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SOLVING STRONGLY NONLINEAR FRACTIONAL FREDHOLM INTEGRAL-DIFFERENTIAL EQUATIONS IN CAPUTO'S SENSE USING THE SBA METHOD

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ABSTRACT. The work addressed in this article consists in constructing the exact solutions, where they exist, of fractional Fredholm-type integro-differential equations in the sense of Caputo. Our results are obtained using the SBA method. The simplification of the approach, the analysis of its convergence, and the generalization of this method to these types of highly non-linear equations constitute our scientific contribution.

Key words and phrases: SOME BLAISE ABBO (SBA) method; Riemann-Liouville fractional integral; Caputo fractional derivative; fractional Fredholm integro-differential equations in the sense of Caputo.

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1. INTRODUCTION

Fractional derivation is one of the most widely used mathematical tools in modeling today. But due to its non-local structure and memory effect, it is a high information storage. As a result, its use in the construction of mathematical models for a given phenomenon comes with a high cost in terms of numerical resolution. When using a discretization algorithm for non-integer derivatives, this structure must be taken into account, resulting in high algorithm complexity. Most existing classical numerical methods also come up against difficulties linked to its non-linearity. There is therefore a strong interest in developing new methods for solving nonlinear functional equations of fractional order. In this work, we propose an iterative method, the SOME BLAISE ABBO (SBA) method, capable of taking into account the complex structure of fractional derivation and easily handling nonlinearity.

Our major contribution to this work is twofold. On the one hand, we simplify the method's approach, when the approximate solution at the first iteration is a root of nonlinearity ($Nu^1 = 0$). While giving a sufficient condition for the convergence of the approximate solution at the first iteration, we propose a new analysis of the method's convergence. On the other hand, we demonstrate the strength of the SBA method in a very important aspect. In the SBA approach, when the solution approximated at the first iteration doesn't cancel out the nonlinearity (i.e. $Nu^1 \neq 0$), we replace the initial problem by an equivalent transformation, with $\overline{N}$ the new nonlinear term, so that by repeating the algorithm we can obtain $\overline{N}u^1 = 0$. Here, this work shows that when $Nu^1 \neq 0$, we can by successive iterations determine the convergent series of general term u^k and deduce the solution to the problem.

After recalling some basic notions of fractional calculus in Section 2 and describing the SBA method in Section 3, we devote Section 4 to illustrating the method's effectiveness on a few examples of fractional Fredholm integro-differential equations in the Caputo sense. Section 5 is the conclusion.

2. PRELIMINAIRES

Most of the definitions and properties we present for our work can be found in [9, 15, 17, 22]. We invite the reader to refer to them for further details.

2.1. Fractional integral in the Riemann-Liouville sense.

Definition 2.1. : Riemann-Liouville fractional integral
Let $f \in C([0; +\infty[)$. The fractional Riemann-Liouville integral (on the left) of order $\alpha \geq 0$ of the function f denoted $\mathcal{I}^\alpha f$ is defined by:

$$\begin{cases} \mathcal{I}^\alpha f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, t > 0. \\ \mathcal{I}^0 f(t) = f(t) \end{cases} \tag{2.1}$$

where $\Gamma(\alpha)$ is the Gamma function.

2.2. Fractional derivation in the Caputo sense.

Definition 2.2. : Fractional derivative in the sense of Caputo
Let $f \in C^m([0; +\infty[)$, $\alpha > 0$ and $n = [\alpha] + 1$. The fractional derivative in Caputo's sense (on

the left) of order α of the function f denoted ${}^{c}\mathcal{D}^{\alpha} f$ is defined by:

(2.2)
$$\begin{cases} {}^{c}\mathcal{D}^{\alpha} f(t) = \mathcal{I}^{n-\alpha} \circ \dfrac{d^n}{dt^n} f(t) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ if \ n-1 < \alpha < n \\ {}^{c}\mathcal{D}^{\alpha} f(t) = \dfrac{d^n}{dt^n} f(t), \ if \ \alpha = n \end{cases}$$

where $[\alpha]$ denotes the integer part of α.

Proposition 2.1. *Let* $\alpha > 0, n = [n] + 1$ *and* $f \in AC^n([a,b])$

$$ {}^{c}\mathcal{D}^{\alpha} \, \mathcal{I}^{\alpha} f(t) = f(t); \tag{2.3}$$

$$\mathcal{I}^{\alpha} \, {}^{c}\mathcal{D}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} f^{(k)}(a). \tag{2.4}$$

3. SBA METHOD

3.1. **Description of the SBA method applied to nonlinear fractional Fredholm integro-differential equations.** Consider the following nonlinear fractional Fredholm-type integro-differential equation:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha} u(x) = g(x) + \displaystyle\int_0^1 K(x,t) F(u(t)) dt \\ u^{(i)}(0) = b_i, (i = 0, 1, 2, ..., h-1) \end{cases} \tag{3.1}$$

in a suitable functional space V, where ${}^{c}\mathcal{D}^{\alpha}$ is the fractional derivative of Caputo of order $\alpha > 0$, F and K are continuous functions defined in V, F is a non-linear operator and u is the unknown function defined in V; $h = [\alpha] + 1$ where $[\alpha]$ is the integer part of α

By posing

$$F = L + N \tag{3.2}$$

where L is a linear operator and N is a non-linear operator, equation (3.1) becomes:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha} u(x) = g(x) + \displaystyle\int_0^1 K(x,t) L(u(t)) dt + \int_0^1 K(x,t) N(u(t)) dt \\ u^{(i)}(0) = b_i, (i = 0, 1, 2, ..., h-1). \end{cases} \tag{3.3}$$

By composing the two members of the first line of (3.3) by $\mathcal{I}^{\alpha}(.)$, we obtain thanks to the proposition (2.1) and the following relation:

$$u(x) = \theta + \mathcal{I}^{\alpha}(g(x)) + \mathcal{I}^{\alpha}\left(\int_0^1 K(x,t) L(u(t)) dt\right) + \mathcal{I}^{\alpha}\left(\int_0^1 K(x,t) N(u(t)) dt\right) \tag{3.4}$$

where

$$\theta = \sum_{i=0}^{h-1} \frac{x^i}{i!} f_i. \tag{3.5}$$

Using the successive approximation method, the first line of (3.1) becomes:

$$u^k(x) = \theta^k + \mathcal{I}^{\alpha}(g(x)) + \mathcal{I}^{\alpha}\left(\int_0^1 K(x,t) L(u^k(t)) dt\right) + \mathcal{I}^{\alpha}\left(\int_0^1 K(x,t) N(u^{k-1}(t)) dt\right), k \geq 1.$$

Assuming $u^k = \sum_{n=0}^{+\infty} u_n^k$, we derive the following SBA algorithm:

$$\begin{cases} u_0^k = \theta^k + \mathcal{I}^\alpha(g(x)) + \mathcal{I}^\alpha\left(\int_0^1 K(x,t)N(u^{k-1}(t))dt\right) \\ u_{n+1}^k = \mathcal{I}^\alpha\left(\int_0^1 K(x,t)L(u^k(t))dt\right) \end{cases}, k \geq 1, n \geq 0. \tag{3.6}$$

Explicitly, the development of the 3.6 algorithm consists in first calculating the terms of the sequence $(u_n^k)_n$ for $k \geq 1$ fixed, and deducing u^k if the series $\sum_{n=0}^{+\infty} u_n^k$ converges.
So for the first iteration, $k = 1$, we choose u^0 such that $Nu^0 = 0$, calculate the terms $u_0^1, u_1^1, u_2^1, u_3^1 ... u_n^1$ of the sequence $(u_n^1)_n$ and deduce

$$u^1 = \sum_{n=0}^{+\infty} u_n^1; \tag{3.7}$$

Then we evaluate Nu^1. If $Nu^1 = 0$ then u^1 is the general solution of the problem. Otherwise, we can:

- calculate $u^2, u^3, u^4, ..., u^k$ and deduce $\lim_{k \longrightarrow +\infty} u^k$;
- if possible, replace the initial problem by an equivalent transformation, with $\overline{N}$ the new nonlinear term, so that by repeating the algorithm we can obtain $\overline{N}u^1 = 0$.

3.2. **Convergence of the SBA method.** By posing

$$\begin{cases} L =^c \mathcal{D}^\alpha(.); L^{-1}(.) = \mathcal{I}^\alpha(.); p^k = \theta^k + \mathcal{I}^\alpha(g(x)); \\ Ru_n^k = \int_0^1 K(x,t)L(u_n^k(t))dt \\ Nu^{k-1} = \int_0^1 K(x,t)N(u^{k-1}(t))dt \end{cases} \tag{3.8}$$

the algorithm **SBA** defined by (3.6) is written:

$$\begin{cases} u_0^k = p^k + L^{-1}Nu^{k-1} \\ u_{n+1}^k = L^{-1}Ru_n^k \end{cases}, k \geq 1, n \geq 0; \tag{3.9}$$

with the assumption that the operator $L^{-1}R$ is contracting.
For convergence, we need to show that:

(1) if $L^{-1}R$ is contracting then the series $\sum_{n=0}^{+\infty} u_n^1$ converges;
(2) if $Nu^1 = 0$ then the solution of the problem is u^1.

Let's show that if $L^{-1}R$ is contracting, then the series $\sum_{n=0}^{+\infty} u_n^1$ converges.
For $k = 1$ and for a good choice of u^0 such that $Nu^0 = 0$ and assuming $\Psi = -L^{-1}R$, the

algorithm (3.9) is written:

$$\begin{cases} u_0^1 = p^1 \\ u_{n+1}^1 = \Psi(u_n^1) \end{cases} \quad or \quad \begin{cases} u_0^1 = p^1 \\ u_1^1 = \Psi(u_0^1) \\ \vdots \\ u_{n+1}^1 = \Psi(u_n^1) \end{cases} \tag{3.10}$$

We can see that (3.10) can be reduced to the search for a sequence (S_n) defined by

$$\begin{cases} S_0 = u_0^1 \\ S_n = u_0^1 + u_1^1 + u_2^1 + u_3^1 + ... + u_n^1 \end{cases} \tag{3.11}$$

and verifying the recurrence relation:

$$\begin{cases} S_0 = u_0^1 \\ S_{n+1} = \Psi(S_n), n \geq 0. \end{cases} \tag{3.12}$$

Theorem 3.1. *If the operator Ψ is contracting (i.e. its norm verifies $||\Psi|| < \lambda < 1$) then the sequence $(S_n)_{n\in\mathbb{N}}$ satisfying the recurrence relation*

$$\begin{cases} S_0 = u_0^1 \\ S_{n+1} = \Psi(S_n), n \geq 0 \end{cases}$$

converges to S where S is the solution of the equation $S = \Psi(S)$ and the series $\sum_{n=0}^{+\infty} u_n^1$ is convergent.

Proof: The proof of this theorem as well as the demonstration of the second part of convergence can be found in our previous article [11].

4. APPLICATIONS

Example 4.1.

$$\begin{cases} {}^c\mathcal{D}^\alpha u(x) = g(x) + \int_0^1 xtNu(t)dt \\ u(0) = 0, \end{cases} \tag{4.1}$$

where $x \geq 0$; $0 < \alpha \leq 1$; $u \in C^1([0;T])$; ${}^c\mathcal{D}^\alpha(.)$ the derivative in Caputo's sense; $\mathcal{I}^\alpha(.)$ the integral in Riemann-Liouville's sense, $g(x) = \dfrac{6x^{3-\alpha}}{\Gamma(4-\alpha)} - \dfrac{1}{8}x; Nu(x) = u^2(x)$.

From the above description, we derive the following SBA algorithm:

$$\begin{cases} u_0^k = u^k(0,x) + \mathcal{I}^\alpha(g(x)) + \mathcal{I}^\alpha\left(\int_0^1 xtNu^{k-1}dt\right) \\ u_{n+1}^k = 0 \end{cases}, k \geq 1, n \geq 0. \tag{4.2}$$

Calculation of u^k

For $k = 1$, (4.2) becomes:

$$\begin{cases} u_0^1 = \mathcal{I}^\alpha(g(x)) + \mathcal{I}^\alpha\left(\int_0^1 xtNu^0dt\right) \\ u_{n+1}^1 = 0 \end{cases}, n \geq 0. \tag{4.3}$$

Let's take u^0 such that $\overline{N}u^0 = 0$ and expand (4.3). We obtain:

$$u^1 = x^3 + \psi_1 x^{\alpha+1}, \quad with\ \psi_1 = -\frac{1}{8\Gamma(\alpha+2)}. \tag{4.4}$$

For $k = 2$, we have

$$\begin{aligned} u^2 &= x^3 + \psi_1(1-8\delta_1)x^{\alpha+1} \\ &= x^3 + \psi_2 x^{\alpha+1}, \end{aligned}$$

with $\delta_1 = \frac{1}{8} + \frac{2\psi_1}{\alpha+6} + \frac{\psi_1^2}{2\alpha+4}$ and $\psi_2 = \psi_1(1-8\delta_1)$.
For $k = 3$, we have

$$\begin{aligned} u^3 &= x^3 + \psi_1(1-8\delta_2)x^{\alpha+1} \\ &= x^3 + \psi_3 x^{\alpha+1} \end{aligned}$$

with $\delta_2 = \frac{1}{8} + \frac{2\psi_2}{\alpha+6} + \frac{\psi_2^2}{2\alpha+4}$ and $\psi_3 = \psi_1(1-8\delta_2)$.
Recursively, we find

$$u^k = x^3 + \psi_k x^{\alpha+1}, \tag{4.5}$$

where

$$\begin{cases} \psi_k = \psi_1(1-8\rho_{k-1}) \\ \delta_{k-1} = \frac{1}{8} + \frac{2\psi_{k-1}}{\alpha+6} + \frac{\psi_{k-1}^2}{2\alpha+4} \\ \delta_0 = 0. \end{cases} \tag{4.6}$$

By doing a formal calculation on matlab we find that ψ_k tends to 0 when k tends to $+\infty$.

α	0.75	0.5	0.25
ψ_1	-0.0777	-0.0940	-0.1103
ψ_2	-0.0136	-0.0204	-0.0288
ψ_3	-0.0025	-0.0047	-0.0080
ψ_4	-0.0004	-0.0011	-0.0022
ψ_5	-0.0000	-0.0002	-0.0006
ψ_6	-0.0000	-0.0000	-0.0001
$\vdots$	$\vdots$	$\vdots$	$\vdots$
ψ_{10}	0	0	0

Therefore, the exact solution of the problem is :

$$u = x^3. \tag{4.7}$$

Example 4.2.

$$\begin{cases} {}^c\mathcal{D}^\alpha u(x) = g(x) - \int_0^1 xtu(t)dt + \int_0^1 xtNu(t)dt \\ u(0) = 0, \end{cases} \tag{4.8}$$

where $x \geq 0$; $0 < \alpha \leq 1$; $u \in C^1([0;T])$; ${}^c\mathcal{D}^\alpha(.)$ the derivative in Caputo's sense; $\mathcal{I}^\alpha(.)$ the integral in Riemann-Liouville's sense, $g(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{12}x; Nu(x) = u^2(x)$.

From the approach described above, we derive the following SBA algorithm:

$$\begin{cases} u_0^k = u^k(0) + \mathcal{I}^\alpha(g(x)) + \mathcal{I}^\alpha\left(\int_0^1 xtNu^{k-1}(t)dt\right) \\ u_{n+1}^k = -\mathcal{I}^\alpha\left(\int_0^1 xtu^k(t)dt\right) \end{cases}, k \geq 1, n \geq 0. \tag{4.9}$$

Calculation of u^k
For $k = 1$, (4.9) becomes:

$$\begin{cases} u_0^1 = u^1(0,x) + \mathcal{I}^\alpha(g(x)) + \mathcal{I}^\alpha\left(\int_0^1 xtNu^0(t)dt\right) \\ u_{n+1}^1 = -\mathcal{I}^\alpha\left(\int_0^1 xtu^1(t)dt\right) \end{cases}, n \geq 0 \tag{4.10}$$

Let's take u^0 such that $\overline{N}u^0 = 0$ and expand (4.10). We obtain:

$$\begin{cases} u_0^1 = x^2 + Kx^{\alpha+1} \\ u_1^1 = -12K\left(\frac{1}{4} + \frac{K}{\alpha+3}\right)x^{\alpha+1} \\ u_2^1 = \left(\frac{1}{4} + \frac{K}{\alpha+3}\right)\frac{(12K)^2}{\alpha+3}x^{\alpha+1} \\ \vdots \\ u_n^1 = -12K\left(\frac{1}{4} + \frac{K}{\alpha+3}\right)\left(\frac{-12K}{\alpha+3}\right)^{n-1}x^{\alpha+1}, n \geq 1; \end{cases} \tag{4.11}$$

with $K = \dfrac{1}{12\Gamma(\alpha+2)}$.
Then posing $M = -12K\left(\dfrac{1}{4} + \dfrac{K}{\alpha+3}\right)$ and $\lambda = -\dfrac{12K}{\alpha+3}$, we deduce

$$\begin{aligned} u^1 &= \sum_{n=0}^{+\infty} u_n^1 \\ &= x^2 + Kx^{\alpha+1} - Mx^{\alpha+1}\lim_{n\longrightarrow+\infty}\left(\frac{1-(\lambda)^n}{1-\lambda}\right) \\ &= x^2 + \left(K - \frac{M}{1-\lambda}\right)x^{\alpha+1} \\ &= x^2 + \varphi_1 x^{\alpha+1} \end{aligned}$$

with $\varphi_1 = K - \dfrac{M}{1-\lambda}$
For $k = 2$, we have:

$$\begin{aligned} u_0^2 &= \mathcal{I}^\alpha(g(x)) + \mathcal{I}^\alpha\left(\int_0^1 xtNu^1(t)dt\right) \\ &= x^2 + Kx^{\alpha+1} + 12K\left(\frac{1}{6} + \frac{2\varphi_1}{\alpha+5} + \frac{\varphi_1^2}{2\alpha+4}\right)x^{\alpha+1} \\ &= x^2 + K(1+12E_1)x^{\alpha+1} \end{aligned}$$

by posing $E_1 = \frac{1}{6} + \frac{2\varphi_1}{\alpha+5} + \frac{\varphi_1^2}{2\alpha+4}$

$$\begin{cases} u_0^2 = x^2 + K(1+12E_1)x^{\alpha+1} \\ u_1^2 = -12K\left(\frac{1}{4} + \frac{K(1+12E_1)}{\alpha+3}\right)x^{\alpha+1} \\ u_2^2 = \left(\frac{1}{4} + \frac{K(1+12E_1)}{\alpha+3}\right)\frac{(12K)^2}{\alpha+3}x^{\alpha+1} \\ \vdots \\ u_n^2 = -12K\left(\frac{1}{4} + \frac{K(1+12E_1)}{\alpha+3}\right)\left(\frac{-12K}{\alpha+3}\right)^{n-1}x^{\alpha+1}, n \geq 1 \end{cases} \tag{4.12}$$

$$\begin{aligned} u^2 &= \sum_{n=0}^{+\infty} u_n^2 \\ &= x^2 + \left(K(1+12E_1) - \left(\frac{1}{4} + \frac{K(1+12E_1)}{\alpha+3}\right)\frac{12K}{1-\lambda}\right)x^{\alpha+1} \\ &= x^2 + \varphi_2 x^{\alpha+1} \end{aligned}$$

by posing $\varphi_2 = K(1+12E_1) - \left(\frac{1}{4} + \frac{K(1+12E_1)}{\alpha+3}\right)\frac{12K}{1-\lambda}$.

For $k = 3$, we have

$$\begin{cases} u_0^3 = x^2 + K(1+12E_2)x^{\alpha+1} \\ u_1^3 = -12K\left(\frac{1}{4} + \frac{K(1+12E_2)}{\alpha+3}\right)x^{\alpha+1} \\ u_2^3 = \left(\frac{1}{4} + \frac{K(1+12E_2)}{\alpha+3}\right)\frac{(12K)^2}{\alpha+3}x^{\alpha+1} \\ \vdots \\ u_n^3 = -12K\left(\frac{1}{4} + \frac{K(1+12E_2)}{\alpha+3}\right)\left(\frac{-12K}{\alpha+3}\right)^{n-1}x^{\alpha+1}, n \geq 1 \end{cases} \tag{4.13}$$

by posing $E_2 = \frac{1}{6} + \frac{2\varphi_2}{\alpha+5} + \frac{\varphi_2^2}{2\alpha+4}$
The result is

$$\begin{aligned} u^3 &= \sum_{n=0}^{+\infty} u_n^3 \\ &= x^2 + \left(K(1+12E_2) - \left(\frac{1}{4} + \frac{K(1+12E_2)}{\alpha+3}\right)\frac{12K}{1-\lambda}\right)x^{\alpha+1} \\ &= x^2 + \varphi_3 x^{\alpha+1} \end{aligned}$$

by posing $\varphi_3 = K(1+12E_2) - \left(\frac{1}{4} + \frac{K(1+12E_2)}{\alpha+3}\right)\frac{12K}{1-\lambda}$.

Recursively, we find

$$u^k = x^2 + \varphi_k x^{\alpha+1} \tag{4.14}$$

where

$$\begin{cases} \varphi_k = K(1+12E_{k-1}) - \left(\frac{1}{4} + \frac{K(1+12E_{k-1})}{\alpha+3}\right)\frac{12K}{1-\lambda}, k \geq 1 \\ E_k = \frac{1}{6} + \frac{2\varphi_k}{\alpha+5} + \frac{\varphi_k^2}{2\alpha+4} \\ E_0 = 0. \end{cases} \tag{4.15}$$

with

$$\begin{cases} K = \frac{1}{12\Gamma(\alpha+2)} \\ \lambda = -\frac{12K}{\alpha+3} \\ M = -12K\left(\frac{1}{4} + \frac{K}{\alpha+3}\right) \end{cases}, \tag{4.16}$$

By doing a formal calculation on matlab one finds that φ_k tends to 0 when k tends to $+\infty$.

α	0.75	0.5	0.25
φ_1	0.1925	0.2286	O.2628
φ_2	0.0393	0.0579	0.0801
φ_3	0.0074	0.0135	0.0222
φ_4	0.0014	0.0031	0.0059
φ_5	2.10^{-4}	6.10^{-4}	0.0016
$\vdots$	$\vdots$	$\vdots$	$\vdots$
φ_{10}	5.10^{-8}	3.10^{-8}	2.10^{-6}

Therefore, the exact solution of the problem is :

$$u = x^2. \tag{4.17}$$

5. CONCLUSION

In this paper, an improved version of the SOME BLAISE ABBO (SBA) method is presented for finding exact solutions of fractional Fredholm integro-differential equations in the Caputo sense. This technique is used to overcome the difficulties associated with computing Adomian polynomials. Compared with existing classical numerical methods, it is easy to see that this approach is simple, easy to understand and fast, requiring far fewer calculations to find the exact solution to the problem, where it exists. Numerical results obtained with the technique confirm its ease, accuracy and efficiency.

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