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STUDY OF COMPLEX OSCILLATION OF SOLUTIONS TO HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS OF FINITE [P,Q]-φ ORDER

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ABSTRACT. In the present paper, we investigate the growth of meromorphic solutions to higher order homogeneous and nonhomogeneous linear differential equations with meromorphic coefficients of finite $[p,q]$-φ order. We obtain some results about the $[p,q]$-φ order and the $[p,q]$-φ convergence exponent of solutions for such equations.

Key words and phrases: Linear differential equation; Meromorphic function; $[p,q]-\varphi$ order; $[p,q]-\varphi$ exponent of convergence of zeros.

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1. INTRODUCTION AND MAIN RESULTS

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [4], [8]). To define the iterated order and the $[p, q]-$ order of meromorphic functions in the complex plane, we use the same notations as in (see [1], [12],[13],[14],[16],[19]).
As far as we know, in [17] Shen, Tu and Xu firstly introduced the concept of $[p, q] - \varphi$ order of meromorphic functions in the complex plane to investigate the growth and zeros of second order linear differential equations.

Definition 1.1. ([17]) Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function, and p, q be positive integers that satisfy $p \geqslant q \geqslant 1$. Then the $[p, q] - \varphi$ order and the lower $[p, q] - \varphi$ order of a meromorphic function f are respectively defined by

$$\rho_{[p,q]}(f, \varphi) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)},$$

$$\mu_{[p,q]}(f, \varphi) = \liminf_{r\to+\infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)}.$$

Definition 1.2. ([17]) Let f be a meromorphic function. Then, the $[p, q] - \varphi$ exponent of convergence of zero-sequence (distinct zero-sequence) of f are respectively defined by

$$\lambda_{[p,q]}(f, \varphi) = \limsup_{r\to+\infty} \frac{\log_p n\left(r, \frac{1}{f}\right)}{\log_q \varphi(r)}$$

and

$$\overline{\lambda}_{[p,q]}(f, \varphi) = \limsup_{r\to+\infty} \frac{\log_p \overline{n}\left(r, \frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Remark 1.1. If $\varphi(r) = r$ in the Definitions 1.1-1.2 , then we will get the standard definitions of the $[p, q]$-order and the $[p, q]$-exponent of convergence.

Remark 1.2. ([17]) Throughout this paper, we assume that $\varphi : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions:
(i) $\lim_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$;
(ii) $\lim_{r\to+\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = 1$ for some $\alpha_1 > 1$.

Proposition 1.1. *([3]) Suppose that $\varphi(r)$ satisfies the condition $(i) - (ii)$ in Remark 1.2 :*
a) If f is a meromorphic function, then

$$\lambda_{[p,q]}(f, \varphi) = \limsup_{r\to+\infty} \frac{\log_p n\left(r, \frac{1}{f}\right)}{\log_q \varphi(r)} = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q \varphi(r)},$$

$$\overline{\lambda}_{[p,q]}(f, \varphi) = \limsup_{r\to+\infty} \frac{\log_p \overline{n}\left(r, \frac{1}{f}\right)}{\log_q \varphi(r)} = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q \varphi(r)}.$$

b) If f is an entire function, then

$$\rho_{[p,q]}(f, \varphi) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r, f)}{\log_q \varphi(r)},$$

$$\mu_{[p,q]}(f,\varphi)=\liminf_{r\to+\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)}=\liminf_{r\to+\infty}\frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)}.$$

In [15], Liu, Tu and Zhang studied the growth and zeros of solutions of equations

$$f^{(k)}+\sum_{j=1}^{k-1}A_j f^{(j)}+A_0 f=0 \tag{1.1}$$

and

$$f^{(k)}+\sum_{j=1}^{k-1}A_j f^{(j)}+A_0 f=F, \tag{1.2}$$

where $A_0(z)\not\equiv 0,\ A_1(z),...,A_{k-1}(z)$ and $F(z)\not\equiv 0$ are entire functions of $[p,q]-\varphi$ order and they obtained the following results.

Theorem 1.2. *([15]) Let $A_j(z)\ (j=0,1,...,k-1)$ be entire functions satisfying*

$$\max\left\{\rho_{[p,q]}(A_j,\varphi),\ j=1,2,...,k-1\right\}<\rho_{[p,q]}(A_0,\varphi)<\infty.$$

Then every solution $f\not\equiv 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f,\varphi)=\rho_{[p,q]}(A_0,\varphi)$.

In the same paper they obtained the following results in the case of the non-homogeneous equation (1.2).

Theorem 1.3. *([15]) Let $A_j(z)\ (j=0,1,...,k-1)$ and $F(z)\not\equiv 0$ be entire functions, and let $f(z)$ be a solution of (1.2) satisfying*

$$\max\left\{\rho_{[p,q]}(A_j,\varphi),\rho_{[p,q]}(F,\varphi),\ j=0,1,...,k-1\right\}<\rho_{[p,q]}(f,\varphi).$$

Then $\overline{\lambda}_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(f,\varphi)=\rho_{[p,q]}(f,\varphi)$.

Theorem 1.4. *([15]) Let $A_j(z)\ (j=0,1,...,k-1)$ and $F(z)\not\equiv 0$ be entire functions satisfying*

$$\max\left\{\rho_{[p,q]}(A_j,\varphi),\rho_{[p+1,q]}(F,\varphi),\ j=1,2,...,k-1\right\}<\rho_{[p,q]}(A_0,\varphi).$$

Then every solution f of equation (1.2) satisfies $\overline{\lambda}_{[p+1,q]}(f,\varphi)=\lambda_{[p+1,q]}(f,\varphi)=\rho_{[p+1,q]}(f,\varphi)=\rho_{[p,q]}(A_0,\varphi)$, with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0,\varphi)<\rho_{[p,q]}(A_0,\varphi)$.

After this, Saidani and Belaïdi studied some properties of solutions of the higher order linear differential equations

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.3}$$

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z), \tag{1.4}$$

and they obtained the following results.

Theorem 1.5. *([16]) Let $H\subset(1,+\infty)$ be a set with a positive upper logarithmic density (or $m_l(H)=+\infty$) and let $A_j(z)\ (j=0,1,...,k)$ with $A_k(z)\,(\not\equiv 0)$ be meromorphic functions with finite $[p,q]$-order. If there exist a positive constant $\sigma>0$ and an integer $s,\ 0\leqslant s\leqslant k$, such that for sufficiently small $\varepsilon>0$, we have $|A_s(z)|\geqslant\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q r\right\}$ as $|z|=r\in H$, $r\to+\infty$ and $\rho=\max\left\{\rho_{[p,q]}(A_j)\ (j\neq s)\right\}<\sigma$, then every non-transcendental meromorphic solution $f\not\equiv 0$ of (1.3) is a polynomial with $\deg f\leqslant s-1$ and every transcendental meromorphic solution f of (1.3) with $\lambda_{[p,q]}\left(\frac{1}{f}\right)<\mu_{[p,q]}(f)$ satisfies*

$$\rho_{[p,q]}(f)=\mu_{[p,q]}(f)=+\infty,\ \sigma\leqslant\rho_{[p+1,q]}(f)\leqslant\rho_{[p,q]}(A_s).$$

Theorem 1.6. *([16]) Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or $m_l(H) = +\infty$), and let $A_j(z)$ ($j = 0, 1, ..., k$) and $F(z) \not\equiv 0$ be meromorphic functions with finite $[p, q]$-order. If there exist a positive constant $\sigma > 0$ and an integer s, $0 \leqslant s \leqslant k$, such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geqslant \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\}$ as $|z| = r \in H$, $r \to +\infty$ and $\max\left\{\rho_{[p,q]}(A_j)\ (j \neq s),\ \rho_{[p,q]}(F)\right\} < \sigma$, then every non-transcendental meromorphic solution f of (1.4) is a polynomial with $\deg f \leq s - 1$ and every transcendental meromorphic solution f of (1.4) with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \min\left\{\sigma, \mu_{[p,q]}(f)\right\}$ satisfies*

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f) = \mu_{[p,q]}(f) = +\infty$$

and

$$\sigma \leqslant \overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) \leqslant \rho_{[p,q]}(A_s).$$

A natural question which arises: How about the growth of meromorphic solutions of equations (1.3) and (1.4) with meromorphic coefficients of finite $[p, q] - \varphi$ order when the dominant coefficient is an arbitrary coefficient A_s?
The main purpose of this paper is to give an answer to the above question. We now present our main results, so for the homogeneous linear differential equation (1.3), we obtain the following results.

Theorem 1.7. *Let G be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in G\} > 0$, p, q be integers such that $p \geqslant q \geqslant 1$ and let $A_j(z)$ $(j = 0, 1, ..., k)$ such that $A_k \not\equiv 0$ be meromorphic functions with finite $[p, q] - \varphi$ order. Suppose there exist a positive constant $\sigma > 0$ and an integer s, $0 \leqslant s \leqslant k$ such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geqslant \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\}$ as $z \in G$, $|z| = r \to +\infty$ and $\rho = \max\left\{\rho_{[p,q]}(A_j, \varphi)\ (j \neq s)\right\} < \sigma$. Then every non-transcendental meromorphic solution $f \not\equiv 0$ of (1.3) is a polynomial with $\deg f \leqslant s - 1$ and every transcendental meromorphic solution f of (1.3) with $\lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \mu_{[p,q]}(f, \varphi)$ satisfies*

$$\rho_{[p,q]}(f, \varphi) = \mu_{[p,q]}(f, \varphi) = +\infty, \sigma \leqslant \rho_{[p+1,q]}(f, \varphi) \leqslant \rho_{[p,q]}(A_s, \varphi).$$

Corollary 1.8. *Under the hypotheses of Theorem 1.7, suppose further that ψ is a transcendental meromorphic function satisfying $\rho_{[p+1,q]}(\psi, \varphi) < \sigma$. Then, every transcendental meromorphic solution f of equation (1.3) with $\lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \mu_{[p,q]}(f, \varphi)$ satisfies*

$$\sigma \leqslant \overline{\lambda}_{[p+1,q]}(f - \psi, \varphi) = \lambda_{[p+1,q]}(f - \psi, \varphi)$$
$$= \rho_{[p+1,q]}(f - \psi, \varphi) = \rho_{[p+1,q]}(f, \varphi) \leqslant \rho_{[p,q]}(A_s, \varphi).$$

Considering nonhomogeneous linear differential equation (1.4), we obtain the following results.

Theorem 1.9. *Let G be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in G\} > 0$, and let $A_j(z)$ $(j = 0, 1, ..., k)$ and $F(z) \not\equiv 0$ be meromorphic functions with finite $[p, q]$-φ order. If there exist a positive constant $\sigma > 0$ and an integer s, $0 \leqslant s \leqslant k$, such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geqslant \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\}$ as $z \in G$, $|z| = r \to +\infty$ and $\rho_1 = \max\left\{\rho_{[p,q]}(A_j, \varphi)\ (j \neq s),\ \rho_{[p,q]}(F, \varphi)\right\} < \sigma$, then every non-transcendental meromorphic solution f of (1.4) is a polynomial with $\deg f \leqslant s - 1$ and every transcendental meromorphic solution f of (1.4) with $\lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \min\left\{\sigma, \mu_{[p,q]}(f, \varphi)\right\}$ satisfies*

$$\overline{\lambda}_{[p,q]}(f, \varphi) = \lambda_{[p,q]}(f, \varphi) = \rho_{[p,q]}(f, \varphi) = \mu_{[p,q]}(f, \varphi) = +\infty$$

and

$$\sigma \leqslant \overline{\lambda}_{[p+1,q]}\left(f,\varphi\right) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}\left(f,\varphi\right) \leqslant \rho_{[p,q]}\left(A_s,\varphi\right).$$

Corollary 1.10. *Let $A_j\left(z\right)$ $(j=0,1,...,k)$, $F\left(z\right)$, G satisfy all the hypotheses of Theorem 1.9, and let ψ be a transcendental meromorphic function satisfying $\rho_{[p+1,q]}\left(\psi,\varphi\right) < \sigma$. Then, every transcendental meromorphic solution f with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \min\{\sigma, \mu_{[p,q]}\left(f,\varphi\right)\}$ of equation (1.4) satisfies $\sigma \leqslant \overline{\lambda}_{[p+1,q]}\left(f-\psi,\varphi\right) = \lambda_{[p+1,q]}\left(f-\psi,\varphi\right) = \rho_{[p+1,q]}\left(f-\psi,\varphi\right) = \rho_{[p+1,q]}\left(f,\varphi\right) \leqslant \rho_{[p,q]}\left(A_s,\varphi\right)$.*

2. AUXILIARY LEMMAS

In order to prove our theorems, we need the following proposition and lemmas. The Lebesgue linear measure of a set $E \subset [0,+\infty)$ is $m\left(E\right) = \int_E dt$, and the logarithmic measure of a set $F \subset [1,+\infty)$ is $m_l\left(F\right) = \int_F \frac{dt}{t}$. The upper density of $E \subset [0,+\infty)$ is given by

$$\overline{dens}\left(E\right) = \limsup_{r\to+\infty} \frac{m\left(E\cap[0,r]\right)}{r}$$

and the upper logarithmic density of the set $F \subset [1,+\infty)$ is defined by

$$\overline{\log dens}\left(F\right) = \limsup_{r\longrightarrow+\infty} \frac{m_l\left(F\cap[1,r]\right)}{\log r}.$$

Proposition 2.1. *([1]) For all $H \subset (1,+\infty)$ the following statements hold:*
(i) If $m_l\left(H\right) = +\infty$, then $m\left(H\right) = +\infty$;
(ii) If $\overline{dens}\left(H\right) > 0$, then $m\left(H\right) = +\infty$;
(iii) If $\overline{\log dens}\left(H\right) > 0$, then $m_l\left(H\right) = +\infty$.

Lemma 2.2. *([5]) Let f be a transcendental meromorphic function in the plane, and let $\alpha > 1$ be a given constant. Then, there exist a set $E_1 \subset (1,+\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on α and (i,j) $((i,j)$ positive integers with $i > j))$ such that for all z with $|z| = r \notin [0,1]\cup E_1$, we have*

$$\left|\frac{f^{(i)}(z)}{f^{(j)}(z)}\right| \leqslant B\left(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)\log T(\alpha r,f)\right)^{i-j}.$$

Lemma 2.3. *(Wiman-Valiron, [7], [18]) Let f be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f\left(z\right)| = M\left(r,f\right)$. Then the estimation*

$$\frac{f^{(j)}\left(z\right)}{f\left(z\right)} = \left(\frac{\nu_f\left(r\right)}{z}\right)^j \left(1+o\left(1\right)\right) \quad (j \geqslant 1 \textit{ is an integer})$$

holds for all $|z|$ outside a set E_2 of r of finite logarithmic measure, where $\nu_f\left(r\right)$ is the central index of f.

Lemma 2.4. *([17]) Let p,q be positive integers that satisfy $p \geqslant q \geqslant 1$. Let f be an entire function of $[p,q]$-φ order and let $\nu_f\left(r\right)$ be the central index of f. Then*

$$\limsup_{r\to+\infty} \frac{\log_p \nu_f\left(r\right)}{\log_q \varphi\left(r\right)} = \rho_{[p,q]}\left(f,\varphi\right), \ \liminf_{r\to+\infty} \frac{\log_p \nu_f\left(r\right)}{\log_q \varphi\left(r\right)} = \mu_{[p,q]}\left(f,\varphi\right).$$

Lemma 2.5. *([3]) Let f and g be non-constant meromorphic functions of $[p,q]-\varphi$ order. Then we have*

$$\rho_{[p,q]}(f+g,\varphi) \leqslant \max\left\{\rho_{[p,q]}(f,\varphi), \rho_{[p,q]}(g,\varphi)\right\}$$

and

$$\rho_{[p,q]}(fg,\varphi) \leqslant \max\left\{\rho_{[p,q]}(f,\varphi), \rho_{[p,q]}(g,\varphi)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(g,\varphi)$, then we obtain

$$\rho_{[p,q]}(f+g,\varphi) = \rho_{[p,q]}(fg,\varphi) = \rho_{[p,q]}(f,\varphi).$$

Lemma 2.6. *([3]) Let $p \geqslant q \geqslant 1$ be integers, and let f and g be non-constant meromorphic functions with $\rho_{[p,q]}(f,\varphi)$ as $[p,q]-\varphi$ order and $\mu_{(p,q)}(g,\varphi)$ as lower $[p,q]-\varphi$ order. Then we have*

$$\mu_{[p,q]}(f+g,\varphi) \leqslant \max\left\{\rho_{[p,q]}(f,\varphi), \mu_{[p,q]}(g,\varphi)\right\}$$

and

$$\mu_{[p,q]}(fg,\varphi) \leqslant \max\left\{\rho_{[p,q]}(f,\varphi), \mu_{[p,q]}(g,\varphi)\right\}.$$

Furthermore, if $\mu_{[p,q]}(g,\varphi) > \rho_{[p,q]}(f,\varphi)$, then we obtain

$$\mu_{[p,q]}(f+g,\varphi) = \mu_{[p,q]}(fg,\varphi) = \mu_{[p,q]}(g,\varphi).$$

By using Lemma 3.6 in ([2]) and mathematical induction, we easily obtain the following lemma.

Lemma 2.7. *Let $f(z)$ be a meromorphic function of $[p,q]-\varphi$ order. Then $\rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f^{(k)},\varphi), \quad (k \in \mathbb{N})$.*

Lemma 2.8. *([6]) Let $\varphi : [0,+\infty) \to \mathbb{R}$ and $\psi : [0,+\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $\varphi(r) \leqslant \psi(r)$ for all $r \notin (E_3 \cup [0,1])$, where E_3 is a set of finite logarithmic measure. Let $\nu > 1$ be a given constant. Then, there exists an $r_1 = r_1(\nu) > 0$ such that $\varphi(r) \leqslant \psi(\nu r)$ for all $r > r_1$.*

Lemma 2.9. *([8]) Let f be a transcendental meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log(rT(r,f))\right),$$

possibly outside a set $E_4 \subset (0,+\infty)$ with a finite linear measure, and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log r\right).$$

Lemma 2.10. *([3]) Let f_1, f_2 be meromorphic functions of $[p,q]-\varphi$ order satisfying $\rho_{[p,q]}(f_1,\varphi) > \rho_{[p,q]}(f_2,\varphi)$, where φ only satisfies $\lim\limits_{r\to+\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = 1$ for some $\alpha_1 > 1$. Then there exists a set $E_5 \subset [1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_5$, we have*

$$\lim_{r\to+\infty} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$

Lemma 2.11. *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \leqslant \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) \leqslant +\infty$ and*

$\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu$. Then there exists a set $E_6 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1] \cup E_6)$ and $|g(z)| = M(r,g)$, we have

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)),\ n \in \mathbb{N},$$

where $\nu_g(r)$ denotes be the central index of g.

Proof. We use the mathematical induction to obtain the following expression

$$f^{(n)} = \frac{g^{(n)}}{d} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{d} \sum_{(j_1 \cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \times \cdots \times \left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.1}$$

where $C_{jj_1\cdots j_n}$ are constants and $j + j_1 + 2j_2 + \cdots + nj_n = n$. Then

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{g} \sum_{(j_1 \cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \times \cdots \times \left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.2}$$

By Lemma 2.3, there exists a set $E_2 \subset [1,+\infty)$ with finite logarithmic measure such that for a point z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r,g)$, we get

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1+o(1)) \quad (j = 1,2,...,n), \tag{2.3}$$

where $\nu_g(r)$ is the central index of g. By replacing (2.3) into (2.2), we obtain

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n [(1+o(1))$$

$$+\sum_{j=0}^{n-1} \left(\frac{\nu_g(r)}{z}\right)^{j-n} (1+o(1)) \sum_{(j_1 \cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \times \cdots \times \left(\frac{d^{(n)}}{d}\right)^{j_n} \Bigg]. \tag{2.4}$$

From the fact that $\rho_{[p,q]}(d,\varphi) = \beta < \mu$, for any given ε $(0 < 2\varepsilon < \mu - \beta)$ and for sufficiently large r, we have

$$T(r,d) \leqslant \exp_p\left\{\left(\beta + \frac{\varepsilon}{2}\right)\log_q \varphi(r)\right\}.$$

By Lemma 2.2 for some α_1 $(1 < \alpha_1 < \alpha)$ with α is a given constant, there exist a set $E_1 \subset (1,+\infty)$ with $m_l(E_1) < \infty$ and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{d^{(m)}(z)}{d(z)}\right| \leqslant B\left[T(\alpha_1 r, d)\right]^{m+1}$$

$$\leqslant B\left[\exp_p\left\{\left(\beta + \frac{\varepsilon}{2}\right)\log_q \varphi(\alpha_1 r)\right\}\right]^{m+1}. \tag{2.5}$$

By (2.5) and Remark 1.2 ($\lim\limits_{r\to+\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = 1$ $(1 < \alpha_1 < \alpha)$), we obtain

$$\left|\frac{d^{(m)}(z)}{d(z)}\right| \leqslant B\left[\exp_p\left\{\left(\beta + \frac{\varepsilon}{2}\right)\frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)}.\log_q \varphi(r)\right\}\right]^{m+1}$$

$$\leqslant \exp_p\left\{(\beta + \varepsilon)\log_q \varphi(r)\right\}^m,\ m = 1,2,...,n. \tag{2.6}$$

By using Lemma 2.4 and $\mu_{[p,q]}(g,\varphi)=\mu_{[p,q]}(f,\varphi)=\mu$, we have

$$\nu_g(r)>\exp_p\left\{(\mu-\varepsilon)\log_q\varphi(r)\right\}$$

for sufficiently large r. Then, since $j_1+2j_2+\cdots+nj_n=n-j$, we get

$$\left|\left(\frac{\nu_g(r)}{z}\right)^{j-n}\left(\frac{d'}{d}\right)^{j_1}\times\cdots\times\left(\frac{d^{(n)}}{d}\right)^{j_n}\right|\leqslant\left[\frac{\exp_p\left\{(\mu-\varepsilon)\log_q\varphi(r)\right\}}{r}\right]^{j-n}$$

$$\times\left[\exp_p\left\{(\beta+\varepsilon)\log_q\varphi(r)\right\}\right]^{n-j}$$

$$=\left[\frac{r\exp_p\left\{(\beta+\varepsilon)\log_q\varphi(r)\right\}}{\exp_p\left\{(\mu-\varepsilon)\log_q\varphi(r)\right\}}\right]^{n-j}\to 0 \tag{2.7}$$

as $r\to+\infty$, where $|z|=r\notin[0,1]\cup E_6$, $E_6=E_1\cup E_2$ and $|g(z)|=M(r,g)$. From (2.4) and (2.7), we obtain our assertion. ∎

Lemma 2.12. *Let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g,\varphi)=\mu_{[p,q]}(f,\varphi)=\mu\leqslant\rho_{[p,q]}(f,\varphi)=\rho_{[p,q]}(g,\varphi)\leqslant+\infty$ and $\lambda_{[p,q]}(d,\varphi)=\rho_{[p,q]}(d,\varphi)=\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right)<\mu$. Then, there exists a set $E_7\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin([0,1]\cup E_7)$ and $|g(z)|=M(r,g)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right|\leqslant r^{2s},\quad (s\in\mathbb{N}).$$

Proof. By Lemma 2.11, there exists a set E_6 of finite logarithmic measure such that the estimation

$$\frac{f^{(s)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^s(1+o(1))\quad (s\geqslant 1 \text{ is an integer}) \tag{2.8}$$

is verified for all $|z|=r\notin[0,1]\cup E_6$ and $|g(z)|=M(r,g)$, where $\nu_g(r)$ is the central index of g. Then again, from Lemma 2.4, for any given ε $(0<\varepsilon<1)$, there exists $R>1$ such that for all $r>R$, we have

$$\nu_g(r)>\exp_p\left\{(\mu-\varepsilon)\log_q(\varphi(r))\right\}. \tag{2.9}$$

If $\mu=+\infty$, then we can replace $\mu-\varepsilon$ by a large enough real number M. Let $E_7=[1,R]\cup E_6$. Then $m_l(E_7)<+\infty$. Finally, by (2.8) and (2.9), we get

$$\left|\frac{f(z)}{f^{(s)}(z)}\right|=\left|\frac{z}{\nu_g(r)}\right|^s\frac{1}{|1+o(1)|}\leqslant\frac{r^s}{\left(\exp_p\left\{(\mu-\varepsilon)\log_q(\varphi(r))\right\}\right)^s}\leqslant r^{2s},$$

where $|z|=r\notin[0,1]\cup E_7, r\to+\infty$ and $|g(z)|=M(r,g)$. ∎

Lemma 2.13. *Let f be an entire function such that $\rho_{[p,q]}(f,\varphi)<+\infty$. Then, there exist entire functions $h(z)$ and $L(z)$ such that*

$$f(z)=h(z)e^{L(z)},$$

$$\rho_{[p,q]}(f,\varphi)=\max\left\{\rho_{[p,q]}(h,\varphi),\rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}$$

and

$$\rho_{[p,q]}(h,\varphi)=\limsup_{r\to+\infty}\frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q\varphi(r)}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$|h(z)| \geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(h,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\right\}(r \notin E_8),$$

where $E_8 \subset (1,+\infty)$ is a set of r of finite linear measure.

Proof. By using Theorem 12.4 in ([10]) and Theorem 2.2 in ([11]), f can be represented by

$$f(z) = h(z)e^{L(z)},$$

with

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(h,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}.$$

On the other hand, by a similar proof of Proposition 6.1 in ([9]), for any given $\varepsilon > 0$, we obtain

$$|h(z)| \geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(h,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\right\}(r \notin E_8),$$

where $E_8 \subset (1,+\infty)$ is a set of r of finite linear measure with

$$\rho_{[p,q]}(h,\varphi) = \limsup_{r\to+\infty} \frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

∎

Lemma 2.14. *Suppose that f is a meromorphic function such that $\rho_{[p,q]}(f,\varphi) < +\infty$. Then, there exist entire functions $h_1(z)$, $h_2(z)$ and $L(z)$ such that*

$$f(z) = \frac{h_1(z)e^{L(z)}}{h_2(z)} \tag{2.10}$$

and

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(h_1,\varphi), \rho_{[p,q]}(h_2,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}. \tag{2.11}$$

Moreover, for any given $\varepsilon > 0$, we have

$$\exp\left\{-\exp_p\left\{(\rho_{[p,q]}(f,\varphi)+\varepsilon)\log_q \varphi(r)\right\}\right\} \leqslant |f(z)|$$

$$\leqslant \exp_{p+1}\left\{(\rho_{[p,q]}(f,\varphi)+\varepsilon)\log_q \varphi(r)\right\} \quad (r \notin E_9), \tag{2.12}$$

where $E_9 \subset (1,+\infty)$ is a set of r of finite linear measure.

Proof. By Hadamard factorization theorem, f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \leqslant \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) < +\infty$$

and

$$\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu.$$

By using Lemma 2.13, we can find entire functions $h(z)$ and $L(z)$ such that

$$g(z) = h(z)e^{L(z)},$$

$$\rho_{[p,q]}(g,\varphi) = \max\left\{\rho_{[p,q]}(h,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}.$$

Then, there exist entire functions $h(z)$, $L(z)$ and $d(z)$ such that

$$f(z) = \frac{h(z)e^{L(z)}}{d(z)}$$

and

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(h,\varphi), \rho_{[p,q]}(d,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}.$$

Therefore (2.10) and (2.11) hold. Set $f(z) = \frac{h_1(z)e^{L(z)}}{h_2(z)}$, where $h_1(z), h_2(z)$ are the canonical products formed with the zeros and poles of f respectively. By using the definition of $[p,q]-\varphi$ order, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$|h_1(z)| \leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(h_1,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}, \tag{2.13}$$

$$|h_2(z)| \leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(h_2,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}. \tag{2.14}$$

From $\max\left\{\rho_{[p,q]}(h_1,\varphi), \rho_{[p,q]}(h_2,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\} = \rho_{[p,q]}(f,\varphi)$, we get

$$|h_1(z)| \leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}, \tag{2.15}$$

$$|h_2(z)| \leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}, \tag{2.16}$$

$$|e^{L(z)}| \leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}. \tag{2.17}$$

Through the use of Lemma 2.13, there exists a set $E_9 \subset (1,+\infty)$ of r of finite linear measure such that for any given $\varepsilon > 0$, we have

$$|h_1(z)| \geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(h_1,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}$$

$$\geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}, (r \notin E_9), \tag{2.18}$$

$$|h_2(z)| \geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(h_2,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}$$

$$\geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}, (r \notin E_9). \tag{2.19}$$

By using (2.15), (2.17) and (2.19), for any given $\varepsilon > 0$ and sufficiently large $r \notin E_9$, we have

$$|f(z)| = \frac{|h_1(z)||e^{L(z)}|}{|h_2(z)|}$$

$$\leqslant \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}}{\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}}$$

$$\leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\}. \tag{2.20}$$

On the other hand, we know $\rho_{[p-1,q]}(L,\varphi) = \rho_{[p,q]}\left(e^L,\varphi\right) \leqslant \rho_{[p,q]}(f,\varphi)$ and $|e^{L(z)}| \geqslant e^{-|L(z)|}$. From the definition of $[p,q]-\varphi$ order, we get

$$\begin{aligned} |L(z)| &\leqslant M(r,L) \leqslant \exp_p\left\{\left(\rho_{[p-1,q]}(L,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\} \\ &\leqslant \exp_p\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}. \end{aligned}$$

Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$\left|e^{L(z)}\right| \geqslant e^{-|L(z)|} \geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi) + \frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}. \tag{2.21}$$

By making use of (2.16), (2.18) and (2.21), for any given $\varepsilon > 0$ and sufficiently large $r \notin E_9$, we can easily obtain

$$|f(z)| = \frac{|h_1(z)||e^{L(z)}|}{|h_2(z)|}$$

$$\begin{aligned}
&\geqslant \frac{\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)+\frac{\varepsilon}{3}\right)\log_q\varphi\left(r\right)\right\}\right\}}{\exp_{p+1}\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)+\frac{\varepsilon}{3}\right)\log_q\varphi\left(r\right)\right\}}\\
&\times\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)+\frac{\varepsilon}{3}\right)\log_q\varphi\left(r\right)\right\}\right\}\\
&= \exp\left\{-3\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)+\frac{\varepsilon}{3}\right)\log_q\varphi\left(r\right)\right\}\right\}\\
&\geqslant \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)+\varepsilon\right)\log_q\varphi\left(r\right)\right\}\right\}.
\end{aligned}$$

Finally Lemma 2.14 is proved. ∎

Lemma 2.15. *Under the assumptions of Theorem 1.7 or Theorem 1.9, we have* $\rho_{[p,q]}\left(A_s,\varphi\right)=\delta\geqslant\sigma$.

Proof. By using the proof by contradiction, we assume that $\rho_{[p,q]}\left(A_s,\varphi\right)=\delta<\sigma$. From the hypotheses of Theorems 1.7 or 1.9, there exist a set G with $\overline{\log dens}\{|z|:z\in G\}>0$ and a positive constant $\sigma>0$ such that for sufficiently small $\varepsilon>0$, we have

$$|A_s\left(z\right)|\geqslant\exp_{p+1}\left\{\left(\sigma-\varepsilon\right)\log_q\left(\varphi\left(r\right)\right)\right\}, \tag{2.22}$$

as $z\in G$, $|z|=r\to+\infty$. By the definition of $[p,q]-\varphi$ order, for any given $\varepsilon\left(0<2\varepsilon<\sigma-\delta\right)$ and sufficiently large r, we have

$$|A_s\left(z\right)|\leqslant\exp_{p+1}\left\{\delta+\varepsilon\right)\log_q\varphi\left(r\right)\right\}. \tag{2.23}$$

Set $G_1=\{|z|:z\in G\}$, so by Proposition 2.1, we know that $m_l\left(G_1\right)=\infty$. Using (2.22) and (2.23), we obtain for $|z|=r\in G_1$, $r\to+\infty$

$$\exp_{p+1}\left\{\left(\sigma-\varepsilon\right)\log_q\left(\varphi\left(r\right)\right)\right\}\leqslant|A_s\left(z\right)|\leqslant\exp_{p+1}\left\{\left(\delta+\varepsilon\right)\log_q\varphi\left(r\right)\right\}$$

which is a contradiction with the fact that $0<2\varepsilon<\sigma-\delta$. Then $\rho_{[p,q]}\left(A_s,\varphi\right)=\delta\geqslant\sigma$. ∎

Lemma 2.16. *Let* $f\left(z\right)=\frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g\left(z\right)$, $d\left(z\right)$ *are entire functions. If* $0\leqslant\rho_{[p,q]}\left(d,\varphi\right)<\mu_{[p,q]}\left(f,\varphi\right)$, *then* $\mu_{[p,q]}\left(g,\varphi\right)=\mu_{[p,q]}\left(f,\varphi\right)$ *and* $\rho_{[p,q]}\left(g,\varphi\right)=\rho_{[p,q]}\left(f,\varphi\right)$. *Moreover, if* $\rho_{[p,q]}\left(f,\varphi\right)=+\infty$, *then* $\rho_{[p+1,q]}\left(g,\varphi\right)=\rho_{[p+1,q]}\left(f,\varphi\right)$.

Proof. **Case 1.** $\rho_{[p,q]}\left(f,\varphi\right)<+\infty$. Using the definition of the $[p,q]$-φ order, there exist an increasing sequence $\{r_n\}$, $(r_n\to+\infty)$ and a positive integer n_0 such that for all $n>n_0$ and for any given $\varepsilon\in\left(0,\frac{\rho_{[p,q]}(f,\varphi)-\rho_{[p,q]}(d,\varphi)}{2}\right)$ $\left(\text{as } 0\leqslant\rho_{[p,q]}\left(d,\varphi\right)<\mu_{[p,q]}\left(f,\varphi\right)\leqslant\rho_{[p,q]}\left(f,\varphi\right)\right)$, we have

$$T\left(r_n,d\right)\leqslant\exp_p\left\{\left(\rho_{[p,q]}\left(d,\varphi\right)+\varepsilon\right)\log_q\varphi\left(r_n\right)\right\}, \tag{2.24}$$

and

$$T\left(r_n,f\right)\geqslant\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)-\varepsilon\right)\log_q\varphi\left(r_n\right)\right\}. \tag{2.25}$$

Using the properties of the characteristic function, we get

$$T(r,f)\leqslant T(r,g)+T(r,d)+O(1). \tag{2.26}$$

By substituting (2.24) and (2.25) into (2.26), for all sufficiently large n, we obtain

$$\begin{aligned}
&\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)-\varepsilon\right)\log_q\varphi\left(r_n\right)\right\}\leqslant T(r_n,g)\\
&+\exp_p\left\{\left(\rho_{[p,q]}\left(d,\varphi\right)+\varepsilon\right)\log_q\varphi\left(r_n\right)\right\}+O(1).
\end{aligned} \tag{2.27}$$

Since $\varepsilon\in\left(0,\frac{\rho_{[p,q]}(f,\varphi)-\rho_{[p,q]}(d,\varphi)}{2}\right)$, then from (2.27), we obtain

$$\left(1-o\left(1\right)\right)\exp_p\left\{\left(\rho_{[p,q]}\left(f,\varphi\right)-\varepsilon\right)\log_q\varphi\left(r_n\right)\right\}\leqslant T(r_n,g)+O(1),$$

for all sufficiently large n. Then $\rho_{[p,q]}(f,\varphi) \leqslant \rho_{[p,q]}(g,\varphi)$. On the other hand, we have $T(r,g) \leqslant T(r,f)+T(r,d)$ and from $\rho_{[p,q]}(d,\varphi) < \rho_{[p,q]}(f,\varphi)$, we get $\rho_{[p,q]}(g,\varphi) \leqslant \rho_{[p,q]}(f,\varphi)$. Hence, $\rho_{[p,q]}(g,\varphi) = \rho_{[p,q]}(f,\varphi)$. Similarly, using the definition of lower $[p,q]$-φ order $\mu_{[p,q]}(f,\varphi)$ and $\mu_{[p,q]}(g,\varphi)$, we can prove $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi)$.

Case 2. $\mu_{[p,q]}(f,\varphi) = +\infty$. By $T(r,g) \leqslant T(r,f) + T(r,d)$ and Lemma 2.6, we have

$$\mu_{[p,q]}(g,\varphi) \leqslant \max\left\{\mu_{[p,q]}(f,\varphi), \rho_{[p,q]}(d,\varphi)\right\} = \mu_{[p,q]}(f,\varphi).$$

Now, we prove $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi)$. We suppose that $\mu_{[p,q]}(g,\varphi) < \mu_{[p,q]}(f,\varphi)$. Using the definition of the $[p,q]$-φ order and the lower $[p,q]$-φ order, there exist an increasing sequence $\{r_n\}$, ($r_n \to +\infty$) and a positive integer n_1 such that for all $n > n_1$ and for any given $\varepsilon > 0$

$$\begin{aligned} T(r_n,d) &\leqslant \exp_p\left\{\left(\rho_{[p,q]}(d,\varphi)+\varepsilon\right)\log_q \varphi(r_n)\right\}, \\ T(r_n,g) &\leqslant \exp_p\left\{\left(\mu_{[p,q]}(g,\varphi)+\varepsilon\right)\log_q \varphi(r_n)\right\}. \end{aligned}$$

From the fact that $T(r_n,f) \leqslant T(r_n,g) + T(r_n,d) + O(1)$, for all sufficiently large n, we obtain

$$T(r_n,f) \leqslant \exp_p\left\{\left(\mu_{[p,q]}(g,\varphi)+\varepsilon\right)\log_q \varphi(r_n)\right\}$$
$$+\exp_p\left\{\left(\rho_{[p,q]}(d,\varphi)+\varepsilon\right)\log_q \varphi(r_n)\right\} + O(1),$$

then $\mu_{[p,q]}(f,\varphi) \leqslant \max\left\{\mu_{[p,q]}(g,\varphi), \rho_{[p,q]}(d,\varphi)\right\}$ and this is a contradiction. Hence $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi)$. Similarly, we can prove $\rho_{[p,q]}(g,\varphi) = \rho_{[p,q]}(f,\varphi)$.

Case 3. $\mu_{[p,q]}(f,\varphi) < +\infty$ and $\rho_{[p,q]}(f,\varphi) = +\infty$. We can prove Case 3 by using the similar method we used to prove Cases 1 and 2.

As last, we will prove $\rho_{[p+1,q]}(g,\varphi) = \rho_{[p+1,q]}(f,\varphi)$. We assume that $\rho_{[p,q]}(f,\varphi) = +\infty$. Then, there exists an increasing sequence $\{r_n\}$, ($r_n \to +\infty$), such that

$$\rho_{[p+1,q]}(f,\varphi) = \lim_{n\to\infty}\frac{\log_{p+1} T(r_n,f)}{\log_q \varphi(r_n)}.$$

Using $\rho_{[p,q]}(d,\varphi) < \mu_{[p,q]}(f,\varphi)$ and the definitions of $[p,q]$-φ order and the lower $[p,q]$-φ order, we obtain

$$\lim_{n\to+\infty}\frac{T(r_n,d)}{T(r_n,f)} = 0,$$

then

$$T(r_n,d) = o(T(r_n,f))$$

as $n \to +\infty$. Therefore, by using $T(r_n,f) \leqslant T(r_n,g) + T(r_n,d) + O(1)$, there exists a positive integer n_2, such that for $n > n_2$

$$(1 - o(1))\, T(r_n,f) \leqslant T(r_n,g) + O(1)$$

which implies $\rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p+1,q]}(g,\varphi)$. By using the same arguments as in the proof of Case 1, from $T(r,g) \leqslant T(r,f) + T(r,d)$, we can find a positive integer $n > n_3$, such that for $n > n_3$, we have

$$T(r_n,g) \leqslant (1 + o(1))\, T(r_n,f) \leqslant 2T(r_n,f).$$

Then, $\rho_{[p+1,q]}(g,\varphi) \leqslant \rho_{[p+1,q]}(f,\varphi)$. Thus $\rho_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(g,\varphi)$. ∎

Lemma 2.17. *Let $A_j(z)$ $(j = 0,1,...,k)$, $A_k(z)$ $(\not\equiv 0)$, $F(z)$ $(\not\equiv 0)$ be meromorphic functions and let $f(z)$ be a meromorphic solution of* (1.4) *of infinite $[p,q]$-φ order satisfying the following condition*

$$b = \max\left\{\rho_{[p+1,q]}(F,\varphi),\ \rho_{[p+1,q]}(A_j,\varphi)\,(j = 0,1,...,k)\right\} < \rho_{[p+1,q]}(f,\varphi).$$

Then

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi).$$

Proof. Assume that $f(z)$ is a meromorphic solution of (1.4) that has infinite $[p,q]$-φ order. We can rewrite (1.4) as

$$\frac{1}{f}=\frac{1}{F}\left(A_k(z)\frac{f^{(k)}}{f}+A_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots++A_1(z)\frac{f'}{f}+A_0(z)\right). \tag{2.28}$$

By Lemma 2.9 and (2.28), for $|z|=r$ outside a set $E_4\subset(0,+\infty)$ of finite linear measure, we get

$$m\left(r,\frac{1}{f}\right)\leqslant m\left(r,\frac{1}{F}\right)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k}m(r,A_j)+O(1)$$

$$\leqslant m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}m(r,A_j)+O(\log rT(r,f)). \tag{2.29}$$

From (1.4), it is easy to see that if f has a zero at z_0 of order $m\,(m>k)$, and A_0, A_1, ..., A_k $(\not\equiv 0)$ are all analytic at z_0, then F must have a zero at z_0 of order at least $m-k$. Hence

$$n\left(r,\frac{1}{f}\right)\leqslant k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}n(r,A_j),$$

and

$$N\left(r,\frac{1}{f}\right)\leqslant k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}N(r,A_j). \tag{2.30}$$

Combining (2.29) with (2.30), for all sufficiently large $r\notin E_4$, we get

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)$$

$$\leqslant T(r,F)+\sum_{j=0}^{k}T(r,A_j)+k\overline{N}\left(r,\frac{1}{f}\right)+O(\log rT(r,f)). \tag{2.31}$$

For sufficiently large r, we have

$$O(\log rT(r,f))\leqslant\frac{1}{2}T(r,f). \tag{2.32}$$

From the definition of the $[p,q]-\varphi$ order, for any given $\varepsilon\left(0<2\varepsilon<\rho_{[p+1,q]}(f,\varphi)-b\right)$ and for sufficiently large r, we have

$$T(r,F)\leqslant\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(r)\right\}, \tag{2.33}$$

$$T(r,A_j)\leqslant\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(r)\right\},\ j=0,1,...,k. \tag{2.34}$$

By substituting (2.32), (2.33), (2.34) into (2.31), for $r\notin E_4$ sufficiently large, we obtain

$$T(r,f)\leqslant 2k\overline{N}\left(r,\frac{1}{f}\right)+2(k+2)\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(r)\right\}. \tag{2.35}$$

By using Lemma 2.8 and (2.35), for any given $\nu>1$ there exists a $r_1=r_1(\nu)$ and sufficiently large $r>r_1$, such that

$$T(r,f)\leqslant 2k\overline{N}\left(\nu r,\frac{1}{f}\right)+2(k+2)\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(\nu r)\right\} \tag{2.36}$$

which gives

$$\rho_{[p+1,q]}(f,\varphi) \leqslant \overline{\lambda}_{[p+1,q]}(f,\varphi)$$

and therefore

$$\rho_{[p+1,q]}(f,\varphi) \leqslant \overline{\lambda}_{[p+1,q]}(f,\varphi) \leqslant \lambda_{[p+1,q]}(f,\varphi).$$

Since by definition we have $\overline{\lambda}_{[p+1,q]}(f,\varphi) \leqslant \lambda_{[p+1,q]}(f,\varphi) \leqslant \rho_{[p+1,q]}(f,\varphi)$, then we obtain

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi).$$

∎

Lemma 2.18. *Let G be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in G\} > 0$, and let $A_j(z)$ $(j = 0, 1, ..., k)$ with $A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite $[p,q]$-φ order. If there exist a positive constant $\sigma > 0$ and an integer s, $0 \leqslant s \leqslant k$, such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geqslant \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\}$ as $z \in G$, $|z| = r \to +\infty$ and*

$$\max\left\{\rho_{[p,q]}(A_j,\varphi)\ (j \neq s),\ \rho_{[p,q]}(F,\varphi)\right\} < \sigma,$$

then every transcendental meromorphic solution f of equation (1.4) satisfies $\rho_{[p,q]}(f,\varphi) \geqslant \sigma$.

Proof. Suppose the contrary. Let f be a transcendental meromorphic solution of the equation (1.4) such that $\rho_{[p,q]}(f,\varphi) < \sigma$. From (1.4), we get

$$A_s = \frac{F}{f^{(s)}} - \sum_{\substack{j=0 \\ j\neq s}}^{k} A_j \frac{f^{(j)}}{f^{(s)}}. \tag{2.37}$$

From the hypotheses of Lemma 2.18, we have

$$\max\left\{\rho_{[p,q]}(A_j,\varphi)\ (j \neq s),\ \rho_{[p,q]}(F,\varphi)\right\} < \sigma.$$

Then by using the assumption $\rho_{[p,q]}(f,\varphi) < \sigma$ and Lemma 2.7, from (2.37) we get

$$\rho_2 = \rho_{[p,q]}(A_s,\varphi)$$

$$\leqslant \max\left\{\rho_{[p,q]}(A_j,\varphi)\ (j \neq s),\ \rho_{[p,q]}(F,\varphi),\ \rho_{[p,q]}(f,\varphi)\right\} < \sigma.$$

Then, for any given $\varepsilon(0 < 2\varepsilon < \sigma - \rho_2)$ and sufficiently large r, we have

$$|A_s(z)| \leq \exp_{p+1}\left\{(\rho_{(p,q)}(A_s,\varphi) + \varepsilon)\log_q \varphi(r)\right\} = \exp_{p+1}\left\{(\rho_2 + \varepsilon)\log_q \varphi(r)\right\}. \tag{2.38}$$

By the hypotheses of Lemma 2.18, we have

$$|A_s(z)| \geqslant \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\} \tag{2.39}$$

holds for all z satisfying $z \in G$, $|z| = r \to +\infty$. Set $G_2 = \{|z| : z \in G\}$, so $m_l(G_2) = \infty$. By combining (2.38) with (2.39), for all z satisfying $|z| = r \in G_2$, $r \to +\infty$, we obtain

$$\exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\} \leqslant \exp_{p+1}\left\{(\rho_2 + \varepsilon)\log_q \varphi(r)\right\},$$

hence

$$\sigma - \varepsilon < \rho_2 + \varepsilon$$

and this contradicts the fact that $0 < 2\varepsilon < \sigma - \rho_2$. Consequently, any transcendental meromorphic solution f of the equation (1.4) satisfies $\rho_{[p,q]}(f,\varphi) \geqslant \sigma$. ∎

Lemma 2.19. *Let $A_0, A_1, ..., A_k \not\equiv 0, F \not\equiv 0$ be finite $[p,q]$-φ order meromorphic functions. If f is a meromorphic solution of the equation (1.4) with $\rho_{[p,q]}(f,\varphi) = +\infty$ and $\rho_{[p+1,q]}(f,\varphi) = \rho < +\infty$, then*

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi) = \rho.$$

Proof. Assume that f is a meromorphic solution of (1.4) that has infinite $[p,q]$-φ order and $\rho_{[p+1,q]}(f,\varphi) = \rho < +\infty$. The equation (1.4) can be rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(A_k(z)\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right). \tag{2.40}$$

By Lemma 2.9 and (2.40), for $|z| = r$ outside a set E_4 of a finite linear measure, we get

$$m\left(r,\frac{1}{f}\right) \leqslant m\left(r,\frac{1}{F}\right) + \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k} m(r,A_j) + O(1)$$

$$\leqslant m\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k} m(r,A_j) + O(\log rT(r,f)). \tag{2.41}$$

On the other, from (1.4), if f has a zero at z_0 of order $\alpha\,(\alpha > k)$, and $A_0,\ A_1, ...,\ A_k$ are all analytic at z_0, then F must have a zero at z_0 of order at least $\alpha - k$. Then

$$n\left(r,\frac{1}{f}\right) \leqslant k\overline{n}\left(r,\frac{1}{f}\right) + n\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k} n(r,A_j)$$

and

$$N\left(r,\frac{1}{f}\right) \leqslant k\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k} N(r,A_j). \tag{2.42}$$

By (2.41) and (2.42), for all sufficiently large $r \notin E_4$, we get

$$T(r,f) = T\left(r,\frac{1}{f}\right) + O(1) \leqslant T(r,F) + \sum_{j=0}^{k} T(r,A_j)$$

$$+ k\overline{N}\left(r,\frac{1}{f}\right) + O(\log rT(r,f)). \tag{2.43}$$

From the hypotheses of Lemma 2.19, we have

$$\rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(F,\varphi), \rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(A_j,\varphi),\ j = 0,1,...,k.$$

Then by using Lemma 2.10, there exists a set $E_5 \subset [1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_5$, we have

$$\max\left\{\frac{T(r,F)}{T(r,f)}, \frac{T(r,A_j)}{T(r,f)},\ j = 0,1,...,k\right\} \to 0 \text{ for } r \to +\infty,$$

hence as $r \in E_5$, $r \to +\infty$

$$T(r,F) = o(T(r,f)), T(r,A_j) = o(T(r,f)), j = 0,1,...,k. \tag{2.44}$$

Since f is transcendental, then for sufficiently large r, we have

$$O(\log rT(r,f)) = o(T(r,f)). \tag{2.45}$$

Substituting (2.44) and (2.45) into (2.43) , for $r \in E_5 \setminus E_4$, we get

$$T(r,f) \leqslant k\overline{N}\left(r, \frac{1}{f}\right) + o\left(T(r,f)\right).$$

Hence

$$(1 - o(1))\left(T(r,f)\right) \leqslant k\overline{N}\left(r, \frac{1}{f}\right). \tag{2.46}$$

Then, by making use of Proposition 1.1, Lemma 2.8, Definition 1.1, Remark 1.2 and (2.46) , for any f with $\rho_{[p,q]}(f,\varphi) = +\infty$ and $\rho_{[p+1,q]}(f,\varphi) = \rho$, we obtain

$$+\infty = \rho_{[p,q]}(f,\varphi) \leqslant \overline{\lambda}_{[p,q]}(f,\varphi),\ \rho_{[p+1,q]}(f,\varphi) \leqslant \overline{\lambda}_{[p+1,q]}(f,\varphi),$$

hence

$$\rho_{[p+1,q]}(f,\varphi) \leqslant \overline{\lambda}_{[p+1,q]}(f,\varphi) \leqslant \lambda_{[p+1,q]}(f,\varphi).$$

On the other hand, we know that by definition, we have

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) \leqslant \lambda_{[p+1,q]}(f,\varphi) \leqslant \rho_{[p+1,q]}(f,\varphi),$$

and therefore

$$\rho_{[p+1,q]}(f,\varphi) = \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho.$$

∎

Lemma 2.20. *Assume that* $k \geqslant 2$ *and* $A_0,\ A_1, ..., A_k \not\equiv 0,\ F$ *are meromorphic functions. Let* $\rho_3 = \max\left\{\rho_{[p,q]}(A_j,\varphi), (j = 0,1,...,k), \rho_{[p,q]}(F,\varphi)\right\} < \infty$ *and let* f *be a meromorphic solution of infinite* $[p,q]$-φ *order of equation* (1.4) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *. Then,* $\rho_{[p+1,q]}(f,\varphi) \leqslant \rho_3$*.*

Proof. Suppose that f is a meromorphic solution of equation (1.4) of infinite $[p,q]$-φ order with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$. By using the Hadamard factorization theorem, f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions such that

$$\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \le \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) = +\infty$$

and

$$\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) \leqslant \mu.$$

By Lemma 2.14, there exists a set $E_9 \subset (1,+\infty)$ of r of finite linear measure such that for all $|z| = r \notin E_9$ and any given $\varepsilon > 0$, by using the hypotheses of Lemma 2.20, we get

$$\begin{aligned} |A_k(z)| &\geqslant \exp\left\{-\exp_p\left\{\left(\rho_{(p,q)}(A_k,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\}\right\} \\ &\geqslant \exp\left\{-\exp_p\left\{(\rho_3 + \varepsilon)\log_q \varphi(r)\right\}\right\}. \end{aligned} \tag{2.47}$$

For any given $\varepsilon > 0$ and sufficiently large r, we have

$$\begin{aligned} |A_j(z)| &\leqslant \exp_{p+1}\left\{\left(\rho_{(p,q)}(A_j,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\} \\ &\leqslant \exp_{p+1}\left\{(\rho_3 + \varepsilon)\log_q \varphi(r)\right\},\ j = 0,1,...,k-1, \end{aligned} \tag{2.48}$$

and

$$|F(z)| \leqslant \exp_{p+1}\left\{\left(\rho_{(p,q)}(F,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\} \leqslant \exp_{p+1}\left\{(\rho_3 + \varepsilon)\log_q \varphi(r)\right\}. \tag{2.49}$$

From the definition of the $[p,q]-\varphi$ order, the lower $[p,q]-\varphi$ order and (2.49), for any given $\varepsilon\left(0<2\varepsilon<\mu_{[p,q]}\left(f,\varphi\right)-\rho_{[p,q]}\left(d,\varphi\right)\right)$, and for all z satisfying $|z|=r$ sufficiently large at which $|g(z)|=M(r,g)$, we obtain

$$\left|\frac{F\left(z\right)}{f\left(z\right)}\right|=\frac{\left|F\left(z\right)\right|}{|g(z)|}\left|d\left(z\right)\right|$$

$$\leqslant\frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}\left(d,\varphi\right)+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}\exp_{p+1}\left\{\left(\rho_{3}+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}}{\exp_{p+1}\left\{\left(\mu_{[p,q]}\left(f,\varphi\right)-\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}}$$

$$\leqslant\exp_{p+1}\left\{\left(\rho_{3}+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}. \tag{2.50}$$

From Lemma 2.11, there exists a set $E_6\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin[0,1]\cup E_6$ and $\left|g\left(z\right)\right|=M\left(r,g\right)$, we have

$$\frac{f^{(j)}\left(z\right)}{f\left(z\right)}=\left(\frac{\nu_{g}\left(r\right)}{z}\right)^{j}\left(1+o\left(1\right)\right),\ j=1,...,k. \tag{2.51}$$

By equation (1.4), we have

$$\left|\frac{f^{(k)}\left(z\right)}{f\left(z\right)}\right|\leqslant\frac{1}{\left|A_{k}\left(z\right)\right|}\left(\left|A_{0}\left(z\right)\right|+\left|\frac{F\left(z\right)}{f\left(z\right)}\right|+\sum_{j=1}^{k-1}\left|A_{j}\left(z\right)\right|\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right|\right). \tag{2.52}$$

Replacing (2.47), (2.48), (2.50) and (2.51) into (2.52), we get

$$\left|\frac{\nu_{g}\left(r\right)}{z}\right|^{k}\left|1+o\left(1\right)\right|\leqslant\frac{1}{\exp\left\{-\exp_{p}\left\{\left(\rho_{3}+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}\right\}}\times$$

$$\left(\left\{2+\sum_{j=1}^{k-1}\left|\frac{\nu_{g}\left(r\right)}{z}\right|^{j}\left|1+o\left(1\right)\right|\right\}\exp_{p+1}\left\{\left(\rho_{3}+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}\right)$$

$$=\left\{2+\sum_{j=1}^{k-1}\left|\frac{\nu_{g}\left(r\right)}{z}\right|^{j}\left|1+o\left(1\right)\right|\right\}\exp\left\{2\exp_{p}\left\{\left(\rho_{3}+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}\right\}.$$

Then

$$\left|\nu_{g}\left(r\right)\right|\left|1+o\left(1\right)\right|\leqslant\left(k+1\right)r\left|1+o\left(1\right)\right|\exp\left\{2\exp_{p}\left\{\left(\rho_{3}+\varepsilon\right)\log_{q}\varphi\left(r\right)\right\}\right\} \tag{2.53}$$

holds for all z satisfying $|z|=r\notin([0,1]\cup E_6\cup E_9)$ and $\left|g\left(z\right)\right|=M\left(r,g\right)$, $r\to+\infty$. From (2.53), we obtain

$$\limsup_{r\to+\infty}\frac{\log_{p+1}\nu_{g}\left(r\right)}{\log_{q}\varphi\left(r\right)}\leqslant\rho_{3}+\varepsilon. \tag{2.54}$$

Using the fact that $\varepsilon>0$ is arbitrary, by (2.54) and Lemma 2.4, we obtain $\rho_{[p+1,q]}\left(g,\varphi\right)\leqslant\rho_3$. Since $\rho_{[p,q]}\left(d,\varphi\right)<\mu_{[p,q]}\left(f,\varphi\right)$, so by Lemma 2.16, we get $\rho_{[p+1,q]}\left(g,\varphi\right)=\rho_{[p+1,q]}\left(f,\varphi\right)$. Finally, $\rho_{[p+1,q]}\left(f,\varphi\right)\leqslant\rho_3$. Therefore, Lemma 2.20 is proved.

3. PROOF OF THEOREM 1.7

Proof. Let $f \not\equiv 0$ be a rational solution of (1.3) . First, we will prove that f must be a polynomial with $\deg f \leqslant s-1$. If either f is a rational function, which has a pole at z_0 of degree $m \geqslant 1$, or f is a polynomial with $\deg f \geqslant s$, then $f^{(s)}(z) \not\equiv 0$. From equation (1.3) we have

$$A_s(z) f^{(s)}(z) = -\sum_{\substack{j=0 \\ j\neq s}}^{k} A_j(z) f^{(j)}(z).$$

By Lemma 2.5 and Lemma 2.15, we obtain

$$\begin{aligned} \sigma &\leqslant \rho_{[p,q]}(A_s,\varphi) = \rho_{[p,q]}\left(A_s f^{(s)},\varphi\right) \\ &= \rho_{[p,q]}\left(-\sum_{\substack{j=0 \\ j\neq s}}^{k} A_j f^{(j)},\varphi\right) \\ &\leqslant \max_{j=0,1,\dots,k,j\neq s}\left\{\rho_{[p,q]}(A_j,\varphi)\right\}, \end{aligned}$$

which is a contradiction. Therefore, f must be a polynomial with $\deg f \leqslant s-1$. In the second part, we assume that f is a transcendental meromorphic solution of (1.3) such that $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$. For any given $\varepsilon\,(0<2\varepsilon<\sigma-\rho)$ and sufficiently large r, we have

$$\begin{aligned} |A_j(z)| &\leqslant \exp_{p+1}\left\{\left(\rho_{[p,q]}(A_j,\varphi)+\varepsilon\right)\log_q\varphi(r)\right\} \\ &\leqslant \exp_{p+1}\left\{(\rho+\varepsilon)\log_q\varphi(r)\right\},\ j=0,1,...,k,\ j\neq s. \end{aligned} \tag{3.1}$$

By making use of Lemma 2.12, there exists a set $E_7 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1]\cup E_7)$ sufficiently large and $|g(z)| = M(r,g)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leqslant r^{2s},\quad s\geqslant 1 \text{ is an integer.} \tag{3.2}$$

From Lemma 2.2, there exist a set $E_1 \subset (1,+\infty)$ that has a finite logarithmic measure, and a constant $B>0$, such that for all z satisfying $|z| = r \notin ([0,1]\cup E_1)$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\left[T(2r,f)\right]^{k+1},\ j=1,2,...,k,\ j\neq s. \tag{3.3}$$

From the hypotheses of Theorem 1.7, there exist a set G with $\overline{\log dens}\{|z| : z\in G\} > 0$ (or by Proposition 2.1, $m_l(\{|z| : z\in G\}) = \infty$) and a positive constant $\sigma > 0$ such that for sufficiently small $\varepsilon > 0$, we have

$$|A_s(z)| \geqslant \exp_{p+1}\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\} \tag{3.4}$$

as $z\in G$, $|z| = r\to+\infty$. By (1.3) , we can write

$$|A_s \leqslant \left|\frac{f}{f^{(s)}}\right|\left(|A_0| + \sum_{\substack{j=1 \\ j\neq s}}^{k}|A_j|\left|\frac{f^{(j)}}{f}\right|\right). \tag{3.5}$$

Substituting (3.1) , (3.2) , (3.3) and (3.4) into (3.5), for all z satisfying $|z| = r \in \{|z| : z\in G\}\setminus([0,1]\cup E_1\cup E_7)$, $r\to+\infty$, we obtain

$$\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\} \leqslant Bkr^{2s}\exp_{p+1}\left\{(\rho+\varepsilon)\log_q\varphi(r)\right\}\left[T(2r,f)\right]^{k+1}.$$

From $0 < 2\varepsilon < \sigma - \rho$, we obtain

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\}\right\} \leqslant Bkr^{2s}\left[T(2r,f)\right]^{k+1}. \tag{3.6}$$

Using Lemma 2.8 and (3.6), for any given $\nu > 1$ there exists an $r_1 = r_1(\nu)$ and sufficiently large $r > r_1$, $r \in \{|z| : z \in G\}$ such that

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\}\right\} \leqslant Bk(\nu r)^{2s}\left[T(2\nu r,f)\right]^{k+1}.$$

By making use of Definition 1.1 and Remark 1.2, we get

$$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty,\ \sigma \leqslant \rho_{[p+1,q]}(f,\varphi). \tag{3.7}$$

In view of Lemma 2.15, we have

$$\max\left\{\rho_{[p,q]}(A_j,\varphi) : j = 0,1,...,k\right\} = \rho_{[p,q]}(A_s,\varphi) = \delta < +\infty.$$

Since f is of infinite $[p,q]$-φ order meromorphic solution of equation (1.3) satisfying $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$, then by Lemma 2.20, we obtain

$$\rho_{[p+1,q]}(f,\varphi) \leqslant \rho_{[p,q]}(A_s,\varphi). \tag{3.8}$$

By (3.7) and (3.8), we get $\mu_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty$ and

$$\sigma \leqslant \rho_{[p+1,q]}(f,\varphi) \leqslant \rho_{[p,q]}(A_s,\varphi).$$

∎

4. PROOF OF COROLLARY 1.8

Proof. Let ψ be a transcendental meromorphic function such that $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$. Putting $\eta = f - \psi$. By Lemma 2.5, we obtain $\rho_{[p+1,q]}(\eta,\varphi) = \rho_{[p+1,q]}(f,\varphi)$. By making use of Theorem 1.7, we have $\sigma \leqslant \rho_{[p+1,q]}(\eta,\varphi) \leqslant \rho_{[p,q]}(A_s,\varphi)$. Replacing $f = \eta + \psi$ into (1.3), we get

$$A_k(z)\eta^{(k)} + A_{k-1}(z)\eta^{(k-1)} + \cdots + A_1(z)\eta' + A_0(z)\eta$$

$$= -\left(A_k(z)\psi^{(k)} + A_{k-1}(z)\psi^{(k-1)} + \cdots + A_1(z)\psi' + A_0(z)\psi\right) = U(z). \tag{4.1}$$

Since $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$, then according to Theorem 1.7, we can see that ψ is not a solution of eqaution (1.3), hence the right side $U(z)$ of equation (4.1) is non-zero. Furthermore, by Lemma 2.5 and Lemma 2.7, we get

$$\rho_{[p+1,q]}(U,\varphi) \leqslant \max\left\{\rho_{[p+1,q]}(\psi,\varphi),\ \rho_{[p+1,q]}(A_j,\varphi)\,(j=0,1,...,k)\right\} < \sigma.$$

As a consequence

$$\max\left\{\rho_{[p+1,q]}(U,\varphi),\ \rho_{[p+1,q]}(A_j,\varphi)\,(j=0,1,...,k)\right\} < \sigma \leqslant \rho_{[p+1,q]}(\eta,\varphi).$$

From Lemma 2.17, we get

$$\sigma \leqslant \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi)$$
$$= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leqslant \rho_{[p,q]}(A_s,\varphi).$$

∎

5. PROOF OF THEOREM 1.9

Proof. Let $f \not\equiv 0$ be a rational solution of (1.4) . First, we will prove that f must be a polynomial with $\deg f \leqslant s-1$. If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m \geqslant 1$, or f is a polynomial with $\deg f \geqslant s$, then $f^{(s)}(z) \not\equiv 0$. By (1.4) we have

$$A_s f^{(s)} = F - \sum_{\substack{j=0 \\ j\neq s}}^{k} A_j(z) f^{(j)}$$

and by Lemma 2.5 and Lemma 2.15, we obtain

$$\begin{aligned} \sigma &\leqslant \rho_{[p,q]}(A_s,\varphi) = \rho_{[p,q]}\left(A_s f^{(s)},\varphi\right) \\ &= \rho_{[p,q]}\left(F - \sum_{\substack{j=0 \\ j\neq s}}^{k} A_j(z) f^{(j)},\varphi\right) \\ &\leqslant \max_{j=0,1,\ldots,k,\, j\neq s}\left\{\rho_{[p,q]}(A_j,\varphi),\ \rho_{[p,q]}(F,\varphi)\right\}, \end{aligned}$$

which is a contradiction. Therefore, f must be a polynomial with $\deg f \leqslant s-1$. Assuming now that f is a transcendental meromorphic solution of (1.4) that satisfies $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$. By Lemma 2.18, we know that f satisfies $\rho_{[p,q]}(f,\varphi) \geqslant \sigma$. Since $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \min\{\mu_{[p,q]}(f,\varphi),\sigma\}$, then by Hadamard factorization theorem, there exist entire functions $g(z)$ and $d(z)$ such that $f(z) = \frac{g(z)}{d(z)}$ and

$$\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \leqslant \rho_{[p,q]}(g,\varphi) = \rho_{[p,q]}(f,\varphi),$$

$$\rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \min\{\mu_{[p,q]}(f,\varphi),\sigma\}.$$

From the definition of the lower $[p,q]-\varphi$ order, for any given $\varepsilon > 0$ and sufficiently large r, we get

$$|g(z)| = M(r,g) \geqslant \exp_{p+1}\left\{(\mu_{[p,q]}(g,\varphi)-\varepsilon)\log_q \varphi(r)\right\}. \tag{5.1}$$

Let

$$\rho_1 = \max\left\{\rho_{[p,q]}(A_j,\varphi),\ j\neq s, \rho_{[p,q]}(F,\varphi)\right\} < \sigma.$$

Then, by (5.1), for any given ε satisfying

$$0 < 2\varepsilon < \min\{\sigma-\rho_1, \mu_{[p,q]}(g,\varphi) - \rho_{[p,q]}(d,\varphi)\},$$

and all z satisfying $|z| = r$ sufficiently large at which $|g(z)| = M(r,g)$, we have

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|}{|g(z)|}|d(z)|$$

$$\leqslant \frac{\exp_{p+1}\left\{(\rho_{[p,q]}(d,\varphi)+\varepsilon)\log_q \varphi(r)\right\}\exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q\varphi(r)\right\}}{\exp_{p+1}\left\{(\mu_{[p,q]}(g,\varphi)-\varepsilon)\log_q\varphi(r)\right\}}$$

$$\leqslant \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q\varphi(r)\right\}. \tag{5.2}$$

Using the similar way as in the proof of Theorem 1.7, for any given ε satisfying $0 < 2\varepsilon < \min\{\sigma - \rho_1, \mu_{[p,q]}(g,\varphi) - \rho_{[p,q]}(d,\varphi)\}$ and all z satisfying $|z| = r \in \{|z| : z \in G\} \setminus ([0,1] \cup E_1 \cup E_7)$, $r \to +\infty$ at which $|g(z)| = M(r,g)$, we have $(3.2), (3.3), (3.4)$ and

(5.3) $$|A_j(z)| \leqslant \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\},\ j = 0,1,...,k,\ j \neq s.$$

From (1.4), we have

(5.4) $$|A_s| \leqslant \left|\frac{f}{f^{(s)}}\right| \left(|A_0| + \sum_{\substack{j=1\\ j\neq s}}^{k} |A_j| \left|\frac{f^{(j)}}{f}\right| + \left|\frac{F}{f}\right| \right).$$

Replacing $(3.2), (3.3), (3.4), (5.2)$ and (5.3) into (5.4), for all z satisfying $|z| = r \in \{|z| : z \in G\} \setminus ([0,1] \cup E_1 \cup E_7)$, $r \to +\infty$, at which $|g(z)| = M(r,g)$ and any given ε satisfying

$$0 < 2\varepsilon < \min\{\sigma - \rho_1, \mu_{[p,q]}(g,\varphi) - \rho_{[p,q]}(d,\varphi)\},$$

we obtain

$$\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\} \leqslant r^{2s}\Big(\exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}$$
$$+ \sum_{j=1, j\neq s}^{k} \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\} B\left[T(2r,f)\right]^{k+1}$$
$$+ \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}\Big)$$

(5.5) $$\leqslant B(k+1)\, r^{2s}\left[T(2r,f)\right]^{k+1} \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}.$$

The fact that $0 < 2\varepsilon < \sigma - \rho_1$ gives

(5.6) $$\exp\left\{(1-o(1))\exp_p(\sigma-\varepsilon)\log_q \varphi(r)\right\} \leqslant B(k+1)\, r^{2s}\left[T(2r,f)\right]^{k+1}.$$

Using Lemma 2.8 and (5.6), for any given $\nu > 1$ there exists an $r_2 = r_2(\nu)$ and sufficiently large $r > r_2$, $r \in \{|z| : z \in G\}$ such that

(5.7) $$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\}\right\} \leqslant B(k+1)(\nu r)^{2s}\left[T(2\nu r,f)\right]^{k+1}.$$

By making use of Definition 1.1 and Remark 1.2, we get

(5.8) $$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty,\ \sigma \leqslant \rho_{[p+1,q]}(f,\varphi).$$

According to Lemma 2.15 and the hypotheses of Theorem 1.9, we get

$$\max\left\{\rho_{[p,q]}(A_j,\varphi)\ (j=0,1,...,k),\ \rho_{[p,q]}(F,\varphi)\right\} = \rho_{[p,q]}(A_s,\varphi) = \delta < +\infty.$$

Using Lemma 2.20 and the fact that f is a meromorphic solution of equation (1.4) of $[p,q]$-φ order with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$, we obtain

(5.9) $$\rho_{[p+1,q]}(f,\varphi) \leqslant \max\left\{\rho_{[p,q]}(A_j,\varphi)\ (j=0,1,...,k),\ \rho_{[p,q]}(F,\varphi)\right\} = \rho_{[p,q]}(A_s,\varphi).$$

From Lemma 2.19 and since $F \not\equiv 0$, we get

(5.10) $$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty$$

and

(5.11) $$\sigma \leqslant \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi).$$

Then from $(5.9), (5.10)$ and (5.11), we conclude that

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty$$

and

$$\sigma \leqslant \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leqslant \rho_{[p,q]}(A_s,\varphi).$$

∎

6. PROOF OF COROLLARY 1.10

Let ψ be a transcendental meromorphic function such that $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$. Putting $\vartheta = f - \psi$, then $\rho_{[p+1,q]}(\vartheta,\varphi) = \rho_{[p+1,q]}(f,\varphi)$, and by Theorem 1.9, we have $\sigma \leqslant \rho_{[p+1,q]}(\vartheta,\varphi) \leqslant \rho_{[p,q]}(A_s,\varphi)$. Replacing $f = \vartheta + \psi$ into (1.4), we get

$$A_k(z)\vartheta^{(k)} + A_{k-1}(z)\vartheta^{(k-1)} + \cdots + A_1(z)\vartheta + A_0(z)\vartheta$$

$$= F(z) - \left(A_k(z)\psi^{(k)} + A_{k-1}(z)\psi^{(k-1)} + \cdots + A_1(z)\psi' + A_0(z)\psi\right) = V(z). \tag{6.1}$$

Since $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$, then according to Theorem 1.9, ψ is not a solution of equation (1.4), hence the right side $V(z)$ of equation (6.1) is non zero. Furthermore, by Lemma 2.5 and Lemma 2.7, we obtain

$$\rho_{[p+1,q]}(V,\varphi) \leqslant \max\left\{\rho_{[p+1,q]}(\psi,\varphi),\ \rho_{[p+1,q]}(A_j,\varphi)\,(j=0,1,...,k)\right\} < \sigma.$$

As a consequence

$$\max\left\{\rho_{[p+1,q]}(V,\varphi),\ \rho_{[p+1,q]}(A_j,\varphi)\,(j=0,1,...,k)\right\} < \sigma \leq \rho_{[p+1,q]}(\vartheta,\varphi).$$

Thus, by Lemma 2.17, we get

$$\sigma \leqslant \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi)$$
$$= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).$$

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