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DERIVATION OF THE EXISTENCE THEOREM OF THE SOLUTION OF THE STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATION USING CONDITIONS GIVEN PARTIAL WEIGHTS

YOUNG-HO KIM

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DEPARTMENT OF MATHEMATICS, CHANGWON NATIONAL UNIVERSITY, CHANGWON, GYEONGSANGNAM 51140, KOREA.
yhkim@changwon.ac.kr

ABSTRACT. The main purpose of this note was to demonstrate the solution existence theorem for stochastic functional differential equations under sufficient conditions. As an alternative to the stochastic process theory of the stochastic functional differential equations, we impose a partial weighting condition and a weakened linear growth condition. We first show that the condition guarantees existence and uniqueness and then show some exponential estimates for the solution.

Key words and phrases: Hölder inequality, Willett and Wong's inequality, Stachurska's inequality, moment inequality, stochastic functional differential equation.

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Corresponding Author: Young-Ho Kim(yhkim@changwon.ac.kr).

1. INTRODUCTION

Stochastic functional differential equations (SFDEs) constitute a powerful mathematical framework for modeling dynamic systems influenced by stochastic processes. They find widespread application across various disciplines, including finance, physics, biology, and engineering. In recent decades, the study of SFDEs has gained significant attention due to their ability to capture both deterministic dynamics and stochastic fluctuations, making them invaluable tools for understanding complex phenomena in uncertain environments. See the references to this [2]-[7] and [9]-[15].

The fundamental question addressed in this paper is the existence of solutions to stochastic differential equations. While the existence or uniqueness of solutions to ordinary differential equations (ODEs) have been extensively studied and established under appropriate conditions, the analysis of SFDEs presents additional challenges due to the presence of stochastic terms. The interplay between deterministic and stochastic components in SFDEs necessitates specialized techniques for establishing the existence of solutions.

The existence theorem for solutions of SFDEs plays a central role in ensuring the well-posedness of stochastic models and underpins their applicability in practical contexts. This theorem provides conditions under which solutions to SFDEs exist, allowing researchers and practitioners to confidently utilize these models for prediction, control, and optimization tasks. See the references to this [7], [10], and [11].

In the Itô's classical theory of SFDEs, the condition of Lipsitz continuity was assumed as a condition for confirming the existence of a solution. The solution is constructed by applying a continuous approximation method through a given Brownian motion and the uniqueness of the solution could be seen in this approximate way.

Nowadays, there are many examples where the Lipschitz condition is not satisfied, but we can prove the existence and uniqueness of the solution. See the references to this [2], [6], and [12]-[15]. However, in these cases, the problem of existence and uniqueness has been dealt with in different ways in continuous approximation. Until now, it was not known enough whether the solution could be continuously approximated in these examples.

In this paper, we aim to supply an overview of the existence theorem for solutions of SFDEs, elucidating the key concepts, mathematical techniques, and theoretical results involved. We begin by introducing the basic definitions and properties of SFDEs, followed by a discussion of the challenges inherent in establishing the existence of solutions. Subsequently, we survey some of the fundamental results and approaches in the literature for proving the existence of solutions to various classes of SFDEs.

Mao [10] obtained that if two conditions Lipschiz and linear growth condition are satisfied, then the SDEs

$$dz(t) = f(z(t), t)dt + g(z(t), t)dB(t) \tag{1.1}$$

had a unique solution $z(t)$, moreover, $z(t) \in \mathcal{M}^2([t_0, T]; R^{d\times m})$ which means that we denoted by $\mathcal{M}^2$ the set of processes $\{z(t)\}$ in $\mathcal{L}^p$ such that $E\int_{t_0}^T |z(t)|^2 dt < \infty$.

Especially, Wei at al.[15] obtained that if two conditions (1.2) and (1.3) are satisfied: For all $y_1, y_2 \in R^d$ and $t \in [t_0, T]$, it follows that

$$|f(y_1, t) - f(y_2, t)|^2 \vee |g(y_1, t) - g(y_2, t)|^2 \le \kappa\,(|y_1 - y_2|^2), \tag{1.2}$$

where the function $\kappa(\cdot)$ is a concave non-decreasing. For all $t \in [t_0, T]$, it follows that $f_1(0, t), f_2(0, t) \in R^d \times [t_0, T]$ such that

$$|f_1(0, t)|^2 \vee |f_2(0, t)|^2 \le K, \tag{1.3}$$

then there exists a unique solution $z(t)$ to stochastic differential equation (1.1) on the closed interval $[t_0, T]$.

And, Mao [10] had established the existence and uniqueness theorems and discussed the properties of the solution for the SFDEs in his book. More specifically, he derived the solution of equation:

$$dy(t) = f(y_t, t)dt + g(y_t, t)dB(t), \tag{1.4}$$

on the closed interval $[x_0, T]$. And he obtained that if there exist two positive constants $\overline{K}$ and K such that

(i) (Uniform Lipschiz condition) for all $y_1, y_2 \in R^d$ and $t \in [x_0, T]$

$$|f(y_1, t) - f(y_2, t)|^2 \vee |g(y_1, t) - g(y_2, t)|^2 \leq \overline{K}||y_1 - y_2||^2; \tag{1.5}$$

(ii) (Linear growth condition) for any $(\varphi, t) \in R^d \times [x_0, T]$

$$|f(\varphi, t)|^2 \vee |g(\varphi, t)|^2 \leq K(1 + ||\varphi||^2), \tag{1.6}$$

then the SFDEs (1.4) had a unique solution $y(t)$, moreover, $y(t) \in \mathcal{M}^2([x_0 - \tau, T]; R^{d\times m})$ which means that we denoted by $\mathcal{M}^2$ the set of processes $\{y(t)\}$ in $\mathcal{L}^p$ such that $E\int_{x_0}^T |y(t)|^2 dt < \infty$.

However, the Lipschiz condition and linear growth condition only ensure the existence and uniqueness of the solution. In general, the solution has no explicit expression except for the linear case discussed in previous researchers. See the references to this [10]. Therefore, in practice, we often explore new conditions that provide exact solutions or approximate solutions. In the book [10], by using the Picard iteration procedure, authors established the theorem on the existence and uniqueness of the solution for d-dimensional stochastic differential equation. As the by-product, authors also obtained the Picard approximate solution for the equation and following Theorem 1.1 which gives an estimate on the difference, called the error, between the approximate and the accurate solution.

Theorem 1.1. *Assume that (1.5) and (1.6) hold. Let $y(t)$ be the unique solution of equation (1.4) and $y_n(t)$ be the Picard iteration. Then*

$$E\left(\sup_{x_0 \leq t \leq T} |y_n(t) - y(t)|^2\right) \leq \gamma_1 \exp\left(2M(T - x_0)\right) \tag{1.7}$$

for all $n \geq 1$.

In practice, given the error $\epsilon > 0$, one can determine n for left-hand side of (1.7) to be less than ϵ, and then compute $y_0(t), y_1(t), \cdots, y_n(t)$ by the Picard iteration. According to Theorem 1.1, we have

$$E\left(\sup_{x_0 \leq t \leq T} |y_n(t) - y(t)|^2\right) \leq \epsilon.$$

In the paper [2], by employing non-Lipschitz condition and non-linear growth condition, authors established the results for d-dimensional stochastic differential equations. Motivated by [2], [10], and [15], we will investigate the existence and uniqueness theorem of the solution for SFDEs at a phase space $\mathcal{M}^2\left([x_0 - \tau, T]; R^d\right)$ in this paper.

By elucidating the theoretical underpinnings of the existence theorem, this paper seeks to contribute to the foundational understanding of stochastic differential equations and facilitate their effective utilization in diverse scientific and engineering applications. Furthermore, we aim to highlight avenues for future research and development in the field of stochastic analysis, with a focus on advancing our understanding of the existence and behavior of solutions to SDEs in complex and high-dimensional settings.

2. PRELIMINARY

For the smooth development of the main theorems, it is necessary to introduce analytical inequalities introduced in the following lemmas.

Lemma 2.1. ([1, 10]) *(Hölder's inequality) If* $\frac{1}{p}+\frac{1}{q}=1$ *for any* $p, q>1$, $f_1 \in \mathcal{L}^p$, *and* $f_2 \in \mathcal{L}^q$, *then* $f_1 f_2 \in \mathcal{L}^1$ *and* $\int_a^b f_1 f_2 dx \leq \left(\int_a^b |f_1|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |f_2|^q dx\right)^{\frac{1}{q}}$.

Lemma 2.2. ([1]) *(Willett and Wong, 1965) Let* $v(t), b(t)$, *and* $k(t)$ *be nonnegative continuous functions in* $J=[\alpha, \beta]$, *and let* $p \geq 0, p \neq 1$, *and* $a>0$ *be constants. Suppose that*

$$v(t) \leq a+\int_\alpha^t b(s) v(s) ds+\int_\alpha^t k(s) v^p(s) ds,\ t \in J.$$

Then

$$v(t) \leq \exp \left(\int_\alpha^t b(s) ds\right)\left[a^{1-p}+(1-p) \int_\alpha^t k(s) \exp \left((p-1) \int_\alpha^s b(\tau) d\tau\right) ds\right]^{\frac{1}{p-1}}.$$

Lemma 2.3. ([1]) *(Stachurska's inequality) Let* v, a_1, a_2, *and* $k(t)$ *be nonnegative continuous functions in* $J=[\alpha, \beta)$, *and let* $p>0, p \neq 1$,

$$v(t) \leq a_1(t)+a_2(t) \int_\alpha^t k(s) v^p(s) ds,\ t \in J,$$

where $\frac{a_1}{a_2}$ *is nondecreasing function* $t \in J$. *Then*

$$v(t) \leq a_1(t)\left(1-(p-1)\left[\frac{a_1(t)}{a_2(t)}\right]^{p-1} \int_\alpha^t k(s) a_2^p(s) ds\right)^{\frac{1}{1-p}},\ \alpha \leq t<\beta_1,$$

where $\beta_1=\beta$ *if* $0<p<1$, *and for* $p>1$, β_1 *is the smallest value of* $t \geq \alpha$ *such that the expression between brackets vanishes.*

Next, we will introduce the symbols and the functions of the stochastic functional differential equations needed to understand the main theorems introduced in the next chapter.

Throughout this paper unless otherwise specified, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ and let the filtration is satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null events), and $B(t)$ is an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$, that is $B(t)=(B_1(t), B_2(t), \ldots, B_m(t))^T$.

Let $\tau>0$ and $|\cdot|$ denote by $C([-\tau, 0]; R^d)$ the family of continuous functions φ from $[-\tau, 0]$ to R^d with the norm $||\varphi||=sup_{-\tau \leq \theta \leq 0}|\varphi|$. Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A|=\sqrt{\operatorname{trace}(A^T A)}$.

Let $0 \leq t_0 \leq T<\infty$, and let

$$f:=C([-\tau, 0]; R^d) \times [t_0, T] \rightarrow R^d \quad on \quad g:=C([-\tau, 0]; R^d) \times [t_0, T] \rightarrow R^{d \times m}$$

be both Borel measurable. Consider the d-dimensional stochastic functional differential equation

$$dx(t)=f(x_t, t) dt+g(x_t, t) dB(t) \quad on\ t_0 \leq t \leq T, \tag{2.1}$$

where $x_t=\{x(t+\theta): -\tau \leq \theta \leq 0\}$ is regarded and a $C([-\tau, 0]; R^d)$-valued stochastic process.

The next definition introduced is the definition of the solution of the above equation (2.1).

Definition 2.1. ([10]) An R^d-valued stochastic process $x(t)$ on $t_0 - \tau \le t \le T$ is called a solution of equation (2.1) if it has the following properties:
(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;
(ii) $\{f(x(t),t)\} \in \mathcal{L}^1([t_0,T];R^d)$ and $\{g(x(t),t)\} \in \mathcal{L}^2([t_0,T];R^{d\times m})$;
(iii) $x_{t_0} = \xi$ and, for every $t_0 \le t \le T$,

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x_s,s)ds + \int_{t_0}^{t} g(x_s,s)dB(s). \tag{2.2}$$

A solution $x(t)$ is said to be unique if any other solution $\bar{x}(t)$ is indistinguishable from it, that is

$$P\{x(t) = \bar{x} \ \ for\ all\ t_0 - \tau \le t \le T\} = 1.$$

The two well-known theorems of moment inequalities, introduced next, are important inequalities that dictate the order between Itô integrals and general integrals. These inequalities will be used to prove the main theorems.

Lemma 2.4. ([10])*(moment inequality) Let $p \ge 2$. Let $f_1 \in \mathcal{M}^2([0,T];R^{d\times m})$ such that*

$$E\int_0^T |f_1(s)|^p ds < \infty.$$

Then

$$E\left|\int_0^T f_1(s)dB(s)\right|^p \le \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |f_1(s)|^p ds.$$

Lemma 2.5. ([10])*(moment inequality) If $p \ge 2$, $g \in \mathcal{M}^2\left([0,T];R^{d\times m}\right)$ such that*

$$E\int_0^T |g(s)|^p ds < \infty,$$

then

$$E\left(\sup_{0\le t\le T}\left|\int_0^t g(s)dB(s)\right|^p\right) \le \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{p}} E\int_0^T |g(s)|^p ds.$$

3. MAIN RESULTS

Let us now begin to establish the theory of the existence and uniqueness of the solution. We first show that the partial weighting Lipschitz condition and the weakened linear growth condition guarantee the existence and uniqueness.

Now,we consider the d-dimensional stochastic functional differential equation

$$dx(t) = f(x_t,t)dt + g(x_t,t)dB(t) \quad on\ t_0 \le t \le T, \tag{3.1}$$

where $x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$ is regarded and a $C([-\tau,0];R^d)$-valued stochastic process. Here we have to think about what is the initial value problem for a given equation (3.1). More specifically, it is necessary to specify the minimum amount of initial data to satisfy the definition of a stochastic process $x(t)$ in the equation (3.1). In other words, it indicates that we need to specify the stochastic process displayed in the entire interval $[t_0 - \tau, t_0]$. We therefore impose the initial data:

$$x_{t_0} = \xi = \{\xi(\theta) : -\tau < \theta \le 0\} \tag{3.2}$$

is a $\mathcal{F}_{t_0}$-measurable, $C((-\tau,0];R^d)$-value random variable such that $E||\xi||^2 < \infty$.

In order to obtain an existence and uniqueness solution to stochastic functional differential equation (3.1), It is necessary to check the following lemma.

Lemma 3.1. *Let the following two conditions partial weighted Lipschitz condition and weakened linear growth condition hold:*

(i) *(Partial weighted Lipschitz condition) For all* $\varphi, \psi \in C((-\tau, 0]; R^d)$ *and* $t \in [t_0, T]$, *we assume that*

$$|f(\varphi, t) - f(\psi, t)|^2 \vee |g(\varphi, t) - g(\psi, t)|^2 \leq \overline{K}(||\varphi - \psi||^2)^{\alpha}, \tag{3.3}$$

where $\overline{K}$ *is a positive constant and* $0 < \alpha < 1$ *is a constant.*

(ii) *(Weakened linear growth condition) For all* $t \in [t_0, T]$, *it follows that* $f_1(0, t), f_2(0, t) \in L^2$ *such that*

$$|f_1(0, t)|^2 \vee |f_2(0, t)|^2 \leq K, \tag{3.4}$$

where K *is a positive constant.*

If $x(t)$ *is the solution of (3.1) with initial data (3.2), then*

$$E\Big(\sup_{t_0-\tau \leq t \leq T} |x(t)|^2\Big) \leq \Big[C_1^{1-\alpha} + (1-\alpha)6\overline{K}(T - t_0 + 4)(T - t_0)\Big]^{1/(1-\alpha)}, \tag{3.5}$$

where $C_1 = 4E||\xi||^2 + 6K(T - t_0 + 4)(T - t_0)$.

Proof. For each number $n \geq 1$, define the stopping time

$$\eta_n = T \wedge \inf\{t \in [t_0, T] : ||x_t|| \geq n\}.$$

Obviously, as $n \to \infty, \eta_n \uparrow T$ a.s. Let $x^n(t) = x(t \wedge \eta_n), t \in [t_0 - \tau, T]$. Then, for $t_0 \leq t \leq T$, $x^n(t)$ satisfy the following equation

$$x^n(t) = \xi(0) + \int_{t_0}^t f(x_s^n, s) I_{[t_0, \eta_n]}(s) ds + \int_{t_0}^t g(x_s^n, s) I_{[t_0, \eta_n]}(s) dB(s).$$

Using the elementary inequality $(y + z + w)^2 \leq 3(y^2 + z^2 + w^2)$ and Hölder's inequality, we have

$$|x^n(t)|^2 \leq 3|\xi(0)|^2 + 3(t - t_0) \int_{t_0}^t \Big| f(x_s^n, s) I_{[t_0, \eta_n]}(s)\Big|^2 ds + 3\Big| \int_{t_0}^t g(x_s^n, s) I_{[t_0, \eta_n]}(s) dBs\Big|^2.$$

Taking the expectation on both sides and applying Doob's martingale inequality, we get the following

$$\begin{aligned} &E\Bigg(\sup_{t_0 \leq s \leq t} |x^n(s)|^2\Bigg) \\ &\leq 3E|\xi(0)|^2 + 3(T - t_0)E \int_{t_0}^t \Big| f(x_s^n, s)(s)\Big|^2 ds + 12E\Big| \int_{t_0}^t g(x_s^n, s) I_{[t_0, \eta_n]}(s) dBs\Big|^2. \end{aligned}$$

By Lemma 2.4, partial weighted Lipschitz condition (3.3) and weakened linear growth condition (3.4), we then show that

$$\begin{aligned} &E\Bigg(\sup_{t_0 \leq s \leq t} |x^n(s)|^2\Bigg) \\ &\leq 3E|\xi(0)|^2 + 6K(T - t_0 + 4)(T - t_0) + 6\overline{K}(T - t_0 + 4) \int_{t_0}^t E\big(||x_s^n||\big)^{\alpha} ds. \end{aligned}$$

Noting that $\sup_{t_0-\tau\le s\le t}|x^n(s)|^2 \le ||\xi||^2 + \sup_{t_0\le s\le t}|x^n(s)|^2$, we obtain

$$E\Big(\sup_{t_0-\tau\le s\le t}|x^n(s)|^2\Big) \le C_1 + 6\overline{K}(T-t_0+4)\int_{t_0}^t E\big(\sup_{t_0-\tau\le r\le s}|x^n(r)|\big)^\alpha ds,$$

where $C_1 = 4E||\xi||^2 + 6K(T-t_0+4)(T-t_0)$.

Now the Jensenpś inequality and Lemma 2.2 yields that

$$E\Big(\sup_{t_0-\tau\le s\le t}|x^n(s)|^2\Big) \le \Big[C_1^{1-\alpha} + 6\overline{K}(1-\alpha)(T-t_0+4)(T-t_0)\Big]^{1/(1-\alpha)}.$$

It then follows that

$$E\Big(\sup_{t_0-\tau\le t\le \eta_n}|x(t)|^2\Big) \le \Big[C_1^{1-\alpha} + 6\overline{K}(1-\alpha)(T-t_0+4)(T-t_0)\Big]^{1/(1-\alpha)}.$$

Consequently the required inequality (3.5) follows by letting $n\to\infty$. ∎

The next two theorems are the main theorem in this section, which states that if conditions (3.3) and (3.4) are satisfied, the unique solution to Stochastic functional differential equation (3.1) exists. It is also intended to show that conditions (3.3) and (3.4) guarantee the existence of the solution to Stochastic functional differential equation (3.1).

Theorem 3.2. *Suppose that the condition (3.3) and (3.4) are valid. If there is a solution to equation (3.1), then there is only one solution to the equation.*

Proof. Let $x(t), \bar{x}(t)$ be any two solutions of the equation. By Lemma 3.1, both of $x(t)$ and $\bar{x}(t)$ belong to $\mathcal{M}^2([t_0-\tau,T];R^d)$. Note that

$$x(t)-\bar{x}(t) = \int_{t_0}^t [f(x_s,s)-f(\bar{x}_s,s)]ds + \int_{t_0}^t [g(x_s,s)-g(\bar{x}_s,s)]dB(s).$$

By the elementary inequality, we can easily show that sees that

$$\begin{aligned}&|x(t)-\bar{x}(t)|^2\\ &\le 2\Big|\int_{t_0}^t [f(x_s,s)-f(\bar{x}_s,s)]ds\Big|^2 + 2\Big|\int_{t_0}^t [g(x_s,s)-g(\bar{x}_s,s)]dB(s)\Big|^2.\end{aligned}$$

Taking the expectation on both sides and applying Hölder inequality, we get the following

$$\begin{aligned}&E\Big(\sup_{t_0\le s\le t}|x(s)-\bar{x}(s)|^2\Big)\\ &\le 2(t-t_0)E\int_{t_0}^t |f(x(s),s)-f(\bar{x}(s),s)|^2 ds + 2E\sup_{t_0\le s\le t}\int_{t_0}^s |g(x_r,s)-g(\bar{x}_r,s)|^2 dr.\end{aligned}$$

By Lemma 2.5 and the condition (3.3), one can show that

$$E\Big(\sup_{t_0\le s\le t}|x(s)-\bar{x}(s)|^2\Big) \le 2\overline{K}(T-t_0+4)\int_{t_0}^t E\Big(\sup_{t_0\le r\le s}(|x(r)-\bar{x}(r)|^2\Big)^\alpha ds.$$

By the Jensen inequality, we have

$$E\Big(\sup_{t_0\le s\le t}|x(s)-\bar{x}(s)|^2\Big) \le 2\overline{K}(T-t_0+4)\int_{t_0}^t \Big(E\sup_{t_0\le r\le s}(|x(r)-\bar{x}(r)|^2\Big)^\alpha ds.$$

By the Stachurska's inequality(Lemma 2.3), one deduces that

$$E\Big(\sup_{t_0\le s\le t}|x(s)-\bar{x}(s)|^2\Big) = 0.$$

This implies that $x(t)=\overline{x}(t)$ for $t_0\le t\le T$. The uniqueness has been proved. ∎

Theorem 3.3. *Suppose that the condition (3.3) and (3.4) are valid. If* $[C_2\{M(t-t_0)\}^n]/n! \geq 1$ *is satisfied, then there exists a solution to the equation (3.1), where* $M = 2\overline{K}(T - t_0 + 4)$ *and* $C_2 = 8K(T - t_0 + 4)(T - t_0) + 8\overline{K}(T - t_0 + 4)E(||\xi||^2)^\alpha(T - t_0)$. *Moreover, the solution belongs to* $\mathcal{M}^2\big([t_0 - \tau, T]; R^d\big)$.

Proof. Define $x^0_{t_0} = \xi$ and $x^0(t) = \xi(0)$ for $t_0 \leq t \leq T$. For each $n = 1, 2, \cdots$, set $x^n_{t_0} = \xi$ and define, by the Picard iterations,

$$x^n(t) = \xi(0) + \int_{t_0}^{t} f(x^{n-1}_s, s)ds + \int_{t_0}^{t} g(x^{n-1}_s, s)dB(s) \tag{3.6}$$

for $t_0 \leq t \leq T$.

It is easy to show that $x^n(\cdot) \in \mathcal{M}^2([t_0 - \tau, T]; R^d)$ (The more detailed verification process is similar to the proof process in Lemma 3.1. The details are left to reader to check).

We claim that for all $n \geq 0$,

$$E\Big(\sup_{t_0 \leq s \leq t} |x^{n+1}(s) - x^n(s)|^2\Big) \leq \frac{C_2[M(t - t_0)]^n}{n!} \tag{3.7}$$

on $t_0 \leq t \leq T$, where $M = 2\overline{K}(T - t_0 + 4)$ and C_2 will defined below. First we compute $E(\sup_{t_0 \leq s \leq t} |x^1(s) - x^0(s)|^2)$. By the Hölder inequality and Doob's martingale inequality, we have

$$\begin{aligned} &E\Big(\sup_{t_0 \leq t \leq T} |x^1(t) - x^0(t)|^2\Big) \\ &\leq 2(T - t_0)E\int_{t_0}^{T} |f(x^0_s, s)|^2 ds + 8E\Big|\int_{t_0}^{T} g(x^0_s, s)dB(s)\Big|^2. \end{aligned}$$

Using the moment inequality(Lemma 2.4), the condition (3.3) and (3.4), one can show that

$$\begin{aligned} &E\Big(\sup_{t_0 \leq t \leq T} |x^1(t) - x^0(t)|^2\Big) \\ &\leq 2(T - t_0 + 4)(4K(T - t_0)) + 2(T - t_0 + 4)\Big(4E\int_{t_0}^{T} \overline{K}(||x^0_s||^2)^\alpha ds\Big) \\ &\leq 8K(T - t_0 + 4)(T - t_0) + 8\overline{K}(T - t_0 + 4)E(||\xi||^2)^\alpha(T - t_0) \equiv C_2. \end{aligned}$$

So the inequality (3.7) holds for $n = 0$. Next, assume the inequality (3.7) holds for some $n \geq 0$. Then, by the Hölder inequality and Doob's martingale inequality, we have

$$\begin{aligned} &E\Big(\sup_{t_0 \leq s \leq t} |x^{n+2}(s) - x^{n+1}(s)|^2\Big) \\ &\leq 2(T - t_0)E\int_{t_0}^{t} |f(x^{n+1}_s, s) - f(x^n_s, s)|^2 ds + 8E\Big|\int_{t_0}^{t} (g(x^{n+1}_s, s) - g(x^n_s, s))dB(s)\Big|^2. \end{aligned}$$

Using the moment inequality(Lemma 2.4), the condition (3.3) and (3.4), one can show that

$$\begin{aligned} &E\Big(\sup_{t_0 \leq s \leq t} |x^{n+2}(s) - x^{n+1}(s)|^2\Big) \leq 2\overline{K}(T - t_0 + 4)E\int_{t_0}^{t} (||x^{n+1}_s - x^n_s||^2)^\alpha ds \\ &\leq M\int_{t_0}^{t} \Big(E\sup_{t_0 \leq r \leq s} |x^{n+1}(r) - x^n(r)|^2\Big)^\alpha ds, \end{aligned}$$

where $M = 2\overline{K}(T-t_0+4)$. Here, the following can be obtained from the induction assumption (3.7) and the inequality assumption $[C_2\{M(t-t_0)\}^n]/n! \geq 1$.

$$E\Big(\sup_{t_0\leq s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2\Big) \leq M\int_{t_0}^{t}\Big(\frac{C_2[M(s-t_0)]^n}{n!}\Big)^{\alpha}ds$$
$$\leq M\int_{t_0}^{t}\Big(\frac{C_2[M(s-t_0)]^n}{n!}\Big)ds = \frac{C_2[M(t-t_0)]^{n+1}}{(n+1)!}.$$

That is, the assumption (3.7) holds for $n+1$. Hence, by induction, the inequality (3.7) holds for all $n \geq 0$. From the inequality (3.7), we can then show in the same way as in the proof of the Theorem([10], p.51, Theorem 2.3.1) that $x^n(\cdot)$ converges to $x(t)$ in $\mathcal{M}^2\left([t_0-\tau, T]; R^d\right)$ in sense of L^2 as well as probability 1, and $x(t)$ is a solution to equation (3.1) satisfying the initial condition (3.2). The existence has also been proved. ∎

In the above two theorem we have shown that the Picard Iterations $x^n(t)$ converge to the unique solution $x(t)$ of equation (3.1). The following two theorems provide estimates for the difference between $x^n(t)$ and $x(t)$, and clearly show that an approximate solution to the equation (3.1) can be obtained using the Picard iterative procedure.

Theorem 3.4. *Let the assumptions of Theorem 3.3 hold. Let $x(t)$ be the unique solution of equation (3.1) with initial data (3.2) and $x^n(t)$ be the Picard iterations defined by (3.6). Then, for all $n \leq 1$,*

$$E\Big(\sup_{t_0\leq t\leq T}|x^n(t)-x(t)|^2\Big) \leq \Big(C_3^{1-\alpha}+(1-\alpha)\overline{M}(t-t_0)\Big)^{1/(1-\alpha)}, \tag{3.8}$$

where $C_3 = \overline{M}\big(C_2[M(t-t_0)]^{n-1}/(n-1)!\big)^{\alpha}(t-t_0)$ and $\overline{M} = 2^{1+\alpha}\overline{K}(T-t_0+4)$.

Proof. By the Hölder inequality and Doob's martingale inequality, it is easy to derive that

$$E\Big(\sup_{t_0\leq s\leq t}|x^n(s)-x(s)|^2\Big)$$
$$\leq 2(T-t_0)E\int_{t_0}^{t}|f(x_s^{n-1},s)-f(x_s,s)|^2ds + 8E\Big|\int_{t_0}^{t}(g(x_s^{n-1},s)-g(x_s,s))dB(s)\Big|^2.$$

Using the moment inequality(Lemma 2.4), the condition (3.3) and (3.4), one can show that

$$E\Big(\sup_{t_0\leq s\leq t}|x^n(s)-x(s)|^2\Big)$$
$$\leq 2\overline{K}(T-t_0+4)E\int_{t_0}^{t}(||x_s^{n-1}-x_s||^2)^{\alpha}ds$$
$$\leq 2\overline{K}(T-t_0+4)\int_{t_0}^{t}E\Big(\sup_{t_0\leq r\leq s}|x^{n-1}(r)-x(r)|^2\Big)^{\alpha}ds$$
$$\leq 2\overline{K}(T-t_0+4)\int_{t_0}^{t}E\Big(2\sup_{t_0\leq r\leq s}|x^n(r)-x^{n-1}(r)|^2+2\sup_{t_0\leq r\leq s}|x^n(r)-x(r)|^2\Big)^{\alpha}ds.$$

Here, applying the meaning that inequality (3.7) is established, the following can be obtained

$$E\Big(\sup_{t_0\leq s\leq t}|x^n(s)-x(s)|^2\Big)$$
$$\leq \overline{M}\Big(\frac{C_2[M(t-t_0)]^{n-1}}{(n-1)!}\Big)^{\alpha}(t-t_0)+\overline{M}\int_{t_0}^{t}\Big(E\sup_{t_0\leq r\leq s}|x^n(r)-x(r)|^2\Big)^{\alpha}ds, \tag{3.9}$$

where $\overline{M} = 2^{1+\alpha}\overline{K}(T - t_0 + 4)$. Now, applying the Lemma 2.2., we can obtain the required inequality (3.8) as follows.

$$E\Big(\sup_{t_0\le s\le t}|x^n(s)-x(s)|^2\Big) \le \Big(C_3^{1-\alpha} + (1-\alpha)\overline{M}(t-t_0)\Big)^{1/(1-\alpha)},$$

where $C_3 = \overline{M}\big(C_2[M(t-t_0)]^{n-1}/(n-1)!\big)^{\alpha}(t-t_0)$. The proof is complete. ∎

Theorem 3.5. *Let the assumptions of Theorem 3.3 hold. Let $x(t)$ be the unique solution of equation (3.1) with initial data (3.2) and $x^n(t)$ be the Picard iterations defined by (3.6). Then, for all $n \le 1$,*

$$E\Big(\sup_{t_0\le t\le T}|x^n(t)-x(t)|^2\Big) \le C_3\Big(1 - (\alpha-1)C_3^{1-\alpha}(t-t_0)\Big)^{1/(1-\alpha)}, \tag{3.10}$$

where $C_3 = \overline{M}\big(C_2[M(t-t_0)]^{n-1}/(n-1)!\big)^{\alpha}(t-t_0)$ and $\overline{M} = 2^{1+\alpha}\overline{K}(T-t_0+4)$.

Proof. The following inequality that applies Lemma 2.3 to the inequality (3.9) that appears in the process of proving Theorem 3.4 can be obtained.

$$E\Big(\sup_{t_0\le s\le t}|x^n(s)-x(s)|^2\Big) \le C_3\Big(1 - (\alpha-1)C_3^{1-\alpha}(t-t_0)\Big)^{1/(1-\alpha)},$$

where $C_3 = \overline{M}\big(C_2[M(t-t_0)]^{n-1}/(n-1)!\big)^{\alpha}(t-t_0)$. The proof is complete. ∎

4. CONCLUSION

Using the partial weighted Lipschitz condition and weakened linear growth condition, in the Theorem 3.3, we have shown that the Picard's approximate solution $x^n(t)$ converge to the unique solution $x(t)$ of equation (3.1) for rational number α. In practice, given the error $\epsilon > 0$, one can determine for the left-hand side of (3.7) to be less than ϵ by the Picard iteration (3.6). According to Theorem 3.3, we have

$$E\Big(\sup_{t_0\le t\le T}|x^{n+1}(t)-x^n(t)|^2\Big) < \epsilon. \tag{4.1}$$

On the other hand, one sees from (4.1) that for every t, $x^n(t)$ is Cauchy sequence in L^2 as well. Hence we also have that $x^n(t)$ is closed enough to the accurate solution $x(t)$ in L^2. Furthermore, the two conditions (3.3) and (3.4) we chose in Theorem 3.3 can be said to be meaningful in providing some advantages on the concave curve over the two conditions (1.5) and (1.6) used in the previous existence theorem study.

In the Theorem 3.4 and 3.5, using the partial weighted Lipschitz condition and weakened linear growth condition, we have shown that a dynamic movement relationship between the approximate solution $x^n(t)$ and the unique solution $x(t)$ of equation (3.1). In this two theorem show that one can use the Picard iteration procedure to obtain the approximate solution of equation (3.1), and (3.8) and (3.10) give the estimate for the error of the approximation. We would like to leave it to the next discussion as to which of the errors in (3.8) and (3.10) are effective.

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