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ON AUTOMORPHISMS AND BI-DERIVATIONS OF SEMIPRIME RINGS

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ABSTRACT. In this article, our goal is to figure out a functional equation involving automorphisms and bi-derivations on certain semiprime ring. Also, we characterize the structure of automorphism, in case of prime rings.

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1. INTRODUCTION

Several authors have looked into the connection between specific unique kinds of mappings on a ring $\mathcal{R}$ and the commutativity of $\mathcal{R}$ over the past several years. Divinsky [5] is responsible for the first accomplishment in this direction by demonstrated that if an automorphism of an artinian ring is nontrivial and commuting, then it must be commutative. Divinsky's argument was extended to prime rings by Luh [8]. Mayne [10] demonstrated that there must be a commutative prime rings, if it owns a non-identity centralizing automorphism. These findings have now been extrapolated in other areas. Posner [12] confirmed that the commutative structure of a prime ring must exist once a derivation occurs on it, which is centralizing and nonzero. Numerous researchers Like Bresar, Luh, Mayne, Kharchenko, Vukman etc. have since modified and refined these results in different directions over the past few decades (see, for example, [1, 4, 6, 7, 9, 14] for further references).

Throughout a ring $\mathcal{R}$ ought to symbolize as associative along with centre $\mathcal{Z}(\mathcal{R})$ and extended centroid $\mathcal{C}(\mathcal{R})$, the centre of over ring $\mathcal{Q}(\mathcal{R})$. Set n to be a constant positive integer. If $nb = 0$ implies $b = 0$ for each $b \in \mathcal{R}$, then $\mathcal{R}$ is termed as n-torsion-free ring.

The commutator of $b, d \in \mathcal{R}$ is represented by the representation $[b, d]$ and defined by $[b, d] = bd - db$. Keep in mind that $\mathcal{R}$ is semiprime if $bRb = 0$ indicates $b = 0$, and supposed to be prime if $cRb = 0$ indicates either $c = 0$ or $b = 0$. A map ζ from $\mathcal{R}$ to $\mathcal{R}$ is referred as (skew)-centralizing on $\mathcal{R}$ if $\zeta(c)c + c\zeta(c) \in \mathcal{Z}(\mathcal{R})$ for each $c \in \mathcal{R}$. A bit more specifically, if $\zeta(c)c + c\zeta(c) = 0$ for every $c \in \mathcal{R}$, then the mapping has become known as (skew)-commuting on $\mathcal{R}$. A mapping η from $\mathcal{R}$ to $\mathcal{R}$ is said to be derivation on $\mathcal{R}$, if it fulfills $\eta(ce) = \eta(c)e + c\eta(e)$, for $c, e \in \mathcal{R}$. Let a ring $\mathcal{R}$ has automorphism be β. If $\mathrm{h}(bd) = \mathrm{h}(b)\beta(d) + b\mathrm{h}(d)$ holds for every pairs b, d in $\mathcal{R}$ and having additivity, then the mapping h on $\mathcal{R}$ will be recognized as β-derivation. If we denote identity map by $\mathcal{I}$ on $\mathcal{R}$, then the combination form like $\mathrm{h} = \beta - I$ functioned as β-derivation.

A function $\mathcal{D} : R \times \mathcal{R} \to \mathcal{R}$ is considered as having symmetry, according to Maksa [9], if $\mathcal{D}(p, q) = \mathcal{D}(q, p)$ for every p, q in $\mathcal{R}$. If a mapping $\mathcal{D}$ from $R \times R$ into R is additive in both slots, it is said to be bi-additive. The bi-derivations theory is now introduced as follows: When the map $q \mapsto \mathcal{D}(p, q)$ and the map $p \mapsto \mathcal{D}(p, q)$ are both derivations of $\mathcal{R}$, the mapping $\mathcal{D}$, additive in each tuple and having symmetric property is named as bi-derivation. For ideational reading in the related matter one can turned to [9, 15]. For a symmetric mapping $\mathcal{D}$, a function $\mathfrak{h}$ on $\mathcal{R}$ shall be called the trace of $\mathcal{D}$ stated as $\mathfrak{h}(p) = \mathcal{D}(p, p)$, $p \in \mathcal{R}$. We can construct such mappings as in example below:

Example 1.1. *Consider a ring* $\mathcal{R} = \left\{ \begin{pmatrix} l & 0 & 0 \\ t & l & 0 \\ o & p & l \end{pmatrix} \mid l, t, o, p \in \mathbb{R} \right\}$. *Then* $\mathcal{R}$ *is a non-commutative associative ring under the usual operations on matrix like addition and multiplication. Next designed a map* $\varrho : \mathcal{R} \to \mathcal{R}$ *by* $\varrho(r) = \begin{pmatrix} l & 0 & 0 \\ 0 & 0 & 0 \\ o & 0 & 0 \end{pmatrix}$ *for all* $r \in \mathcal{R}$. ϱ *must be additive function, that much is certain. Now, introduce a map* $\varpi : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ *by* $\varpi(r, e) = [r, \varrho(e)] + [e, \varrho(r)]$ *for each* $r, e \in \mathcal{R}$*, The symmetry and bi-additivity of* ϖ *can be verified with ease.*

Bresar's [3] finding, according to which each and every map of a prime ring $\mathcal{R}$, that is, additive and commuting on $\mathcal{R}$ has the following structure: $x \longmapsto \mu x + \rho(x)$, where μ a component of $\mathcal{C}$ and ρ is an additive mapping from $\mathcal{R}$ to $\mathcal{C}$. Lanski [7] generalizes the previous result and proved that: If g and f are derivations of $\mathcal{R}$ into itself, where $\mathcal{R}$ is a prime and noncommutative ring such that $[\mathrm{g}(c), \mathrm{f}(c)] = 0$ fulfills for every c in $\mathcal{R}$, then at this instance $\mathrm{g} = \mu\mathrm{f}$, for μ belongs to $\mathcal{C}(\mathcal{R})$. A next fundamental generalization in the sequel presented by Vukman [14] by stating : Assume that $\mathcal{R}$ is a semiprime ring. Let's say a derivation g from $\mathcal{R}$ to $\mathcal{R}$ and an automorphism β exist such that a function $c \longmapsto \mathrm{g}(c) + \beta(c)$ is commuting on $\mathcal{R}$. At this instance, g and $\beta - \mathcal{I}$ transforms to $\mathcal{Z}(\mathcal{R})$.

Let $\mathfrak{L}$ be an inner generalized derivation of a ring $\mathcal{R}$. That is, $\mathfrak{L} = \gamma c + c\nu$ for some fixed $\gamma, \nu \in \mathcal{R}$. Take a note that the statement " $\mathfrak{L}$ is centralizing over some subset $\mathcal{K}_1$ of $\mathcal{R}$" may serve as an alternate of the condition $[\gamma, t]t + t[t, \nu] \in \mathcal{Z}(\mathcal{R})$ for t in $\mathcal{K}_1$. On operator algebra, such mappings have been thoroughly explored. Therefore, analyzing such kinds of mapping on algebraic structures might be appealing for both algebraist and analyst.

An another interesting research finding in [4] states that: let a semiprime ring $\mathcal{R}$ and g be a commuting γ-derivation on it. Then $[b, d]\mathrm{g}(l) = \mathrm{g}(l)[b, d] = 0$ for every $b, d, l \in \mathcal{R}$. Particularly, g transfer from $\mathcal{R}$ to $\mathcal{Z}(\mathcal{R})$. Authors examine some features of γ-derivation on prime and semiprime rings, as noted in [13]. In the same article, authors define several identities for a γ-derivation g that commutes on a semiprime ring $\mathcal{R}$, demonstrate that g transfers into $\mathcal{Z}(\mathcal{R})$ from $\mathcal{R}$. In order to figure out a functional equation of automorphisms on special ring structures, Posner's theorem on the composition of derivations for γ-derivations is extended in this way.

Motivated by the literature review cited above, we study and examine identities combining bi-derivations and automorphisms on (semi)prime rings. The goal of our research is to bring out the conclusion: Let $\mathcal{R}$ be a semiprime ring with 2-torsion freeness, a nonzero ideal be $\mathcal{K}$ of $\mathcal{R}$ and $\mathcal{D}$ be a bi-derivation on $\mathcal{R}$. If $[\mathcal{D}(j, j) + \beta(j), j] = 0$ for every j in $\mathcal{K}$ and an automorphism β on $\mathcal{R}$, then one of these conditions is fulfilled:

(1) $\mathcal{D} = 0$ on $\mathcal{R}$.
(2) A nonzero central ideal contained in $\mathcal{R}$.

Moreover, we establish the structure of β, in case of prime ring.

2. MAIN THEOREMS

We start by listing the following lemmas that the next paragraph will require.

Lemma 2.1. [3] *Let an additive mapping* $\mathrm{f} : \mathcal{R} \longrightarrow \mathcal{R}$ *be centralizing on* $\mathcal{R}$, *a semiprime ring having* 2-*torsion freeness. Subsequently* , f *will be commuting on* $\mathcal{R}$.

Lemma 2.2. [3] *If any map* $\mathrm{f} : \mathcal{R} \longrightarrow \mathcal{R}$ *on a prime ring is commuting and additive on* $\mathcal{R}$, *then there exists* ω *consisting of the form* $\mathrm{f}(r) = \omega r + \zeta(r)$ *for every* $r \in \mathcal{R}$ *for* $\zeta : \mathcal{R} \longrightarrow \mathcal{C}$, *an additive map.*

Lemma 2.3. [8] *Let a ring* $\mathcal{R}$ *be semiprime and* k *be a fixed component in* $\mathcal{R}$. *If* $k[b, d] = 0$ *for every* $b, d \in \mathcal{R}$, *then an ideal* $\mathcal{K}$ *emerges in* $\mathcal{R}$ *such as* $k \in \mathcal{K} \subseteq \mathcal{Z}(\mathcal{R})$.

Lemma 2.4. [14] *Assuming* $\mathcal{R}$ *is a semiprime ring with* 2-*torsion freeness and let an additive map be* f *from* $\mathcal{R}$ *to* $\mathcal{R}$. *If either* $\mathrm{f}(c)c = 0$ *or* $c\mathrm{f}(c) = 0$ *applies for every* $c \in \mathcal{R}$, *then* $\mathrm{f} = 0$.

The lemmas listed below is the refinement of Lemma 2.4.

Lemma 2.5. *Let* $\mathcal{R}$ *be a semiprime ring holding* 2-*torsion freeness, and* $\mathcal{D}$ *be a bi-derivation on* $\mathcal{R}$. *If* $\mathcal{D}(j, j)j = 0$ *(or* $j\mathcal{D}(j, j) = 0$*) for every* j *in* $\mathcal{R}$, *then* $\mathcal{D} = 0$ *on* $\mathcal{R}$.

Proof. By given condition, we have

$$\mathcal{D}(j,j)j = 0 \text{ for each } j \in \mathcal{R}. \tag{2.1}$$

Linearizing (2.1) to obtain

$$\mathcal{D}(j,j)c + \mathcal{D}(c,c)j + 2\mathcal{D}(j,c)j + 2\mathcal{D}(j,c)c = 0 \text{ for each } j,c \in \mathcal{R}. \tag{2.2}$$

Put $-j$ for j in the previous equation, in order to get

$$\mathcal{D}(j,j)c - \mathcal{D}(c,c)j + 2\mathcal{D}(j,c)j - 2\mathcal{D}(j,c)c = 0 \text{ for each } j,c \in \mathcal{R}. \tag{2.3}$$

Summing up (2.2) and (2.3), we get after applying torsion of $\mathcal{R}$

$$\mathcal{D}(j,j)c + 2\mathcal{D}(j,c)j = 0 \text{ for each } j,c \in \mathcal{R}. \tag{2.4}$$

Fill in tc for c in (2.4) to find

$$\mathcal{D}(j,j)tc + 2t\mathcal{D}(j,c)j + 2\mathcal{D}(j,t)cj = 0 \text{ for each } j,c,t \in \mathcal{R}. \tag{2.5}$$

Associating (2.4) and (2.5) with torsion restriction to attain

$$t\mathcal{D}(j,c)j + \mathcal{D}(j,t)[c,j] = 0 \text{ for each } j,c,t \in \mathcal{R}. \tag{2.6}$$

Rephrase the above equation in pt for t and use (2.6) to have

$$\mathcal{D}(j,p)t[c,j] = 0 \text{ for each } j,c,t,p \in \mathcal{R}. \tag{2.7}$$

This indicates a possibility that $\mathcal{D}(j,p)t[c,j] = 0$ for every $j,c,t,p \in \mathcal{R}$. Some suitable replacement for t in last expression hints that $\mathcal{D}(j,p)[c,j]t\mathcal{D}(j,p)[c,j] = 0$ for every $j,c,t,p \in \mathcal{R}$. Significance of the fact that $\mathcal{R}$ is semiprime, we have $\mathcal{D}(j,p)[c,j] = 0$ for $j,c,p \in \mathcal{R}$. Utilizing a lemma 2.3 to achieve $\mathcal{D}(j,p) \subseteq \mathcal{Z}(\mathcal{R})$ for $j,p \in \mathcal{R}$. Since $\mathcal{D}(j,j)j = 0$ and $j \neq 0$, we conclude $\mathcal{D}(j,p) = 0$ for $j,p \in \mathcal{R}$.

Corollary 2.6. *Let a ring $\mathcal{R}$ be semiprime with 2-torsion freeness, and $\mathcal{D}$ be a bi-derivation on $\mathcal{R}$. If $\mathcal{D}(j,j)j = 0$ (or $j\mathcal{D}(j,j) = 0$) for every j in $\mathcal{R}$, then $\mathcal{D}$ transfers into $\mathcal{Z}(\mathcal{R})$ from $\mathcal{R}$.*

Corollary 2.7. *Let a ring $\mathcal{R}$ be prime possess $char(R) \neq 2$, and $\mathcal{D}$ be a bi-derivation on $\mathcal{R}$. If $\mathcal{D}(w,w)w = 0$ (or $w\mathcal{D}(w,w) = 0$) for every w in $\mathcal{R}$, then one of these conditions is fulfilled:*

(1) $\mathcal{D} = 0$ *over* $\mathcal{R}$.
(2) $\mathcal{R}$ *is a commutative ring.*

Theorem 2.8. *Let $\mathcal{R}$ be a semiprime ring with 2-torsion freeness, $\mathcal{K}$ be a nonzero ideal of $\mathcal{R}$ and $\mathcal{D}$ be a bi-derivation on $\mathcal{R}$. If $[\mathcal{D}(j,j) + \beta(j), j] = 0$ for every j in $\mathcal{K}$ and an automorphism β on $\mathcal{R}$, then one of these conditions is fulfilled:*

(1) $\mathcal{D} = 0$ *over* $\mathcal{R}$.
(2) *A nonzero central ideal contained in* $\mathcal{R}$.

Proof. By prescribed condition, we have

$$[\mathcal{D}(j,j), j] + [\beta(j), j] = 0 \text{ for each } j \in \mathcal{K}. \tag{2.8}$$

Linearizing (2.8) and using (2.8), we find

$$\begin{aligned} &[\mathcal{D}(j,j), d] + [\mathcal{D}(d,d), j] + 2[\mathcal{D}(j,d), j] + 2[\mathcal{D}(j,d), d] \\ &+[\beta(j), d] + [\beta(d), j] = 0 \text{ for each } j,d \in \mathcal{K}. \end{aligned} \tag{2.9}$$

Rewrite (2.9) by interchanging $-j$ in place of j to get

$$\begin{aligned} &[\mathcal{D}(j,j), d] - [\mathcal{D}(d,d), j] + 2[\mathcal{D}(j,d), j] - 2[\mathcal{D}(j,d), d] \\ &-[\beta(j), d] - [\beta(d), j] = 0 \text{ for each } j,d \in \mathcal{K}. \end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10) to obtain

$$[\mathcal{D}(j,j),d]+2[\mathcal{D}(j,d),j]=0 \text{ for each } j,d\in\mathcal{K}. \tag{2.11}$$

Substitute ds for d in above equation to get

$$\begin{aligned} &d[\mathcal{D}(j,j),s]+[\mathcal{D}(j,j),d]s+2[\mathcal{D}(j,d),j]s+2\mathcal{D}(j,d)[s,j]\\ &2d[\mathcal{D}(j,s),j]+2[d,j]\mathcal{D}(j,s)=0 \text{ for each } j,d\in\mathcal{K},\ s\in\mathcal{R}. \end{aligned} \tag{2.12}$$

Making use of (2.11) and applying torsion condition in (2.12), we notice that

$$\mathcal{D}(j,d)[s,j]+[d,j]\mathcal{D}(j,s)=0 \text{ for each } j,d\in\mathcal{K},\ s\in\mathcal{R}. \tag{2.13}$$

Particularly, last equation takes the form as below

$$[d,j]\mathcal{D}(j,j)=0 \text{ for each } j,d\in\mathcal{K}. \tag{2.14}$$

Semiprimeness of $\mathcal{R}$ granted the existence of a family of prime ideals say $\mathfrak{P}=\{\mathcal{P}_i \mid i\in\ltimes\}$ such that $\bigcap\mathcal{P}_i=\{0\}$. Let us suppose that $\mathcal{P}_m$ and $\mathcal{P}_l$ are typical member of $\mathfrak{P}$. By (2.14), we have $[j,d]\in\mathcal{P}_m$ and $\mathcal{D}(j,j)\in\mathcal{P}_l$ for all $j,d\in\mathcal{K}$. Now designed the two subsets as $\mathrm{A}=\{d\in\mathcal{K}\mid [j,d]\subseteq\mathcal{P}_m\}$ and $\mathrm{C}=\{j\in\mathcal{K}\mid \mathcal{D}(j,j)\subseteq\mathcal{P}_l\}$. We observe that both A and C are additive subgroup of $\mathcal{R}$ such that $\mathcal{R}=\mathrm{A}\bigcup\mathrm{C}$. Being the property that a group cannot consist of joint of its appropriate subgroups. As a result, we determine either $\mathcal{R}=\mathrm{A}$ or $\mathcal{R}=\mathrm{C}$. First, take the situation $\mathcal{R}\neq\mathrm{A}$, this yields that $\mathcal{R}=\mathrm{C}$. That is, $\mathcal{D}(j,j)\in\mathcal{P}_l$ for all $j\in\mathcal{K}$. A simple manipulation gives that $\mathcal{D}(j,j)t\in\mathcal{P}_l$ for all $j\in\mathcal{K}$ and $t\in\mathcal{R}$. Using primeness of $\mathcal{P}_l$, we find either $\mathcal{D}(j,j)\in\mathcal{P}_l$ or $t\in\mathcal{P}_l$ for each $j\in\mathcal{K}$ and $t\in\mathcal{R}$. If $t\in\mathcal{P}_l$, then $[t,\mathcal{R}]\subseteq\mathcal{P}_l$, a contradiction occur to our expectation $\mathcal{R}\neq\mathrm{A}$. Therefore, we have $\mathcal{D}(j,j)\in\mathcal{P}_l$ for all $j\in\mathcal{R}$. Hence we get $\mathcal{D}(j,j)\subseteq\cap\mathcal{P}_l=\{0\}$ for each $j\in\mathcal{R}$. This implies that $\mathcal{D}(j,j)=0$ for every $j\in\mathcal{R}$. Similarly, we discard the case when $\mathcal{R}\neq\mathrm{C}$ and we obtain $\mathcal{R}=\mathrm{A}$. This implicit that $[j,d]=0$ for each $j,d\in\mathcal{K}$. Hence $\mathcal{R}$ owns an ideal contained in itself, which is central and nonzero.

Corollary 2.9. *Let a ring $\mathcal{R}$ be semiprime with 2-torsion freeness and $\mathcal{D}$ be a bi-derivation over $\mathcal{R}$. If $[\mathcal{D}(j,j)+\beta(j),j]=0$ for every j in $\mathcal{R}$ and an automorphism β on $\mathcal{R}$, then $\mathcal{D}$ and $\beta-\mathfrak{I}$, maps into $\mathcal{Z}(\mathcal{R})$.*

Proof. The detailed proof of this corollary presented in [14].

Corollary 2.10. *Let $\mathcal{R}$ be a noncommutative prime ring possess characteristic not 2, If $[\mathcal{D}(v,v)+\beta(v),v]=0$ for every v in $\mathcal{K}$ and an automorphism β on $\mathcal{R}$, then $\mathcal{D}$ and β contained in $\mathcal{Z}(\mathcal{R})$.*

Our next Theorem is the conclusion of the previous study presented in [3, 13, 14].

Theorem 2.11. *Let $\mathcal{R}$ be a prime ring possess characteristic not 2, If $[\mathcal{D}(b,b)+\beta(b),b]=0$ for every b in $\mathcal{K}$ and an automorphism β on $\mathcal{R}$, then one of these conditions is fulfilled:*

(1) $\mathcal{D}=0$ *on* $\mathcal{R}$.
(2) $\mathcal{R}$ *is a commutative ring.*
(3) *In case* $\mathcal{D}=0$, $\beta(b)$ *has substructure as* $\beta(r)=\omega r+\zeta(r)$ *for every* $r\in\mathcal{R}$, $\omega\in\mathcal{C}$ *and an additive mapping* $\zeta:\mathcal{R}\longrightarrow\mathcal{C}$.

Proof. The proof is straightforward by using Theorem 2.8 and Lemma 2.1, 2.2.

Example 2.1. *The ring $\mathcal{R}=\left\{\begin{pmatrix} l & 0\\ m & n\end{pmatrix} \mid l,m,n\in\mathbb{R}\right\}$ is not a prime ring. Take $b=\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}\in\mathcal{R}$ and an automorphisms $\gamma(c)=bcb^{-1}=\begin{pmatrix} l & 0\\ -m & n\end{pmatrix}$ for $c\in\mathcal{R}$. Designed a* *bi-additive function* $\mathcal{D} : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ *as*

$$\mathcal{D} = \left\{ \begin{pmatrix} l & 0 \\ m & n \end{pmatrix}, \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 \\ mq & 0 \end{pmatrix}.$$

Then $\mathcal{R}$ *owns a bi-derivation say* $\mathcal{D}$*. The preceding theorem's condition is simple to test the condition* $[\mathcal{D}(z,z) + \beta(z), z] = 0$ *for every* $z = \begin{pmatrix} l & 0 \\ m & l \end{pmatrix}$ *in* $\mathcal{R}$*. However, neither* $\mathcal{D} = 0$ *nor* $\mathcal{R}$ *is going to be commutative, emphasizing the significance of* $\mathcal{R}$ *being a prime.*

Theorem 2.12. *Let* $\mathcal{R}$ *be a semiprime ring with* 2*-torsion freeness and* $\mathcal{D}$ *be a bi-derivation on* $\mathcal{R}$*. If* $\mathcal{D}(q,q)q + q(\beta(q) - q) = 0$ *for every* q *in* $\mathcal{R}$ *and an automorphism* β *on* $\mathcal{R}$*, then one of these conditions is fulfilled:*

(1) $\mathcal{D} = 0$ *on* $\mathcal{R}$*.*
(2) $\beta = \mathfrak{I}$*, act as identity operator.*

Proof. We are given that

$$\mathcal{D}(b,b)b + b(\beta(b) - b) = 0 \text{ for each } b \in \mathcal{R}. \tag{2.15}$$

Linearizing (2.15) and using (2.15), we find

$$\begin{aligned} &\mathcal{D}(b,b)d + 2\mathcal{D}(b,d)b + \mathcal{D}(d,d)b + 2\mathcal{D}(b,d)d \\ &b\beta(d) + d\beta(b) = 0 \text{ for each } b, d \in \mathcal{R}. \end{aligned} \tag{2.16}$$

Rearrange (2.16) by putting $-b$ in place of b to get

$$\begin{aligned} &\mathcal{D}(b,b)d + 2\mathcal{D}(b,d)b - \mathcal{D}(d,d)b - 2\mathcal{D}(b,d)d \\ &-b\beta(d) - d\beta(b) = 0 \text{ for each } b, d \in \mathcal{R}. \end{aligned} \tag{2.17}$$

Adding (2.16) and (2.17) and applying torsion of $\mathcal{R}$ to obtain

$$\mathcal{D}(b,b)d + 2\mathcal{D}(b,d)b = 0 \text{ for each } b, d \in \mathcal{R}. \tag{2.18}$$

Fill in ds for d in (2.18) to gain

$$\mathcal{D}(b,b)ds + 2\mathcal{D}(b,d)sb + 2d\mathcal{D}(b,s)b = 0 \text{ for each } b, d, s \in \mathcal{R}. \tag{2.19}$$

Correlating the equations (2.18) and (2.19) to get

$$-2\mathcal{D}(b,d)bs + 2\mathcal{D}(b,d)sb + 2d\mathcal{D}(b,s)b = 0 \text{ for each } b, d, s \in \mathcal{R}. \tag{2.20}$$

This implies that

$$\mathcal{D}(b,d)[s,b] + d\mathcal{D}(b,s)b = 0 \text{ for each } b, d, s \in \mathcal{R}. \tag{2.21}$$

Particularly, we get

$$d\mathcal{D}(b,b)b = 0 \text{ for each } b, d, s \in \mathcal{R}. \tag{2.22}$$

Some basic replacement yielding us that $\mathcal{D}(b,b)b\mathcal{R}\mathcal{D}(b,b)b = 0$ for each $b \in \mathcal{R}$. Being $\mathcal{R}$ Semiprime, means that $\mathcal{D}(b,b)b = 0$ for each $b \in \mathcal{R}$. Proceed with Lemma 2.5 to get $\mathcal{D}(b,b) = 0$ for each $b \in \mathcal{R}$. From (2.15), we hold the condition $z(\beta(z) - z) = 0$ for each z in $\mathcal{R}$. We define a mapping $\varrho(z) = \beta(z) - z$ for each $z \in \mathcal{R}$. Clearly, ϱ is an additive mapping, Lemma 2.4 came into action to gives us $\varrho(b) = \beta(b) - b = 0$ for each $b \in \mathcal{R}$. Therefore, we say $\beta(z) - \mathfrak{I}(z) = 0)$ for each z in $\mathcal{R}$. Hence, $\beta = \mathfrak{I}$, as desired.

In our next investigation, we consider the more general case of skew-commuting identities involving with automorphisms and bi-derivations.

Theorem 2.13. *Let $\mathcal{R}$ be a semiprime ring with 2-torsion freeness and $\mathcal{D}$ be a bi-derivation on $\mathcal{R}$. If $[\mathcal{D}(b,b)b+b(\beta(b)-b),b]=0$ for every b in $\mathcal{R}$ and an automorphism β on $\mathcal{R}$, then one of these conditions is fulfilled:*

(1) $\mathcal{D}=0$ *on* $\mathcal{R}$.
(2) $\beta=\mathfrak{I}$, *act as identity operator.*

Proof. We omit the proof because it uses the same linearization arrangements, even function property like $\mathcal{D}(-b,c)=-\mathcal{D}(b,c)$ for $b,c\in\mathcal{R}$ and the techniques applying in Theorem 2.12.

Corollary 2.14. *Let $\mathcal{R}$ be a non commutative prime ring with $Char(\mathcal{R})\neq 2$ and $\mathcal{D}$ be a bi-derivation on $\mathcal{R}$. If $[\mathcal{D}(b,b)b+b(\beta(b)-b),b]=0$ for every b in $\mathcal{R}$ and an automorphism β on $\mathcal{R}$, then one of these conditions is fulfilled:*

(1) $\mathcal{D}=0$ *on* $\mathcal{R}$.
(2) $\beta=\mathfrak{I}$, *act as identity operator.*

3. CONCLUSION AND CONJECTURE

For two automorphisms β,γ of $\mathcal{R}$ involved with β-derivations and γ-derivations over a ring $\mathcal{R}$ with $char(R)\neq 2$ and prime, and for looking application, we are enable to derive the solution of functional equation

$$\beta+\gamma^{-1}\beta^{-1}\gamma=\gamma+\gamma_{-1}.$$

In [4], a remarkable conclusion has been drawn for above equation. Some other conclusion also be considered as application for nontrivial mappings, which has image in $\mathcal{Z}(\mathcal{R})$. We suggest the reader to look in [6] for further details on the functional equation mentioned above, as well as exclusive information and references.

For Future research we leave an open discussion in this section for readers “To find out the nontrivial solution of functional identities involving with additive and n-additive mappings on different algebraic structures." In my opinion, It would be more appealing to look the behavior of self adjoint operators on algebraic spaces.

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