and a tripotent element of $\alpha(A) + I$. It is clear that $\overline{\alpha(x)} = \overline{\alpha(x) + i_1}$ (resp., $\overline{\alpha(e)}$) and $\overline{\alpha(y)} = \overline{\alpha(y) + i_2}$ (resp., $\overline{\alpha(t)}$), are respectively an idempotent and a tripotent element of $\dfrac{\alpha(A) + I}{I}$, and we have $\overline{\alpha(a)} = \overline{\alpha(e)} + \overline{\alpha(t)} = \overline{\alpha(x)} + \overline{\alpha(y)}$. Thus, $\overline{\alpha(e)} = \overline{\alpha(x)}$ and $\overline{\alpha(t)} = \overline{\alpha(y)}$ since $\dfrac{\alpha(A) + I}{I}$ is an uniquely SIT-ring. So there exist $i_1^*, i_2^* \in K$ such that $\alpha(x) = \alpha(e) + i_1^*$ and $\alpha(y) = \alpha(t) + i_2^*$ and also since $\beta(A) + I'$ is a SIT-ring, $\beta(a) + i' = \beta(x) + i_1' + \beta(y) + i_2'$ with $\beta(x) + i_1'$ and $\beta(y) + i_2'$ are respectively an idempotent and a tripotent element of $\beta(A) + I'$. It is clear that $\overline{\beta(x)} = \overline{\beta(x) + i_1'}$ (resp., $\overline{\beta(e)}$) and $\overline{\beta(y)} = \overline{\beta(y) + i_2'}$ (resp., $\overline{\beta(t)}$) are respectively an idempotent and a tripotent element of $\dfrac{\beta(A) + I'}{I'}$, and we have $\overline{\beta(a)} = \overline{\beta(e)} + \overline{g(t)} = \overline{\beta(x)} + \overline{\beta(y)}$. Thus, $\overline{\beta(e)} = \overline{\beta(x)}$ and $\overline{\beta(t)} = \overline{\beta(y)}$ since $\dfrac{\beta(A) + I'}{I'}$ is an uniquely SIT-ring. So there exist $i_1^{'*}, i_2^{'*} \in I'$ such that $\beta(x) = \beta(e) + i_1^{'*}$ and $\beta(y) = \beta(t_1) + i_2^{'*}$. We have, $(\alpha(a) + i, \beta(a) + i') = (\alpha(e) + i_1^* + i_1, \beta(e) + i_1^{'*} + i_1') + (\alpha(t_1) + i_2^* + i_2, \beta(t_1) + i_2^{'*} + i_2')$, and it is clear that $(\alpha(e) + i_1^* + i_1, \beta(e) + i_1^{'*} + i_1')$ is an idempotent and $(\alpha(t_1) + i_2^* + i_2, \beta(t_1) + i_2^{'*} + i_2')$ tripotent elements of $A \bowtie^{\alpha,\beta} (I, I')$. ∎

Proposition 2.4. *Let $\alpha : A \to B$ and $\beta : A \to C$ be a ring homomorphisms and let (e_1) be an ideal of B generated by the central idempotent element e_1 and (e_2) be an ideal of C generated by the central idempotent element e_2. Assume that A is a SIT-ring. Then $A \bowtie^{\alpha,\beta} ((e_1), (e_2))$ is a SIT-ring if and only if $\alpha(A) + (e_1)$ and $\beta(A) + (e_2)$ are SIT-ring.*

Proof. Suppose that $\alpha(A) + (e_1)$ and $\beta(A) + (e_2)$ are SIT rings. In light of Proposition 2.2, we only have to show that $A \bowtie^{\alpha,\beta} ((e_1), (e_2))$ is a SIT-ring. Let $(\alpha(a) + r_1e_1, \beta(a) + r_2e_2)$ be an element of $A \bowtie^{\alpha,\beta} ((e_1), (e_2))$ (with $a \in A$, $r_1 \in B$ and $r_2 \in C$). Since A is a SIT-ring, we can write $a = s + t$, where $s \in Id(A)$ and $t \in Tr(A)$ and also since $\alpha(A) + (e_1)$ and $\beta(A) + (e_2)$ are SIT-rings, we can write $\alpha(a) + r_1e_1 = s' + t'$, where $s' \in Id(\alpha(A) + (e_1))$ and $t' \in Tr(\alpha(A) + (e_1))$ and $\beta(a) + r_2e_2 = s'' + t''$, where $s'' \in Id(\beta(A) + (e_2))$ and $t'' \in Tr(\beta(A) + (e_2))$. We have $(\alpha(a) + r_1e_1, \beta(a) + r_2e_2) = (\alpha(s) + (s' - \alpha(s))e_1, \beta(s) + (s'' - \beta(s))e_2) + (\alpha(t) + (t' - \alpha(t))e_1, \beta(t) + (t'' - \beta(t))e_2)$. On the other hand,

$$\begin{aligned}
[\alpha(s) + (s' - \alpha(s))e_1]^2 &= [\alpha(s)(1 - e_1) + s'e_1]^2 \\
&= \alpha(s)(1 - e_1) + s'e_1 \\
&= \alpha(s) + (s' - \alpha(s))e_1.
\end{aligned}$$

$$\begin{aligned}
[\alpha(t) + (t' - \alpha(t))e_1]^3 &= [\alpha(t)(1 - e_1) + t'e_1]^3 \\
&= \alpha(t)(1 - e_1) + t'e_1 \\
&= \alpha(t) + (t' - \alpha(t))e_1.
\end{aligned}$$

Similarly, $\beta(s) + (s'' - \beta(s))e_2$ is an idempotent of $\beta(A) + (e_2)$ and $\beta(t) + (t'' - \beta(t))e_2$ is a tripotent of $\beta(A) + (e_2)$. Then, $(\alpha(s) + (s' - \alpha(s))e_1, \beta(s) + (s'' - \beta(s))e_2)$ and $(\alpha(t) + (t' - \alpha(t))e_1, \beta(t) + (t'' - \beta(t))e_2)$ respectively are an idempotent and tripotent in $A \bowtie^{\alpha,\beta} ((e_1), (e_2))$. Consequently, $A \bowtie^{\alpha,\beta} ((e_1), (e_2))$ is a SIT-ring, as desired.

∎

3. SEMIREGULAR PROPERTY IN BI-AMALGAMATED RINGS

Proposition 3.1. [18, Proposition 2.2] *The following are equivalent for an element a of a ring R.*

(1) *There exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$.*
(2) *There exists $e^2 = e \in aR$ such that $a(1-e) \in J(R)$.*
(3) *There exists a regular element $b \in R$ with $a - b \in J(R)$.*
(4) *There exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$.*

By Nicholson [18] , an element a of a ring R is called semiregular, if it satisfies any one of the above conditions. A ring is a semiregular if each of its elements is semiregular. Nicholson in [18, Theorem 2.9], shows that a ring R is semiregular if $\frac{R}{J(R)}$ is von Neumann regular and idempotents lift modulo $J(R)$. The class of semiregular rings is very large. For example every von Neumann regular ring is semiregular.

Lemma 3.2. *Let $a \in J(R)$. Then we have $J(A \bowtie^{\alpha,\beta} (I, I')) = \{(\alpha(a) + i, \beta(a) + i') | \alpha(a) + i \in J(\alpha(A) + I), \beta(a) + i' \in J(\beta(A) + I')\}$.*

Proposition 3.3. *If $A \bowtie^{\alpha,\beta} (I, I')$ is a semiregular, then $\alpha(A)+I$ and $\beta(A)+I'$ are semiregular.*

Proof. Note that $\alpha(A) + I$ and $\beta(A) + I'$ are homomorphic images of $A \bowtie^{\alpha,\beta} (I, I')$. Then the result follows immediately from [18, Corollary 2.3]. ∎

Lemma 3.4. *If I is a nil ideal of B and I' is a nil ideal of C, then $J(\alpha(A)) + I \subseteq J(\alpha(A) + I)$ and $J(\beta(A)) + I' \subseteq J(\beta(A) + I')$.*

Proof. This was proved in [20, Lemma 4.6]. ∎

In view of [18, Corollary 2.3], every homomorphic image of a semiregular ring is semiregular, so if A is semiregular then $\alpha(A)$ is semiregular.

Theorem 3.5. *Let I be a nil ideal of B and I' be a nil ideal of C and A be a semiregular ring. Then $A \bowtie^{\alpha,\beta} (I, I')$ is a semiregular ring.*

Proof. Let $(\alpha(a) + i, \beta(a) + i') \in A \bowtie^{\alpha,\beta} (I, I')$. Since $\alpha(a) \in f(A)$ is semiregular and $\beta(a) \in \beta(A)$ is semiregular, there exists a von-Neumann regular element $b \in A$ with $\alpha(a) - \alpha(b) \in J(\alpha(A))$ and $\beta(a) - \beta(b) \in J(\beta(A))$. So $\alpha(a) - \alpha(b) + i \in J(\alpha(A)) + I \subseteq J(\alpha(A) + I)$ and $g(a) - g(b) + i' \in J(g(A)) + I' \subseteq J(\beta(A) + I')$, by Lemma 3.4. Thus $(\alpha(a) + i) - \alpha(b) \in J(\alpha(A) + I)$ for a von Neumann regular element $\alpha(b) \in \alpha(A) + I$ and $(\beta(a) + i') - \beta(b) \in J(\beta(A) + I')$ for a von Neumann regular element $\beta(b) \in \beta(A) + I'$. So for the von Neumann regular element $(\alpha(b), \beta(b)) \in A \bowtie^{\alpha,\beta} (I, I')$, $(\alpha(a) + i, \beta(a) + i') - (\alpha(b), \beta(b)) \in J(A \bowtie^{\alpha,\beta} (I, I'))$, by the Lemma 3.2 and the result follows. ∎

Corollary 3.6. *Let I be a nil ideal of B and I' be a nil ideal of C and A be a semiregular ring. Then $\alpha(A) + I$ and $\beta(A) + I'$ are semiregular rings.*

Using [19, Proposition 1.1], we conclude the following.

Proposition 3.7. *Let $A \bowtie^{\alpha,\beta} (I, I')$ be an exchange ring. Then $\alpha(A) + I$ and $\beta(A) + I'$ are exchange rings.*

Proof. Note that $\alpha(A) + I$ and $\beta(A) + I'$ are homomorphic images of $A \bowtie^{\alpha,\beta} (I, I')$. The result is an immediate consequence of [19, Proposition 1.4] ∎

Since by [19, Proposition 1.4], every homomorphic image of an exchange ring is exchange, if A is an exchange ring then $\alpha(A)$ is an exchange ring.

Theorem 3.8. *Let I be a nil ideal of B, I' be a nil ideal of C and A be an exchange ring. Then $A \bowtie^{\alpha,\beta} (I, I')$ is an exchange ring.*

Proof. Let $(\alpha(a) + i, \beta(a) + i') \in A \bowtie^{\alpha,\beta} (I, I')$. As A is an exchange ring, $\alpha(A)$ and $\beta(A)$ are exchange rings. So for $(\alpha(e))^2 = \alpha(e) \in \alpha(A)\alpha(a)$, $\alpha(c) \in \alpha(A)$ and $(\beta(e))^2 = \beta(e) \in \beta(A)\beta(a)$, $\beta(c) \in \beta(A)$, we have $(\alpha(1) - \alpha(e)) - \alpha(c)(\alpha(1) - \alpha(a)) \in J(\alpha(A))$ and $(\beta(1) - \beta(e)) - \beta(c)(\beta(1) - \beta(a)) \in J(\beta(A))$. Then $(\alpha(1) - \alpha(e)) - \alpha(c)(\alpha(1) - \alpha(a)) + \alpha(c)i \in J(\alpha(A))+I$ and $(\beta(1)-\beta(e))-\beta(c)(\beta(1)-\beta(a))+\beta(c)i' \in J(\beta(A))+I'$ and hence by Lemma 3.2, $(1 - \alpha(e)) - \alpha(c)(1 - (\alpha(a) + i)) \in J(\alpha(A) + I)$ and $(1 - \beta(e)) - \beta(c)(1 - (\beta(a) + i')) \in J(\beta(A)+I')$, for $(\alpha(e))^2 = \alpha(e) \in (\alpha(A)+I)(\alpha(a)+i)$, $(\beta(e))^2 = \beta(e) \in (\beta(A)+I')(\beta(a)+ i')$ and $\alpha(c) \in \alpha(A) + I, \beta(c) \in \beta(A) + I'$. Thus for $((\alpha(e), \beta(e))^2 = (\alpha(e), \beta(e)) \in A \bowtie^{\alpha,\beta} (I, I')$ and $(\alpha(c), \beta(c)) \in A \bowtie^{\alpha,\beta} (I, I')$, we have $((1, 1) - (\alpha(e), \beta(e)) - (\alpha(c), \beta(c))((1, 1) - (\alpha(a) + i, \beta(a) + i')) \in J(A \bowtie^{\alpha,\beta} (I, I'))$, by Lemma 3.4 and the result follows. ∎

4. SEMICOMMUTATIVE AND SEMIPRIME PROPERTIES IN BI-AMALGAMATED RINGS

In this section, we prove that properties of semicommutativity, nil-semicommutativity and weak semicommutativity in bi-amalgamted rings.

Theorem 4.1. *Let A, B and C be rings, $\alpha : A \longrightarrow B$ and $\beta : A \longrightarrow C$ be ring homomorphisms and let I and I' be two ideals of B and C, respectively. Then the following hold:*

(1) *If $\alpha(A) + I$ and $\beta(A) + I'$ are semicommutative rings, then so is $A \bowtie^{\alpha,\beta} (I, I')$.*
(2) *Assume that $I \cap S \neq \emptyset$, where S is the set of all central regular elements of B and $I' \cap S' \neq \emptyset$, where S' is the set of all central regular elements of C. Then $A \bowtie^{\alpha,\beta} (I, I')$ is a semicommutative ring if and only if $\alpha(A) + I$ and $\beta(A) + I'$ are semicommutative rings.*

Proof. (1) Note that the class of semicommutative rings is closed under finite products and subrings. This implies (1).

(2) To prove that $\alpha(A) + I$ is semicommutative, let $\alpha(a) + i_1, \alpha(b) + i_2 \in \alpha(A) + I$ with $(\alpha(a) + i_1)(\alpha(b) + i_2) = 0$ and let $0 \neq s \in I \cap S$. Using $(s(\alpha(a) + i_1), 0)((\alpha(b) + i_2)s, 0) = 0$ in $A \bowtie^{\alpha,\beta} (I, I')$ one gets $s(\alpha(a) + i_1)(\alpha(A) + I)(\alpha(b) + i_2)s = 0$. By the regularity of s then $(\alpha(a) + i_1)(\alpha(A) + I)(\alpha(b) + i_2) = 0$.
Similarly, for $0 \neq s' \in I' \cap S'$ by the regularity of s' we can prove that $(\beta(a) + i'_1)(\beta(A) + I')(\beta(b) + i'_2) = 0$. Hence, $\alpha(A) + I$ and $\beta(A) + I'$ are semicommutative rings.
∎

Theorem 4.2. *Let A, B and C be rings, $\alpha : A \longrightarrow B$ and $\beta : A \longrightarrow C$ be ring homomorphisms and let I and I' be two ideals of B and C, respectively. Then the following hold:*

(1) *If $\alpha(A) + I$ and $\beta(A) + I'$ are semiprime rings, then $A \bowtie^{\alpha,\beta} (I, I')$ is semiprime.*
(2) *If A, B and C are semiprime rings and let B and C be semicommutative, then $A \bowtie^{\alpha,\beta} (I, I')$ is a semiprime ring.*

Proof. (1) Suppose that $\alpha(A) + I$ and $\beta(A) + I'$ are semiprime rings. We prove that $A \bowtie^{\alpha,\beta} (I, I')$ semiprime ring. Let $(\alpha(a)+i, \beta(a)+i') \in A \bowtie^{\alpha,\beta} (I, I')$. Assume that $(\alpha(a)+i, \beta(a)+ i')(A \bowtie^{\alpha,\beta} (I, I'))(\alpha(a) + i, \beta(a) + i') = 0$. Then $(\alpha(a) + i)(\alpha(A) + I)(\alpha(a) + i) = 0$ and $(\beta(a) + i')(\beta(A) + I')(\beta(a) + i') = 0$. By hypothesis, $\alpha(a) + i = 0$ and $\beta(a) + i' = 0$, proving (1).
(2) Assume that A, B and C are semiprime rings. We prove that $A \bowtie^{\alpha,\beta} (I, I')$ is semiprime. Let $(\alpha(a) + i, \beta(a) + i') \in A \bowtie^{\alpha,\beta} (I, I')$ with $(\alpha(a) + i, \beta(a) + i')A \bowtie^{\alpha,\beta} (I, I')(\alpha(a) +$

$i, \beta(a) + i') = 0$ in $A \bowtie^{\alpha,\beta} (I, I')$. Then $(\alpha(a) + i)(\alpha(A) + I)(\alpha(a) + i) = 0$ and $(\beta(a) + i')(\beta(A) + I')(\beta(a) + i') = 0$. By the assumption, $a = 0$, $\alpha(a) = 0$ and $\beta(a) = 0$. Hence, $i(\alpha(A) + I)i = 0$ and $i'(\beta(A) + I')i' = 0$ and so $i^2 = 0$ and $i'^2 = 0$. By the semicommutative and the semiprimeness of B and C, we have $i = 0$ and $i' = 0$. Thus $\alpha(a) + i = 0$ and $\beta(a) + i' = 0$, proving (2). ∎

Now, we investigate nil-semicommutativity of bi-amalgamated rings. In [17], a ring R is called nil-semicommutative if for every $a, b \in Nil(R), ab = 0$ implies $aRb = 0$. Every semicommutative ring is nil-semicommutative. We study the conditions under which $A \bowtie^{\alpha,\beta} (I, I')$ is nil-semicommutative. We start with the following example for motivation.

Example 4.1. *Let* $A = \mathbb{Z}_2$ *and* $B = \begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ *and* $C = \begin{pmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{pmatrix}$ *the rings and* $I = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ *the ideal of* B *and* $I' = \begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *the ideal of* C.

Let $\alpha : A \longrightarrow B$ *defined by* $\alpha(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ *where* $a \in \mathbb{Z}_2$ *and* $\beta : A \longrightarrow C$ *defined by* $\beta(a) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *where* $a \in \mathbb{Z}_2$.

Also $A \bowtie^{\alpha,\beta} (I, I') = \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \right\}$ *is a nil-semicommutative ring.*

Theorem 4.3. *Let A, B and C be rings, $\alpha : A \longrightarrow B$ and $\beta : A \longrightarrow C$ be two ring homomorphisms and let I and I' be two ideals of B and C, respectively and A is nil-semicommutative ring. Then the following hold:*

1. *If $\alpha(A) + I$ and $\beta(A) + I'$ are nil-semicommutative rings, then so is $A \bowtie^{\alpha,\beta} (I, I')$.*
2. *Assume that α and β are monomorphisms and B and C are semicommutatve. If $\alpha(A)+I$ and $\beta(A)+I'$ are nil-semicommutative rings, then $A \bowtie^{\alpha,\beta} (I, I')$ is a nil-semicommutative.*

Proof. (1) Suppose that $\alpha(A) + I$ and $\beta(A) + I'$ are nil-semicommutative rings. Let $(\alpha(a) + i_1, \beta(a)+i'_1), (\alpha(b)+i_2, \beta(b)+i'_2)$ be nilpotent and $(\alpha(a)+i_1, \beta(a)+i'_1)(\alpha(b)+i_2, \beta(b)+i'_2) = 0 \in A \bowtie^{\alpha,\beta} (I, I')$. Then $\alpha(a) + i_1$ and $\alpha(b) + i_2$ are nilpotents, $(\alpha(a) + i_1)(\alpha(c) + i_3)(\alpha(b) + i_2) = 0$ for all $\alpha(c)+i_3 \in \alpha(A)+I$ and $\beta(a)+i'_1$ and $\beta(b)+i'_2$ are nilpotents, $(\beta(a)+i'_1)(\beta(c)+i'_3)(\beta(b) + i'_2) = 0$ for all $\beta(c) + i'_3 \in \beta(A) + I'$. Then $(\alpha(a) + i_1, \beta(a) + i'_1)(\alpha(c) + i_3, \beta(c) + i'_3)(\alpha(b)+i_2, \beta(b)+i'_2) = ((\alpha(a)+i_1)(\alpha(c)+i_3)(\alpha(b)+i_2), (\beta(a)+i'_1)(\beta(c)+i'_3)(\beta(b)+i'_2)) = 0$ for all $(\alpha(c) + i_3, \beta(c) + i'_3) \in A \bowtie^{\alpha,\beta} (I, I')$. Hence, $A \bowtie^{\alpha,\beta} (I, I')$ is a nil-semicommutative ring.
(2) Let $(\alpha(a) + i_1, \beta(a) + i'_1)$ and $(\alpha(b) + i_2, \beta(b) + i'_2)$ be nilpotents in $A \bowtie^{\alpha,\beta} (I, I')$ with $(\alpha(a) + i_1, \beta(a) + i'_1)(\alpha(b) + i_2, \beta(b) + i'_2) = 0$. Then $(\alpha(a) + i_1)(\alpha(b) + i_2) = 0$ and $(\beta(a) + i'_1)(\beta(b) + i'_2) = 0$. So $\alpha(a)\alpha(b) = 0$ and $\beta(a)\beta(b) = 0$. Semicommutativity of B and C, we have $(\alpha(a) + i_1)B(\alpha(b) + i_2) = 0$ and $(\beta(a) + i'_1)C(\beta(b) + i'_2) = 0$. In particular, $(\alpha(a) + i_1)(\alpha(A) + I)(\alpha(b) + i_2) = 0$ and $(\beta(a) + i'_1)(\beta(A) + I')(\beta(b) + i'_2) = 0$. Since

α, β are monomorphisms, we have $aAb = 0$. It follows that $(\alpha(a) + i_1, \beta(a) + i'_1)(A \bowtie^{\alpha,\beta} (I, I'))(\alpha(b) + i_2, \beta(b) + i'_2) = 0$. Hence, $A \bowtie^{\alpha,\beta} (I, I')$ is a nil-semicommutative. ∎

Here, weakly semicommutativity of bi-amalgamated rings is investigated under some conditions. In [16], weakly semicommutative rings were defined and studied. A ring R is called weakly semicommutative if for any $a, b \in R, ab = 0$ implies arb is nilpotent for any $r \in R$.

Theorem 4.4. *Let A, B and C be rings, $\alpha : A \longrightarrow B$ and $\beta : A \longrightarrow C$ be two ring homomorphisms and let I and I' be two ideals of B and C, respectively and A is a nil-semicommutative ring. Then the following hold:*

(1) *If $\alpha(A) + I$ and $\beta(A) + I'$ are weakly semicommutative rings, then so is $A \bowtie^{\alpha,\beta} (I, I')$.*
(2) *Assume that $I \cap S \neq \emptyset$, where S is the set of all central regular elements of B and $I' \cap S' \neq \emptyset$ where S' is the set of all central regular elements of C. Then $A \bowtie^{\alpha,\beta} (I, I')$ is a weakly semicommutative ring if and only if $\alpha(A) + I$ and $\beta(A) + I'$ are semicommutative rings.*
(3) *Assume that $\alpha(A) \cap I = (0)$ and $\beta(A) \cap I' = (0)$. If $A \bowtie^{\alpha,\beta} (I, I')$ is weakly semicommutative, then $\alpha(A) + I$ and $\beta(A) + I'$ are semicommutative rings.*

Proof. (1) Let $(\alpha(a)+i_1, \beta(a)+i'_1), (\alpha(b)+i_2, \beta(b)+i'_2) \in A \bowtie^{\alpha,\beta} (I, I'$ with $(\alpha(a)+i_1, \beta(a)+i'_1)(\alpha(b) + i_2, \beta(b) + i'_2) = 0$. Then $(\alpha(a) + i_1)(\alpha(b) + i_2) = 0$ and $(\beta(a) + i'_1)(\beta(b) + i'_2) = 0$. Suppose $(\alpha(a) + i_1)(\alpha(c) + i_3)(\alpha(b) + i_2)$ and $(\beta(a) + i'_1)(\beta(c) + i'_3)(\beta(b) + i'_2)$ are nilpotents for each $c \in A$ and $i_3 \in I, i'_3 \in I'$. If $((\alpha(a) + i_1)(\alpha(c) + i_3)(\alpha(b) + i_2))^s = 0$ and $((\beta(a) + i'_1)(\beta(c) + i'_3)(\beta(b) + i'_2))^r = 0$ for some positive integers s and r and let $m = max\{s, r\}$. Then $((\alpha(a)+i_1, \beta(a)+i'_1)(\alpha(c)+i_3, \beta(c)+i'_3)(\alpha(b)+i_2, \beta(b)+i'_2))^m = 0$. So $A \bowtie^{\alpha,\beta} (I, I')$ is weakly semicommutative.
(2) Let $(\alpha(a) + i_1)(\alpha(b) + i_2) = 0$ in $\alpha(A) + I$ and $0 \neq s \in I \cap S$. Then $(s(\alpha(a) + i_1), 0)(s(\alpha(b) + i_2), 0) = 0$. Hence $(s(\alpha(a) + i_1), 0)(\alpha(c) + i_3, \beta(c) + i'_3)(s(\alpha(b) + i_2), 0)$ is nilpotent in $A \bowtie^{\alpha,\beta} (I, I'$ for all $\alpha(c) + i_3 \in \alpha(A) + I$ and $\beta(c) + i'_3 \in \beta(A) + I'$. The element s being central implies that $s^2((\alpha(a) + i_1)(\alpha(c) + i_3)(\alpha(b) + i_2))$ is nilpotent for all $\alpha(c)+i_3 \in \alpha(A)+I$. Thus$\alpha(A)+I$ is weakly semicommutative and let $(\beta(a)+i'_1)(\beta(b)+i'_2) = 0$ in $\beta(A) + I'$ and $0 \neq' s \in I' \cap S'$. Then $(0, s'(\beta(a) + i'_1))(0, s'(\beta(b) + i'_2)) = 0$. Hence $(0, s'(\beta(a) + i'_1))(\alpha(c) + i_3, \beta(c) + i'_3)(0, s'(\beta(b) + i'_2))$ is nilpotent in $A \bowtie^{\alpha,\beta} (I, I')$ for all $\alpha(c) + i_3 \in \alpha(A) + I$ and $\beta(c) + i'_3 \in \beta(A) + I'$. The element s' being central implies that $s'^2((\beta(a) + i'_1)(\beta(c) + i'_3)(\beta(b) + i'_2))$ is nilpotent for all $\beta(c) + i'_3 \in \beta(A) + I'$. Thus $\beta(A) + I'$ is weakly semicommutative.
(3) Assume that $\alpha(A)+I = (0)$ and $\beta(A)+I' = (0)$ and $A \bowtie^{\alpha,\beta} (I, I')$ is weakly semicommutative. We prove $\alpha(A)+I$ and $\beta(A)+I'$ are weakly semicommutative. Let $\alpha(a)+i_1, \alpha(b)+i_2 \in \alpha(A) + I$ with $(\alpha(a) + i_1)(\alpha(b) + i_2) = 0$. Then $\alpha(a)\alpha(b) \in \alpha(A) \cap I = (0)$ implies $\alpha(a)\alpha(b) = 0$ and let $\beta(a) + i'_1, \beta(b) + i'_2 \in \beta(A) + I'$ with $(\beta(a) + i'_1)(\beta(b) + i'_2) = 0$. Then $\beta(a)\beta(b) \in \beta(A) \cap I' = (0)$ implies $\beta(a)\beta(b) = 0$. Hence $(\alpha(a)+i_1, \beta(a)+i'_1)(\beta(b)+i_2, \beta(b)+i'_2) = 0$. Weakly semicommutativity of $A \bowtie^{\alpha,\beta} (I, I')$ implies that $(\alpha(a)+i_1, \beta(a)+i'_1)A \bowtie^{\alpha,\beta} (I, I')(\beta(b) + i_2, \beta(b) + i'_2)$ is nil. It follows that $(\alpha(a) + i_1)(\alpha(A) + I)(\alpha(b) + i_2)$ is nil and $(\beta(a) + i'_1)(\beta(A) + I')(\beta(b) + i'_2)$ is nil. So $\alpha(A) + I$ and $\beta(A) + I'$ are semicommutative rings. ∎

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

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