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GENERALIZED TRIANGULAR AND SYMMETRIC SPLITTING METHOD FOR STEADY STATE PROBABILITY VECTOR OF STOCHASTIC MATRICES

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ABSTRACT. To find the steady state probability vector of homogenous linear system $\pi Q = 0$ of stochastic rate matrix Q, generalized triangular and symmetric (GTS) splitting method is presented. Convergence analysis and choice of parameters are given when the regularized matrix $A = Q^T + \epsilon I$ of the regularized linear system $Ax = b$ is positive definite. Analysis shows that the iterative solution of GTS method converges unconditionally to the unique solution of the regularized linear system. From the numerical results, it is clear that the solution of proposed method converges rapidly when compared to the existing methods.

Key words and phrases: Stochastic matrices; Steady state probability vector; GTS Method; Convergence analysis; Relative and Absolute error.

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1. INTRODUCTION

A class of Markovian Arrival Process (MAP) and Markov modulated Poisson process (MMPP) are generalization of the Poisson process. These are widely used in the modelling of communication systems, automata networks, and manufacturing systems[1, 2, 3, 4, 5, 6, 7, 8]. Many network models, in general, have a regular repetitive structure that fits within the matrix geometric frame work [9]. In most of the cases, when the network nodes are modeled as queueing systems, the problem of computation of performance measures is reduced to that of steady state probability distribution vector of transition rate matrix or transition probability matrix. Therefore, it is the key importance to investigate pertinent linear system. In this direction, many researchers proposed the methods and their convergence criteria [2, 7, 4, 8, 28, 9, 30, 3, 6, 27, 19, 25, 29, 21]. The significant improvements in convergence rates can be achieved from the Krylov subspace methods [29, 20, 22, 32], some preconditioning techniques [14, 18, 16], and two splitting and multi splitting iterative methods [15, 21]. Two alternative methods (Hermitian and Skew-Hermitian) HSS and (Positive definite Skew-Symmetric) PSS methods proposed in the papers [10, 11] which converge unconditionally to the unique solution of the system of equations. Moreover, the triangular and skew-symmetric (TSS) iteration method has been developed and discussed for solving positive-definite linear system of stochastic matrices [21]. It is clear from the papers [21, 10, 11] that the estimation of optimal parameter α is not so easy. In the paper [26], the triangular and symmetric (TS) iteration method has been developed to compute the steady state vector of pertinent linear system of circulant stochastic matrices. In the said paper, the TS method is restricted only for circulant matrices. Hence, in this paper, we generalize the TS method to find the steady state probability vector of positive definite regularized linear system of general stochastic matrices. This method splits the regularized matrix into the triangular and symmetric matrices. Moreover, we modify the splitting matrices for the estimation of optimal parameter, and its convergence criteria.

The rest of the paper is organized as follows: In section 2, the regularized preconditioned linear system of rate matrix is considered. In section 3, the GTS iteration method is applied to solve the regularized linear system and discussed its convergence. In section 4, the choice of the contraction factor α is analyzed and proposed an inexact triangular and symmetric splitting (ITS) iteration method. In section 5, the proposed method is implemented numerically to realize the advantages. Finally, conclusions are drawn in section 6.

2. REGULARIZED PRECONDITIONED LINEAR SYSTEM OF STOCHASTIC MATRICES

In this section, we define some basic definitions and prove that the stochastic rate matrix is singular M-matrix. We shall prove that the coefficient matrix in the preconditioned regularized linear system is positive definite.

Definition 1. Any matrix $A \in R^{n\times n}$ of the form $A = sI - B, s > 0, B \geq 0$ is called an M-matrix if $s \geq \rho(B)$. If $s > \rho(B)$ then A is non-singular M-matrix, otherwise A is singular M-matrix.

Definition 2. A non-symmetric matrix $A \in R^{n\times n}$ is M-matrix if its symmetric part $\frac{A+A^T}{2}$ is M-matrix.

Definition 3. A non-symmetric matrix $A \in R^{n\times n}$ is positive definite if its symmetric part is positive definite.

Consider the stochastic probablity matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3n} \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn} \end{bmatrix} \text{ (say) for}$$

$$0 \leq p_{ij} \leq 1,\ 1 \leq i, j \leq n \tag{2.1}$$

and $\sum_{i,j=1}^{n} p_{ij} = 1$. Since each row and column sum are one, then the matrix P is a doubly stochastic matrix. The steady state distribution vector π satisfies the equation

$$\begin{aligned} \pi &= \pi P, \\ \Rightarrow \pi[I-P] &= 0, \\ \Rightarrow \pi Q &= 0, \end{aligned} \tag{2.2}$$

where

$$Q = I - P = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2n} \\ q_{31} & q_{32} & q_{33} & \cdots & q_{3n} \\ \ldots & \ldots & \ldots & \ddots & \vdots \\ q_{n1} & q_{n2} & q_{n3} & \cdots & q_{nn} \end{bmatrix} \text{ (say) for}$$

$$q_{ii} > 0, q_{ij} \leq 0,\ 1 \leq i, j \leq n, \tag{2.3}$$

and $\sum_{i,j=1}^{n} q_{ij} = 0$. Since each row and column sum are zero, then the matrix Q is a doubly stochastic rate matrix.

Theorem 2.1. [17] *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$ and row sums $r_i(A)$, $i \in \{1, 2, ..., n\}$. Then, $\min_i r_i(A) \leq \rho(A) \leq \max_i r_i(A)$. Moreover, if A is an irreducible matrix, then equality holds on either side if and only if all row sums of A are equal.*

Lemma 2.2. *A stochastic rate matrix $Q \in R^{n \times n}$ is singular M-matrix.*

Proof. Consider the stochastic rate matrix Q given in "Eq. (2.3)".
Since Q stochastic rate matrix then the sum of each row and each column is zero.

$$\text{i.e., } \sum_{i,j=1}^{n} q_{ij} = 0 \text{ with } q_{ii} > 0, q_{ij} \leq 0, 1 \leq i, j \leq n, \tag{2.4}$$

$$\Rightarrow q_{ii} = \sum_{j=1, i \neq j}^{n} q_{ij} \text{ for } i = 1, 2, 3 \ldots, n \tag{2.5}$$

From the definition 2, in order to prove the matrix Q is singular M-matix, it is suffices to prove that symmetric part of the matrix Q, $i.e.,\frac{Q+Q^T}{2}$is singular M-Matrix. Therefore, we consider

$$\frac{Q+Q^T}{2} = \begin{bmatrix} q_{11} & \frac{q_{12}+q_{21}}{2} & \frac{q_{13}+q_{31}}{2} & \dots & \frac{q_{1n}+q_{n1}}{2} \\ \frac{q_{12}+q_{21}}{2} & q_{22} & \frac{q_{23}+q_{32}}{2} & \dots & \frac{q_{2n}+q_{n2}}{2} \\ \frac{q_{13}+q_{31}}{2} & \frac{q_{32}+q_{23}}{2} & q_{33} & \dots & \frac{q_{3n}+q_{n3}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{q_{1n}+q_{n1}}{2} & \frac{q_{2n}+q_{n2}}{2} & \frac{q_{3n}+q_{n3}}{2} & \dots & q_{nn} \end{bmatrix}, \tag{2.6}$$

$$= aI_n - R \text{ (say)}, \text{ where } a = \max_{1\le i\le n} q_{ii} \tag{2.7}$$

$$\text{and } R = \begin{bmatrix} a-q_{11} & \frac{q_{12}+q_{21}}{2} & \frac{q_{13}+q_{31}}{2} & \dots & \frac{q_{1n}+q_{n1}}{2} \\ \frac{q_{12}+q_{21}}{2} & a-q_{22} & \frac{q_{23}+q_{32}}{2} & \dots & \frac{q_{2n}+q_{n2}}{2} \\ \frac{q_{13}+q_{31}}{2} & \frac{q_{32}+q_{23}}{2} & a-q_{33} & \dots & \frac{q_{3n}+q_{n3}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{q_{1n}+q_{n1}}{2} & \frac{q_{2n}+q_{n2}}{2} & \frac{q_{3n}+q_{n3}}{2} & \dots & a-q_{nn} \end{bmatrix} \ge 0, \tag{2.8}$$

and I_n is the identity matrix of order n. Now, consider the row sum of the matrix R,

$$\begin{aligned} r_i(R) &= a - q_{ii} + \frac{q_{i2}+q_{2i}}{2} + \dots\dots + \frac{q_{in}+q_{ni}}{2}, \\ &= a - q_{ii} + \frac{q_{i2}+q_{i3}+\dots q_{in}}{2} + \frac{q_{2i}+q_{3i}+\dots+q_{ni}}{2}, \\ \text{we have } q_{ii} &= q_{i2}+q_{i3}+\dots.q_{in}, \text{and} q_{ii} = q_{2i}+q_{3i}+\dots+q_{ni}, \\ \therefore r_i(R) &= a - q_{ii} + \frac{q_{ii}}{2} + \frac{q_{ii}}{2}, \\ &= a - q_{ii} + q_{ii}, \\ r_i(R) &= a. \end{aligned} \tag{2.9}$$

$$\begin{aligned} \text{From the Theorem 2.1, we have } \underset{i}{min}\, r_i(R) \le \rho(R) &\le \underset{i}{max}\, r_i(R), \\ &\Rightarrow a \le \rho(R) \le a, \\ &\Rightarrow \rho(R) = a. \end{aligned}$$

$\therefore$ Q is singular M-matrix

∎

From the lemma 2.2, the coefficient Q of "Eq. (2.2)" has one dimensional null space, thus GTS iteration method cannot be directly applied to solve the linear system"Eq. (2.2)". Hence, we go for a regularized linear system [21]. The equation "Eq. (2.2)" can be written as $Q^T\pi^T = 0$

$$i.e., \overline{A}x = 0, \tag{2.10}$$

where $\overline{A} = Q^T$, and $x = \pi^T$. There exists a nonnegative constant $\epsilon > 0$ such that the above equation can be put in the following form of preconditioned linear system [21, 23, 10]

$$Ax = (Q^T + \epsilon I)x = e_n, \tag{2.11}$$

where e_n is a unit vector given by $e_n = \begin{bmatrix} 0, & 0, & \dots, & 0, & 1 \end{bmatrix}^T$. The steady-state probability distribution vector is then obtained by normalizing the vector x.

Theorem 2.3. *For any nonsymmetric stochastic rate matrix $Q \in R^{n\times n}$ there exists a constant $\epsilon > 0$ such that $A = Q^T + \epsilon I_n$ is positive definite if all its real eigenvalues are non-negative.*

Proof. Consider the stochastic rate matrix Q.
Then from "Eq. (2.3)", we have

$$Q^T = \begin{bmatrix} q_{11} & q_{21} & q_{31} & \cdots & q_{n1} \\ q_{12} & q_{22} & q_{32} & \cdots & q_{n2} \\ q_{13} & q_{23} & q_{33} & \cdots & q_{n3} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ q_{1n} & q_{2n} & q_{3n} & \cdots & q_{nn} \end{bmatrix} \text{ (say) for}$$

$$q_{ii} > 0, q_{ij} \leq 0, \ 1 \leq i, j \leq n \tag{2.12}$$

From "Eq. (2.11)", we have $A = Q^T + \epsilon I_n$. Suppose that all real eigenvalues of the matrix A are non-negative and let $\epsilon > 0$. Since $q_{ii} > 0, q_{ij} \leq 0, 1 \leq i, j \leq n$, then $q_{ii} + \epsilon$ is the maximum value in the matrix Q. For proving the regularized preconditioned matrix A is positive definite, it suffices to prove that the symmetric part of A, $i.e.$, $\frac{A+A^T}{2}$ is positive definite.
From "Eq. (2.11)", $\frac{A+A^T}{2} = \frac{Q+Q^T}{2} + \epsilon I_n = (a+\epsilon)I_n - R,$ where $a = \max_i q_{ii}$
$\Rightarrow R = (a+\epsilon)I_n - \frac{A+A^T}{2}$,
$\Rightarrow R$ is non-negative matrix.
Let r be the maximal real eigenvalue of the matrix R. Then, we have
$\|rI_n - R\| = 0$ and $\rho(R) = r$.
$\Rightarrow (a+\epsilon - r)$ is the real eigenvalue of $\frac{A+A^T}{2}$,
$\Rightarrow (a+\epsilon - r) > 0$,
$\Rightarrow (a+\epsilon) > r = \rho(R)$,
$\Rightarrow (a+\epsilon) > \rho(R)$,
$\Rightarrow \frac{A+A^T}{2}$ is positive definite,
$\Rightarrow A$ is positive definite.
∎

3. THE CONVERGENCE ANALYSIS OF TRIANGULAR AND SYMMETRIC ITERATION METHOD

In this section, the steady state probability vector of an irreducible stochastic rate matrix (regularized linear system) is computed, and also obtained the condition for the convergence of pertinent iterative solution as in the cases of TS, TSS, HSS, and PSS methods [26, 21, 10, 11]. Consider the coefficient matrix $A = D + L + U$, where D is a diagonal matrix with the diagonal elements of the matrix A and L, U are lower, upper triangular matrix of A respectively of regularized linear system "Eq. (2.11)".
$i.e., A = D + L + U$,

where $$D = \begin{bmatrix} \frac{\epsilon}{3} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\epsilon}{3} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\epsilon}{3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon}{3} \end{bmatrix}, L = \begin{bmatrix} \frac{q_{11}}{2} + \frac{\epsilon}{3} & 0 & 0 & \cdots & 0 \\ q_{12} & \frac{q_{22}}{2} + \frac{\epsilon}{3} & 0 & \cdots & 0 \\ q_{13} & q_{23} & \frac{q_{33}}{2} + \frac{\epsilon}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & q_{3n} & \cdots & \frac{q_{nn}}{2} + \frac{\epsilon}{3} \end{bmatrix},$$

and $U = \begin{bmatrix} \frac{q_{11}}{2}+\frac{\epsilon}{3} & q_{21} & q_{31} & \cdots & q_{n1} \\ 0 & \frac{q_{22}}{2}+\frac{\epsilon}{3} & q_{32} & \cdots & q_{n2} \\ 0 & 0 & \frac{q_{33}}{2}+\frac{\epsilon}{3} & \cdots & q_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{q_{nn}}{2}+\frac{\epsilon}{3} \end{bmatrix}$.

The triangular and symmetric matrix splitting method on the regularized matrix is as follows:

$$A = (L + D - U^T) + (U + U^T) = (U + D - L^T) + (L + L^T) = T + S, \tag{3.1}$$

where $T \in R^{n\times n}$ triangular matrix with nonnegative diagonal elements, and $S \in R^{n\times n}$ is symmetric with positive diagonal elements and negative off diagonal elements. To find the solution of the regularized linear system, we can use the iterative method described [26, 10, 11]. Given an initial guess $x^{(0)}$, compute the next approximations using the following scheme [21, 10, 11]

$$\begin{aligned} (\alpha I + T)x^{(k+1/2)} &= (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} &= (\alpha I - T)x^{(k+1/2)} + b, \end{aligned} \tag{3.2}$$

for $k = 0, 1, 2, \ldots,$ until $x^{(k)}$ converges for the contraction factor α. The above iterative scheme could be written as $x^{(k+1)} = M(\alpha)x^{(k)} + N(\alpha)b$, for $k = 0, 1, 2, \ldots,$ where

$$M(\alpha) = (\alpha I + S)^{-1}(\alpha I - T)(\alpha I + T)^{-1}(\alpha I - S), \tag{3.3}$$

is the iteration matrix of the GTS iteration method, and $N(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + T)^{-1}$. If $\rho(M(\alpha)) < 1$ then the GTS iterative method is convergent. To prove $\rho(M(\alpha)) < 1$, we assume the lemma in the paper [21].

Lemma 3.1. *Let $W(\alpha) = (\alpha I - T)(\alpha I + T)^{-1}$. If $T \in R^{n\times n}$ is a positive-definite matrix, then we have $\|W(\alpha)\|_2 < 1, \forall \alpha > 0$.*

Lemma 3.2. *If the matrix S is the symmetric part of the coefficient matrix A of the regularized linear system “Eq. (2.11)” then there exist $\epsilon > 0$ such that the matrix S is positive definite.*

Proof. Consider the symmetric part S of the coefficient matrix A of the regularized linear system "Eq. (2.11)" as

$$S = (L + L^T)$$
$$= \begin{bmatrix} q_{11} + \frac{2\epsilon}{3} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{12} & q_{22} + \frac{2\epsilon}{3} & q_{23} & \cdots & q_{2n} \\ q_{13} & q_{23} & q_{33} + \frac{2\epsilon}{3} & \cdots & q_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & q_{3n} & \cdots & q_{nn} + \frac{2\epsilon}{3} \end{bmatrix},$$

with $\epsilon > 0, q_{ij} < 0$ for $i \neq j$,

$$= \delta I_n - V,$$

$$\text{where } V = \begin{bmatrix} \delta - q_{11} - \frac{2\epsilon}{3} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{12} & \delta - q_{22} - \frac{2\epsilon}{3} & q_{23} & \cdots & q_{2n} \\ q_{13} & q_{23} & \delta - q_{33} - \frac{2\epsilon}{3} & \cdots & q_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & q_{3n} & \cdots & \delta - q_{nn} - \frac{2\epsilon}{3} \end{bmatrix}$$

is a nonnegative matrix,

$$\text{and } \delta = \max_{1 \leq i \leq n} (q_{ii} + \frac{2\epsilon}{3}) \text{ for } i = 1, 2, \ldots, n.$$

The spectral radius of the matrix V is

$$\sum_{i,j=1, i \neq j}^{n} q_{ij} \text{ and } q_{ii} = \sum_{i,j=1, i \neq j}^{n} q_{ij},$$
$$\Rightarrow \max_{1 \leq i \leq n} (q_{ii} + \frac{2\epsilon}{3}) > \sum_{i,j=1, i \neq j}^{n} q_{ij},$$
$$\Rightarrow \delta > \sum_{i,j=1, i \neq j}^{n} q_{ij} \text{ and } \rho(V) = \delta - \frac{2\epsilon}{3},$$
$$\Rightarrow \delta > \delta - \frac{2\epsilon}{3},$$
$$\Rightarrow \delta > \rho(V).$$

Therefore, the matrix S is positive definite. ∎

Lemma 3.3. [26] *Let $H(\alpha) = (\alpha I - S)(\alpha I + S)^{-1}$. If $S \in R^{n \times n}$ is a positive-definite matrix, then we have $\|H(\alpha)\|_2 < 1, \forall \alpha > 0$.*

Theorem 3.4. *Let $A \in R^{n \times n}$ be the regularized matrix defined in "Eq. (2.11)", and splitting into generalized triangular and symmetric matrices given in "Eq. (3.1)". Then the spectral radius of the iterative matrix $M(\alpha)$ is less than one.*

Proof. We prove this theorem on lines of the theorems [26, 21, 10]. Let $A \in R^{n \times n}$ be the regularized matrix defined in "Eq. (2.11)", and splitting into the form "Eq. (3.1)". Let $M(\alpha)$ be the iterative matrix given in "Eq. (3.3)". Then the iteration matrix $M(\alpha)$ is similar to the matrix $\overline{M(\alpha)} = (\alpha I - T)(\alpha I + T)^{-1}(\alpha I - S)(\alpha I + S)^{-1} = W(\alpha)H(\alpha)$, where $W(\alpha) = (\alpha I - T)(\alpha I + T)^{-1}$, and $H(\alpha) = (\alpha I - S)(\alpha I + S)^{-1}$. Since the triangular matrix T, and symmetric S of the regularized preconditioned matrix A given in "Eq. (2.11)" are positive definite , then from the lemma 3.1 and lemma 3.3, we have $\|W(\alpha)\|_2 < 1, \|H(\alpha)\|_2 < 1, \forall \alpha > 0$.

Therefore, $\rho(M(\alpha)) = \rho(\overline{M(\alpha)}) = \|(\alpha I - T)(\alpha I + T)^{-1}(\alpha I - S)(\alpha I + S)^{-1}\|_2 = \|W(\alpha)H(\alpha)\|_2$, which gives

$$\rho(M(\alpha)) = \rho(\overline{M(\alpha)}) = \|W(\alpha)\|_2\|H(\alpha)\|_2 < 1. \tag{3.4}$$

Therefore, the GTS iteration method converge to unique solution of the regularized linear system "Eq. (2.11)". ∎

4. CONVERGENCE ANALYSIS OF IGTS METHOD AND CONTRACTION FACTOR

In this section, we find the contraction factor and discuss the inexact generalized triangular and symmetric (IGTS) iterative method by using the Krylov subspace method [21, 31]. Here we find the contraction factor α on lines of the papers [26, 21, 10]. From the TS, TSS, HSS, PSS iterative methods and the above theoretical results, it is clear that the iterative solution of preconditioned matrix A converges for any contraction factor α, and it was converge to the exact value as in the case of TS. Along the lines of the papers [26, 21], the contraction factor α is converges to the fixed value in the following results. Therefore, for the fast convergence of the solution of the GTS iteration method, it is the key importance to choose the appropriate values of ϵ. Since the preconditioned matrix A of the regularized linear system "Eq. (2.11)" splits into generalized triangular and symmetric matrices as given by "Eq. (3.1)", we have

$$\begin{aligned} A &= (L + D - U^T) + (U + U^T) = T_1 + S_1, \\ A &= (U + D - L^T) + (L + L^T) = T_2 + S_2, \end{aligned}$$

where T_i and S_i $(i = 1, 2)$ are triangular and symmetric matrices, respectively. On lines of the papers [26, 21, 10], we find out the contraction factor as follows:
Let $G_1 = (L - U^T)$ and $G_2 = (U - L^T)$ then $G_i (i = 1, 2)$ are strictly lower and upper triangular such that $[G_i(\alpha I + D)^{-1}]^n = [(\alpha I + D)^{-1}G_i]^n = 0$ for $i = 1, 2$ Consider

$$\begin{aligned} (\alpha I + T_i)^{-1} &= [(\alpha I + D) + G_i]^{-1}, \\ &= (\alpha I + D)^{-1}[I + G_i(\alpha I + D)^{-1}]^{-1}, \\ &= (\alpha I + D)^{-1}\sum_{j=0}^{n-1}(-1)^j[G_i(\alpha I + D)^{-1}]^j. \end{aligned}$$

Now,

$$\begin{aligned} (\alpha I - T_i)(\alpha I + T_i)^{-1} &= (\alpha I - D - G_i)(\alpha I + T_i)^{-1}, \\ &= (\alpha I - D - G_i)(\alpha I + D)^{-1}\sum_{j=0}^{n-1}(-1)^j[G_i(\alpha I + D)^{-1}]^j, \\ &\approx (\alpha I - D)(\alpha I + D)^{-1}[I - G_i(\alpha I + D)^{-1}]. \end{aligned}$$

Considering the first order approximation and taking the norm on both sides, we obtain

$$\begin{aligned} \|(\alpha I - T_i)(\alpha I + T_i)^{-1}\|_2 &\approx \|(\alpha I - D_i)(\alpha I + D_i)^{-1}\|_2, \\ \|W(\alpha)\|_2 &= \max_{1\le j\le n}\{(\alpha - d_{jj})(\alpha + d_{jj})^{-1}\}. \end{aligned}$$

In above, d_{jj}'s are the diagonal elements of the matrix D_i. Following the theorems from the papers [26, 21, 10, 11], we compute an exact optimal value $\alpha > 0$ for the convergence factor

$\rho(M(\alpha))$ of the GTS iteration method is minimized. If $\overline{\alpha}$ is the minimum point of convergence factor, then it must satisfy

$$
\begin{aligned}
\overline{\alpha} &= \text{arg} \min_{\alpha>0} \|W(\alpha)\|_2, \\
&= \text{arg} \min_{\alpha>0} \max_{1\le j\le n} \{(\alpha - d_{jj})(\alpha + d_{jj})^{-1}\}, \\
&= \sqrt{d_{\min} d_{\max}}, \qquad (4.1)
\end{aligned}
$$

where

$$
\begin{aligned}
d_{\min} &= \min_{1\le j\le n} \{d_{jj}\} = \frac{\epsilon}{3}, \\
\text{and } d_{\max} &= \max_{1\le j\le n} \{d_{jj}\} = \frac{\epsilon}{3},
\end{aligned}
$$

From the above equation, we have

$$
\begin{aligned}
\|W(\alpha)\|_2 &\simeq \overline{\alpha}, \\
&\simeq \sqrt{d_{\min} d_{\max}}, \\
\|W(\alpha)\|_2 &\simeq \sqrt{\frac{\epsilon}{3}\frac{\epsilon}{3}} = \frac{\epsilon}{3}. \qquad (4.2)
\end{aligned}
$$

Now, we find out the strict upper bound for the $\|H(\alpha)\|_2$ as follows:
Consider

$$
\begin{aligned}
(\alpha I + S_i)^{-1} &= (\alpha I + L + L^T)^{-1}, \\
&= (\alpha I + L)^{-1}(I + ((\alpha I + L)^{-1}L^T))^{-1}, \\
&= (\alpha I + L)^{-1} \sum_{j=0}^{\infty} (-1)^j ((\alpha I + L)^{-1}L^T)^j, \\
&= (\alpha I + L)^{-1}(I - ((\alpha I + L)^{-1}L^T) + \ldots),
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
(\alpha I - S_i)(\alpha I + S_i)^{-1} &= (\alpha I - (L + L^T))(\alpha I + L)^{-1}(I - ((\alpha I + L)^{-1}L^T) + \ldots), \\
(\alpha I - S_i)(\alpha I + S_i)^{-1} &= ((\alpha I - L) - L^T)(\alpha I + L)^{-1}(I - ((\alpha I + L)^{-1}L^T) + \ldots).
\end{aligned}
$$

using the first order approximation and taking norm on both sides we get,

$$
\begin{aligned}
\|(\alpha I - S_i)(\alpha I + S_i)^{-1}\|_2 &\approx \|(\alpha I - L)(\alpha I + L)^{-1}\|_2, \\
\|(\alpha I - S_i)(\alpha I + S_i)^{-1}\|_2 &\approx \max_{1\le j\le n} \{(\alpha I - d_{jj})(\alpha I + d_{jj})^{-1}\},
\end{aligned}
$$

where d'_{jj}s are the diagonal elements of the lower triangular matrix L. Following derivation of the equation "Eq. (4.1)" and "Eq. (4.2)", we obtain

$$
\begin{aligned}
\|H(\alpha)\|_2 &\approx \|(\alpha I - (a + \frac{\epsilon}{3}))(\alpha I + (a + \frac{\epsilon}{3}))^{-1}\|, \\
\|H(\alpha)\|_2 &= \sqrt{(a + \frac{\epsilon}{3})(a + \frac{\epsilon}{3})} = a + \frac{\epsilon}{3}. \qquad (4.3)
\end{aligned}
$$

From the "Eq. (3.4)" , "Eq. (4.2)" , and "Eq. (4.3)" we have $\rho(M(\alpha)) = (a + \frac{\epsilon}{3})\frac{\epsilon}{3} < 1$ if $\frac{\epsilon}{3} < 1$ and $(a + \frac{\epsilon}{3}) < 1$. That is, we have a sharp upper bound for $\rho(M(\alpha))$. From the papers [21, 10], it is clear that estimation of the contraction factor is not so easy. On lines of the paper [26, 21, 10], we obtain the contraction factor for GTS method and is $\alpha = \overline{\alpha} = \frac{\epsilon}{3}$. The computational procedure for finding the solution of the given system obtain along the lines of the paper [26].

5. NUMERICAL RESULTS

In this section, we examine the effectiveness of the GTS iteration method for the numerical solution of Markov process and compare them with the TSS and Jacobi methods. For the numerical illustration, we consider the following 4×4 stochastic probability matrix

$$P = \begin{bmatrix} 0.4 & 0.1 & 0.35 & 0.15 \\ 0.3 & 0.3 & 0.15 & 0.25 \\ 0.1 & 0.25 & 0.25 & 0.4 \\ 0.2 & 0.35 & 0.25 & 0.2 \end{bmatrix}.$$

We consider only one case $A = (L + D - U^T) + (U + U^T) = T_1 + S_1$ of GTS splitting method and other methods would follow. Considering the initial distribution $x^{(0)} = [0, 0, 0, 1]^T$ for the system "Eq. (2.11)", relative error and absolute error are computed according to the basic definitions of the error analysis. The steady state distribution vector x of the preconditioned linear system "Eq. (2.11)" is obtained and results are presented in the Figs. 1-3. From these figures, we illustrate the result for the case of contraction factor α which is numerically equivalent to the diagonal elements of the matrix Q, for variant values of ϵ. Also, we conclude that the GTS iterative solution converges rapidly than the TSS and Jacobi's methods. Moreover, error decreases as ϵ value increase.

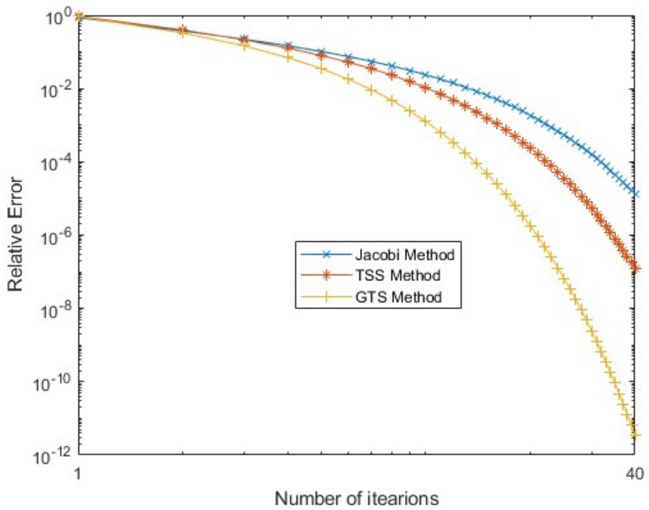

Figure 1: Relative error of the GTS, TSS, and Jacobi methods for the contraction factor $\alpha = 0.4$, and $\epsilon = 0.3$.

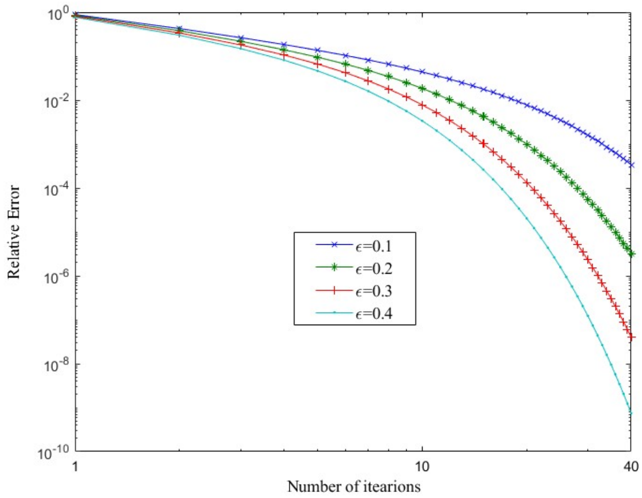

Figure 2: Relative error of the GTS method for the contraction factor $\alpha = 0.4$, and different ϵ values.

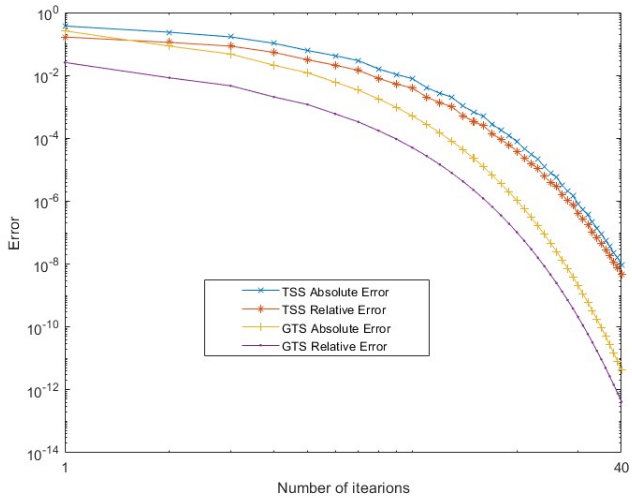

Figure 3: Absolute and Relative error of the GTS, TSS methods for the contraction factor $\alpha = 0.4$, and $\epsilon = 0.2$.

6. CONCLUSIONS

In this paper, we present GTS splitting iterative method for the regularized linear system of stochastic matrix. We conclude that this method unconditionally converges to a unique solution and the convergence rate is rapid when compared to the existing methods. We proved that the regularized matrix is positive definite under specific condition. From the numerical results, it clear that how well the proposed splitting method is efficient when compared to other existing TSS and Jacobi methods.

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